HIGH ENERGY ASTROPHYSICS - Lecture 7

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Wednesday

(Inverse) Compton Scattering

1 Overview

- Compton Scattering, polarised and unpolarised light
- Differential cross-section $d\sigma/d\Omega$ and total cross-section σ
- Compton kinematics $\epsilon_f(\epsilon_i, \theta)$
- Thomson ($\epsilon_f \simeq \epsilon_i$) and Klein-Nishina regime
- Inverse Compton scattering
- Energy change in Inverse Compton scattering

2 Thomson Scattering of Polarized Radiation by an electron

Consider radiation from a free electron in response to *incident linearly polarised electromagnetic wave.*

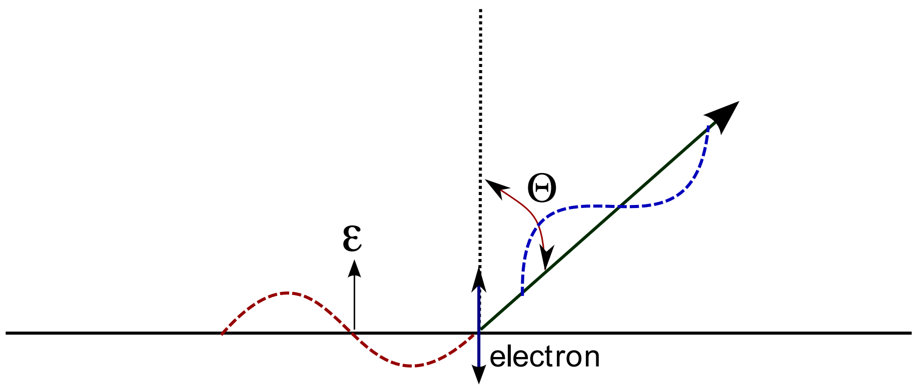
Force on charge (neglecting magnetic force for $v \ll c$):

$$\vec{F} = m_e \ddot{\vec{r}} = e\ E_0 \vec{\epsilon}\ \sin \omega_0 t$$

with $\vec{\epsilon}$ denoting E-field direction.

With dipole moment $\vec{d} := e\vec{r}$:

$$\ddot{\vec{d}} = e\ddot{\vec{r}} = \frac{e^2 E_0}{m_e}\ \vec{\epsilon}\ \sin \omega_0 t\ \Rightarrow\ \vec{d} = -\frac{e^2 E_0}{m_e \omega_0^2}\ \vec{\epsilon}\ \sin \omega_0 t$$



Power of radiating dipole (cf. Larmor's formula, lecture 4):

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\dot{\vec{v}}|^2 \sin^2 \Theta = \frac{\ddot{\vec{d}}^2}{4\pi c^3} \sin^2 \Theta$$

and by intergration over solid angle (Larmor's formula):

$$P = \frac{2q^2}{3c^3} |\dot{\vec{v}}|^2 = \frac{2\ddot{\vec{d}}^2}{3c^3}$$

Time-averaged power: Averaging $\ddot{\vec{d}}^2 = \frac{e^4 E_0^2}{m_e^2} \sin^2 \omega_0 t$ over time t, with $< \sin^2 \omega_0 t >= \frac{1}{T} \int_0^{T=\frac{2\pi}{\omega_0}} \sin^2 \omega_0 t \; dt = 1/2$ gives:

$$\frac{dP}{d\Omega} = \frac{e^4 E_0^2}{8\pi m_e^2 c^3} \sin^2 \Theta \quad \text{and} \quad P = \frac{e^4 E_0^2}{3 m_e^2 c^3}$$

(Note: Θ:= angle between $\ddot{\vec{d}}$ and $\vec{n}$!)

