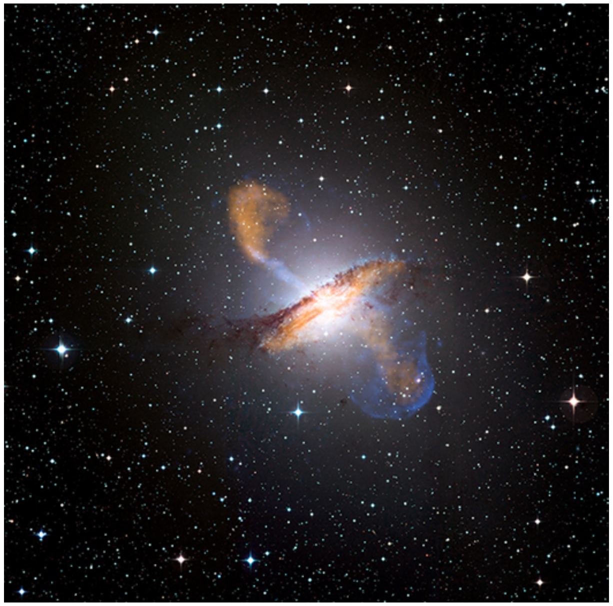
HIGH ENERGY ASTROPHYSICS - Lecture 4



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Wednesday

RADIATION FROM ACCELERATED CHARGES

1 Overview

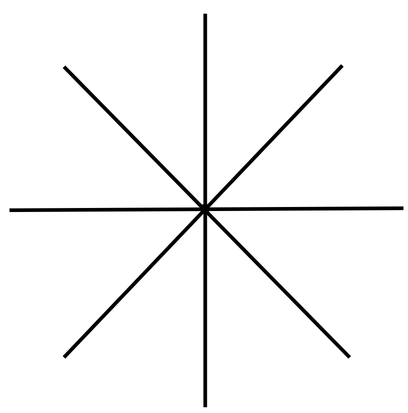
- Radiation from moving charges, Larmor's formula (non-relativistic version)
- Lorentz Transformations and Relativistic Invariants
- Beaming and Relativistic Larmor's formula
- Relativistic Doppler effect (emitted-received)
- Example: Relativistic Jet Sources, Superluminal Motion, de-/beaming, etc

2 Radiation from Accelerated Charges

"Larmor's formula" = accelerated charges emit radiation.

Heuristic derivation following treatment by J.J. Thomson (cf. Longair §6.2):

Charge q accelerated to $\Delta v \ll c$ in short interval Δt. After $t \gg \Delta t$, field lines outside sphere $r = ct$ do not know that charges has moved, still radially centered on origin at $t = 0$. Inside field lines radially centered on moving charge. Transition region $c\Delta t$ where fields have to join up $\Rightarrow$ non-radial component.



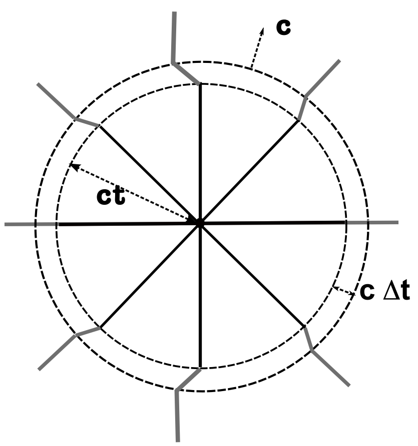


Figure 1: Electric field lines for a charge accelerated to Δv in Δt along x-axis. After some time t, field configuration inside and outside sphere of radius $r = ct$ can be distinguished. Transition layer thickness $c\Delta t$.

Observer in shell measures temporal change in E-field/propagating pulse = electromagnetic radiation.