Incident (time-averaged) radiation flux on electron (with $|\vec{S}| = \frac{c}{4\pi} E_0^2 \sin^2 \omega_0 t$):

$$< |\vec{S}| >= \frac{c}{8\pi} E_0^2$$

Differential cross-section $d\sigma$ for scattering into $d\Omega$, is defined as

$$\frac{dP}{d\Omega} :=< |\vec{S}| > \frac{d\sigma}{d\Omega} = \frac{cE_0^2}{8\pi} \frac{d\sigma}{d\Omega}$$

With $\frac{dP}{d\Omega} = \frac{e^4 E_0^2 \sin^2 \Theta}{8\pi m_e^2 c^3}$:

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{polarized} = \frac{e^4}{m_e^2 c^4} \sin^2 \Theta =: r_0^2 \sin^2 \Theta}$$

with classical electron radius

$$r_0 := \frac{e^2}{m_e c^2} = 2.82 \times 10^{-13} \text{ cm}$$

Visualization: $d\sigma$ gives area presented by electron to a photon that is going to be scattered in direction $d\Omega$.

Total cross-section σ by integrating over solid angle, or immediately from

$$P = <|\vec{S}|> \sigma \quad \Rightarrow \quad \sigma = P\,/<|\vec{S}|>$$

where $P = (e^4 E_0^2)/(3m_e^2 c^3) = E_0^2 r_0^2 c/3$ and $<|\vec{S}|>= cE_0^2/(8\pi)$, giving:

$$\sigma = \frac{8\pi}{3} r_0^2 =: \sigma_T$$

with Thomson cross-section σ_T:

$$\boxed{\sigma_T = 6.652 \times 10^{-25} \quad \mathrm{cm}^2}$$

Note:

1. σ_T applies only to non-relativistic regime; for higher energies, Klein-Nishina cross-section σ_{KN} must be used (see later)
2. Scattered radiation is *linearly polarised* in direction of incident polarisation vector, $\vec{\epsilon}$, and direction of scattering, $\vec{n}$.
3. Single particle (time-averaged) Thomson power $P = \sigma_T <|\vec{S}|>= \sigma_T c\,(E_0^2/8\pi) = \sigma_T c U_{rad}$ with radiation energy density $U_{rad} = E_0^2/8\pi$ =time-averaged energy density in incident wave.

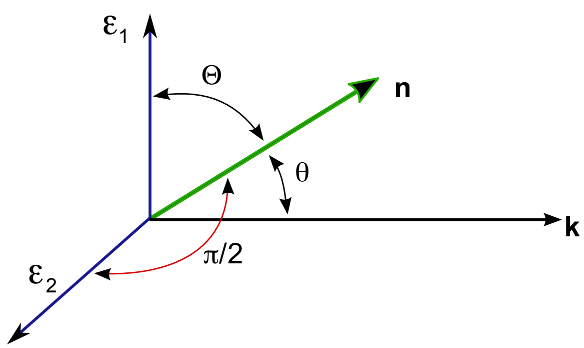
3 Thomson Scattering of Unpolarized Radiation

Unpolarised radiation: can have E-field in any direction = independent superposition of two linearly polarized beams with perpendicular axes.

To scatter non-polarised radiation propagating in direction $\mathbf{k}$ into direction $\mathbf{n}$, need to average two scatterings through angle Θ and $\pi/2$:

$$\left.\frac{d\sigma}{d\Omega}\right|_{unpol} = \frac{1}{2}\left(\left.\frac{d\sigma(\Theta)}{d\Omega}\right|_{pol} + \left.\frac{d\sigma(\pi/2)}{d\Omega}\right|_{pol}\right)$$
$$= \frac{r_0^2}{2}(\sin^2\Theta + 1) = \frac{r_0^2}{2}(\cos^2\theta + 1) = \frac{3\sigma_T}{16\pi}(1+\cos^2\theta)$$

where $\theta = \angle(\mathbf{k}, \mathbf{n})$. **Forward-backward symmetry ($\theta \to \pi + \theta$)!**
Total cross-section, integrated over Ω, is again $\sigma = \sigma_T$.



4 Thomson Optical Depth

=Probability for a photon to experience Thomson scattering while traversing region containing free electrons with density n_e

$$\boxed{\tau_T := \int n_e \sigma_T \, ds}$$

For $\tau_T \ll 1$ source optically-thin to Thomson scattering, for $\tau_T \gg 1$ optically-thick.

Example - *Thomson scattering in black hole accretion flows:*
Ion density = electron density, in terms of dimensionless accretion rate $\dot{m} := \dot{M}/\dot{M}_{Edd}$, where $\dot{M}_{Edd} \simeq 2.2 \; (M_{BH}/10^8 M_\odot) \; M_\odot/\mathrm{yr} \simeq 10^{26} M_{BH,8}$ g/sec

$$n(r) = \frac{\dot{M}}{4\pi r^2 m_p v_r} = 2.5 \times 10^{11} \alpha^{-1} \dot{m} \left(\frac{10^8 M_\odot}{M_{BH}} \right) \left(\frac{r_s}{r} \right)^{3/2} \quad \mathrm{cm}^{-3}$$

assuming radial inflow velocity $v_r = \alpha \; (GM/r)^{1/2} = \alpha \; c \; (r_s/r)^{1/2}/\sqrt{2}$ (fraction $\alpha < 1$ of free infall); Schwarzschild radius $r_s := 2GM/c^2 = 3 \times 10^{13} \; M_{BH,8}$ cm.

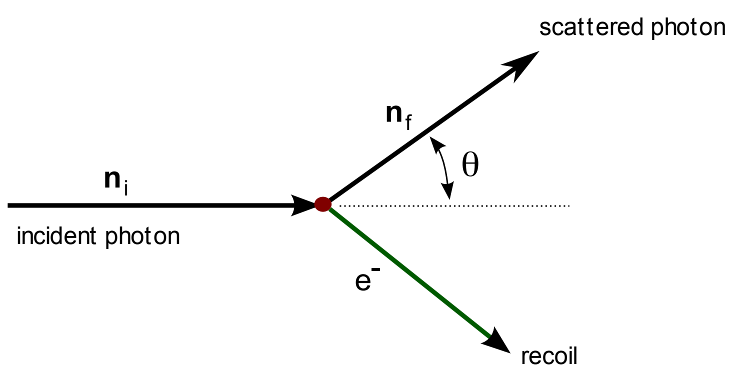
$\Rightarrow$ Thomson optical depth: $\tau_T \simeq n_e(r) \sigma_T r \simeq 5 \; \alpha^{-1} \dot{m} \; \left(\frac{r_s}{r}\right)^{1/2}$
$\Rightarrow$ optically-thin for low accretion rates.

5 Kinematics of Compton Scattering

Scattering of photon off an *electron at rest*; with momentum 4-vector $(E/c, \vec{p})$ notation:

- Initial and final four momentum of photon: $\tilde{P}_i = \frac{h\nu_i}{c}(1, \vec{n}_i), \tilde{P}_f = \frac{h\nu_f}{c}(1, \vec{n}_f)$

- Initial and final four momenta of electron: $\tilde{Q}_i = m_e(c, 0), \tilde{Q}_f = \gamma_f m_e(c, \vec{v}_f)$



- Energy and momentum conservation: $\tilde{P}_i + \tilde{Q}_i = \tilde{P}_f + \tilde{Q}_f$

$$\Rightarrow \tilde{Q}_f^2 = (\tilde{P}_i + \tilde{Q}_i - \tilde{P}_f)^2 = \tilde{P}_i^2 + \tilde{Q}_i^2 + \tilde{P}_f^2 + 2\tilde{P}_i\tilde{Q}_i - 2\tilde{P}_i\tilde{P}_f - 2\tilde{Q}_i\tilde{P}_f$$

With $\tilde{Q}^2 = \tilde{Q}^\nu \tilde{Q}_\nu = \gamma^2 m_e c^2 - \gamma^2 m_e v^2 = \gamma^2 m_e c^2 (1 - \frac{v^2}{c^2}) = m_e c^2$, and $\tilde{P}^2 = 0$:

$$\Rightarrow \tilde{P}_i \cdot \tilde{P}_f = \tilde{Q}_i(\tilde{P}_i - \tilde{P}_f)$$

or:

$$\frac{h\nu_i h\nu_f}{c^2}(1 - \vec{n}_i \vec{n}_f) = m_e(h\nu_i - h\nu_f)$$

With $\vec{n}_i \vec{n}_f = \cos\theta$:

$$h\nu_i \nu_f (1 - \cos\theta) = m_e c^2 (\nu_i - \nu_f)$$

So that

$$\boxed{\nu_f = \frac{\nu_i}{1 + \frac{h\nu_i}{m_e c^2}(1 - \cos\theta)}} \quad \text{or} \quad \boxed{\epsilon_f = \frac{\epsilon_i}{1 + \frac{\epsilon_i}{m_e c^2}(1 - \cos\theta)}}$$

or in terms of wavelength $\lambda = c/\nu$:

$$\lambda_f - \lambda_i = \frac{h}{m_e c}(1 - \cos\theta) =: \lambda_c \, (1 - \cos\theta) \geq 0$$

with Compton wavelength $\lambda_c := h/m_e c = 2.426 \times 10^{-10}$ cm.

Note: $\lambda_f > \lambda_i$ for all angles θ ($\theta \neq 0$), photons always loses energy.

Fractional energy change: averaging over θ, with Taylor expansion for $h\nu_i \ll m_e c^2$ (=Thomson regime):

$$\begin{aligned} \frac{\Delta\epsilon}{\epsilon} &:= \frac{h\Delta\nu}{h\nu_i} = \frac{\nu_f - \nu_i}{\nu_i} \simeq \frac{1}{\nu_i}\left[\nu_i\left(1 - \frac{h\nu_i}{m_e c^2}[1-\cos\theta]\right) - \nu_i\right] = -\frac{h\nu_i}{m_e c^2}\,[1- < \cos\theta >] \\ &= -\frac{h\nu_i}{m_e c^2} \ll 1 \end{aligned}$$

Compton scattering: Scattering of photon off an electron accompanied by energy transfer (decrease in photon energy).

- *Thomson scattering:* in regime $h\nu_i \ll m_e c^2$, transfer small, scattering almost **elastic** ($\epsilon_i = h\nu_i = h\nu_f = \epsilon_f$), Thomson cross-section applies (initial and final wavelength quasi identical).
- *Klein-Nishina scattering:* in regime $h\nu_i > m_e c^2$ transfer large, scattering deeply **inelastic**, need to use cross-section derived from QED.

6 Klein-Nishina Formula

General formula for differential cross-section derived by Klein & Nishina 1929 based on QED:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{3}{16\pi}\sigma_T \left(\frac{\epsilon_f}{\epsilon_i}\right)^2 \left(\frac{\epsilon_i}{\epsilon_f} + \frac{\epsilon_f}{\epsilon_i} - \sin^2\theta\right)}$$

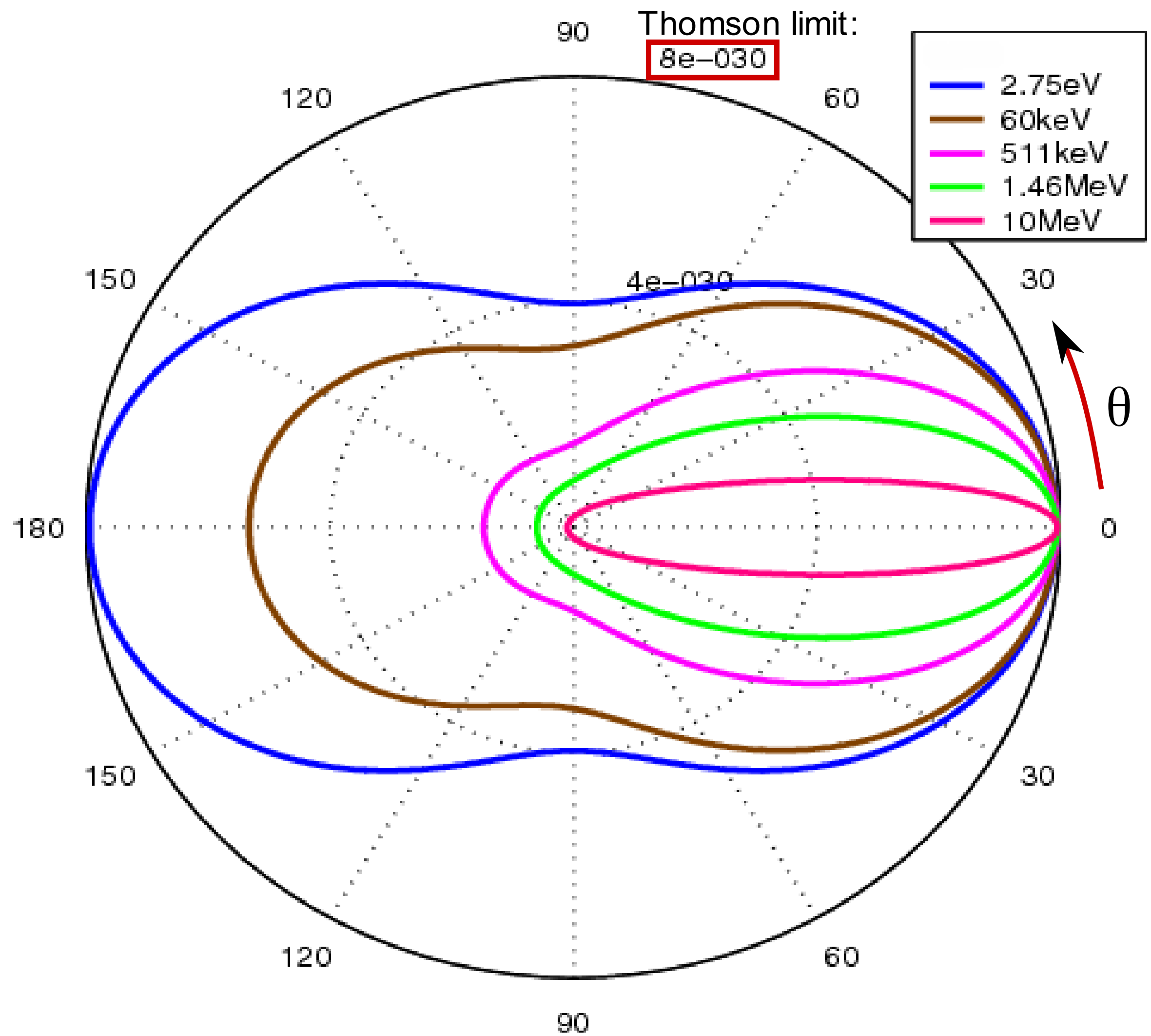
with $\epsilon = h\nu$ and $\epsilon_f = \epsilon_i/(1 + \frac{\epsilon_i}{mc^2}[1 - \cos\theta])$ (kinematics). $d\sigma/d\Omega$ measures probability that photon gets scattered into angle θ.
If $\epsilon_i \simeq \epsilon_f$ (for $\epsilon_i \ll m_e c^2$) *Thomson regime*:

$$\frac{d\sigma}{d\Omega} \to \frac{3}{16\pi}\sigma_T \left(1 + [1 - \sin^2\theta]\right) = \frac{3}{16\pi}\sigma_T \left(1 + \cos^2\theta\right) = \left.\frac{d\sigma}{d\Omega}\right|_{Thomson,unpolarized}$$

Note:

- Principal effects is to reduce cross-section from classical value σ_T as energy increases.
- Increased forward-scattering with increasing energy. At small energies, cross-section is forward-backward $(\theta, \pi + \theta)$ symmetric.

$d\sigma/d\Omega$ [m^2/sr]



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7 Total Compton Cross-section

Integration over solid angle gives total cross-section:

$$\begin{aligned}\sigma &= 2\pi \int_0^{\pi} \frac{d\sigma}{d\Omega} \sin\theta d\theta \\ &= \ldots.. \\ &= \frac{3}{4}\,\sigma_T \left[\frac{1+x}{x^3}\left(\frac{2x(1+x)}{1+2x} - \ln(1+2x)\right) + \frac{1}{2x}\ln(1+2x) - \frac{1+3x}{(1+2x)^2}\right]\end{aligned}$$

where

$$x \equiv \frac{h\nu_i}{m_e c^2}$$

Limits:

$$\sigma(x) \simeq \sigma_T\,(1 - 2x + \ldots) \quad \text{for} \quad x \ll 1 \quad \text{(Thomson)}$$

$$\sigma(x) \simeq \frac{3}{8}\sigma_T \frac{1}{x}\left(\ln 2x + \frac{1}{2}\right) \quad \text{for} \quad x \gg 1 \quad \text{(extreme KN)}$$

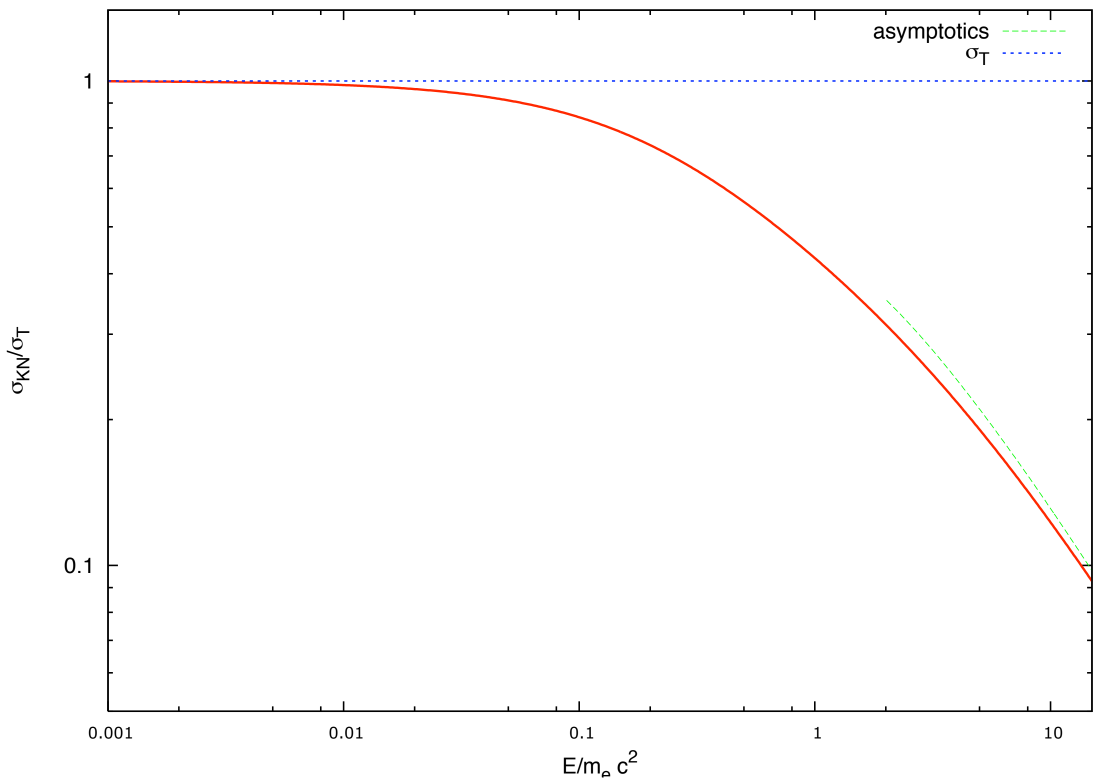

Figure 1: Total Compton cross-section as a function of normalized photon energy $x = h\nu_i/m_e c^2$ along with asymptotics to high energies.

8 Inverse Compton Scattering

If electron moves with velocity v (lab. frame K), energy can be transferred from electron to photon = *Inverse Compton Scattering* (ICS).

In electron rest frame K', previous result holds (written in terms of rest-frame variables = primed), e.g.:

$$\epsilon'_f = \frac{\epsilon'_i}{1 + \frac{\epsilon'_i}{m_e c^2}(1 - \cos\alpha')} \simeq \epsilon'_i \left[1 - \frac{\epsilon'_i}{m_e c^2}(1 - \cos\alpha')\right]$$

with α' scattering angle in K', θ'_i, θ'_f polar angles between electron and photon propagation directions:

$$\cos\alpha' = \cos\theta'_i \cos\theta'_f + \sin\theta'_i \sin\theta'_f \cos(\phi'_i - \phi'_f)$$

where ϕ'_i, ϕ'_f azimuthal angles of incident and scattered photon in electron rest frame [ERF], noting that $\cos\alpha' = \vec{n}'_f \cdot \vec{n}'_i$ and in spherical coordinates $\vec{n}'_i = (\cos\theta'_i, \sin\theta'_i \cos\phi'_i, \sin\theta'_i \sin\phi'_i)$ with $\cos(a-b) = \cos a \cos b + \sin a \sin b$ etc. Last relation on rhs in energy eq. valid in Thomson regime.

Photon energies ϵ's in K' and K are related by Doppler formula

$$\epsilon_i = D\epsilon'_i \quad \leftrightarrow \quad \epsilon'_i = \epsilon_i \gamma(1 - \beta \cos\theta_i) \tag{1}$$

$$\epsilon_f = \frac{\epsilon'_f}{\gamma(1 - \beta\cos\theta_f)} = \epsilon'_f \gamma(1 + \beta \cos\theta'_f) \tag{2}$$

where $D = 1/(\gamma[1 - \beta\cos\theta])$, $\beta = v/c$, $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Thomson regime for $\epsilon'_i \ll m_e c^2$, i.e.,

$$\epsilon_i \ll m_e c^2/\gamma$$

Limits:

(1) for $\theta_i = 0$ (photon approaches from behind): $\epsilon'_i = \epsilon_i \gamma(1 - \beta) \to \epsilon_i/[2\gamma]$.

(2) for $\theta_i = \pi$ (head-on collision): $\epsilon'_i = \epsilon_i \gamma(1 + \beta) \to 2\gamma\epsilon_i$

$\Rightarrow$ Maximum energy in Thomson regime (using equation (2) above):

$\epsilon_{f,max} = 2\epsilon'_f \gamma = 2\epsilon'_i \gamma = 4\gamma^2 \epsilon_i$.

$\Rightarrow$ in Klein-Nishina regime: $\epsilon_{f,max} < \gamma m_e c^2 + \epsilon_i \sim \gamma m_e c^2$ (energy conservation).

Alternatively, using *aberration formula* (lecture 4):

$$\cos\theta'_{i,f} = \frac{\cos\theta_{i,f} - \beta}{1 - \beta\cos\theta_{i,f}}$$

Isotropic distribution in lab. frame K: half the photons have θ_i between π (head-on) and $\pi/2$.

$\Rightarrow$ in electron rest frame K', $\cos\theta'_i = -\beta$ for $\theta_i = \pi/2$.

$\Rightarrow$ For relativistic electrons ($\beta \simeq 1$), most photons are close to head-on in ERF.

In **Thomson regime:** for $\epsilon'_i \ll m_e c^2$, $\epsilon'_i = \epsilon'_f$ with eqs. (1),(2) before:

$$\begin{aligned}
\epsilon_f &= \gamma^2\epsilon_i(1-\beta\cos\theta_i)(1+\beta\cos\theta'_f) = \gamma^2\epsilon_i(1-\beta\cos\theta_i)\left(1+\beta\frac{\cos\theta_f - \beta}{1-\beta\cos\theta_f}\right) \\
&= \gamma^2\epsilon_i\frac{(1-\beta\cos\theta_i)}{(1-\beta\cos\theta_f)}(1-\beta^2) = \epsilon_i\frac{(1-\beta\cos\theta_i)}{(1-\beta\cos\theta_f)}
\end{aligned}$$

For head-on scattering $\theta_i = \pi$ and $\theta_f = 0$ (photons turns around after scattering),

$$\frac{\epsilon_{f,max}}{\epsilon_i} = \frac{(1+\beta)}{(1-\beta)} = \gamma^2(1+\beta)^2 \simeq 4\gamma^2\,.$$

Summary: Scattered photon energy in lab frame:

$$\epsilon_f \simeq \begin{cases} \gamma^2 \epsilon_i & , \epsilon_i \ll m_e c^2/\gamma \quad \text{Thomson regime} \\ \gamma m_e c^2 & , \epsilon_i \gg m_e c^2/\gamma \quad \text{Klein} - \text{Nishina limit} \end{cases}$$

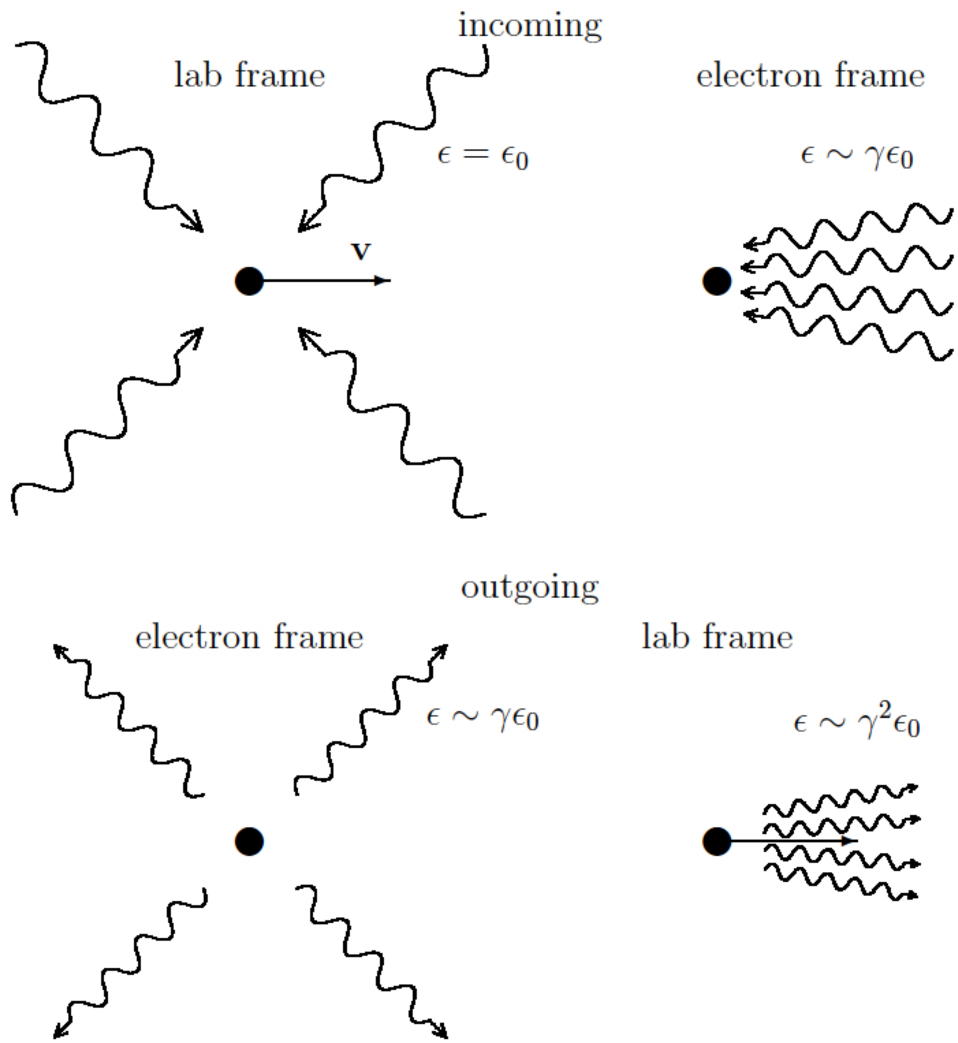

Figure 2: γ^2-energy boost in Thomson regime as a result of relativistic beaming. *Top left:* Electron moving with velocity v in lab frame, incoming photons are isotropically distributed. *Top right:* Incoming photons as seen in the rest frame of the electron. They are now highly anisotropic, electron sees them as nearly head-on, their typical energies are boosted by a factor $\sim \gamma$. *Bottom left:* Photons after scattering in electron rest frame. They are approximately isotropic and have roughly the same energy (Thomson regime) they had before being scattered. *Bottom right:* Scattered photons as seen in the lab-frame. They are now again highly collimated, with their typical energies boosted by a further factor $\sim \gamma$, so that in lab-frame the overall energy is boosted by a factor γ^2 [Credits: N. Kaiser].