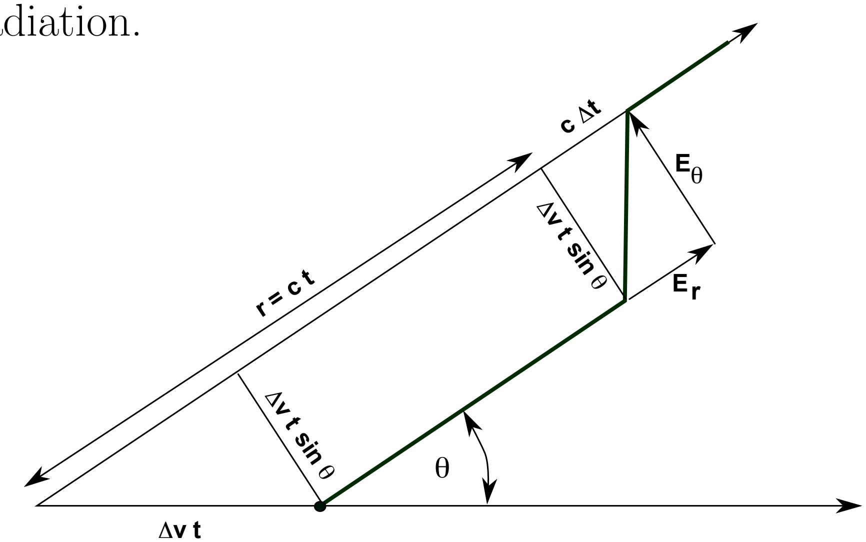


Figure 2: Focus on the pulse related to acceleration phase. Further field modifications due to constant velocity phase ignored (!) as this information has not yet travelled out.

Ratio of E-field components in pulse region for direction θ (angle between acceleration vector and field line):

$$\frac{E_\theta}{E_r} = \frac{\Delta v\ t \sin\theta}{c\Delta t}$$

From Coulomb's law [cgs]:

$$E_r = \frac{q}{r^2}$$

Observing at time t implies $r = ct$, so

$$E_\theta = E_r \frac{\Delta v}{\Delta t} \frac{t \sin\theta}{c} = \frac{q}{r} \frac{\Delta v}{ct} \frac{t \sin\theta}{\Delta t} \frac{}{c}$$

In limit $\Delta t \to 0$, we have $\Delta v / \Delta t \to |\dot{\vec{v}}| = |\vec{a}|$ acceleration, so

$$\boxed{E_\theta = \frac{q}{rc^2} |\dot{\vec{v}}| \sin\theta \propto \frac{1}{r}}$$

E-field in θ-direction changes from 0 to E_θ and back to 0.

Propagating electromagnetic wave carries energy. Rate of energy flow in direction $\vec{n}$ per unit area per sec, $\vec{S}$ [erg cm^{-2} s^{-1}], given by Poynting's theorem (elm plane wave)

$$\boxed{\vec{S} = \frac{c}{4\pi} (\vec{E}_{rad} \times \vec{B}) = \frac{c}{4\pi} E_{rad}^2 \, \vec{n} \simeq \frac{q^2}{4\pi c^3 r^2} |\dot{\vec{v}}|^2 \sin^2\theta \, \vec{n}}$$

Power = Rate of energy flow per sec, multiplied by area $dA = r^2 d\Omega$ subtended by solid angle $d\Omega = \sin\theta d\theta d\phi$ at angle θ and distance r from charge.

$$\frac{dP}{d\Omega} := -\frac{dE}{dt d\Omega} = -\frac{dE}{dt d\Omega} \frac{r^2}{r^2} = |\vec{S}| r^2 \propto \sin^2\theta$$

Total loss rate via integration over solid angle (with $\int_0^\pi \sin^3\theta d\theta = 4/3$)

$$-P = \frac{dE}{dt} = -\frac{q^2}{4\pi c^3}|\dot{\vec{v}}|^2 \int_0^{2\pi} d\phi \int_0^\pi \sin^3\theta d\theta = -\frac{2}{3}\frac{q^2}{c^3}|\dot{\vec{v}}|^2$$

Larmor's formula = power radiated by non-relativistic charge [erg/sec]:

$$\boxed{P = -\frac{dE}{dt} = \frac{2}{3}\frac{q^2}{c^3}|\dot{\vec{v}}|^2}$$

Notes:

1. Total power $P \propto q^2 \dot{v}^2$
2. Dipole pattern: Power radiated per unit solid angle $dP/d\Omega \propto \sin^2\theta$. No radiation emitted $\|$ to $\dot{\vec{v}}$, maximum radiation emitted $\perp$ $\dot{\vec{v}}$.
3. Direction of $\vec{E}_{rad}$ determined by $\dot{\vec{v}}$. For acceleration along a line, observed radiation linearly polarized in plane of $\dot{\vec{v}}$ and $\vec{n} = \vec{R}/R$.
4. Force required to produce acceleration $F = ma$, so $a \propto 1/m$, i.e. $P \propto 1/m^2$. $\Rightarrow$ Electrons are much better at radiating than protons.

3 Lorentz Transformations (recap)

Consider frame K' moving with uniform velocity v with respect to a frame K. Take motion along x-axis:

$$\begin{aligned} x' &= \gamma(x - vt) \quad (1)\\ y' &= y \quad (2)\\ z' &= z \quad (3)\\ ct' &= \gamma\left(ct - \frac{v}{c}x\right) \quad (4) \end{aligned}$$

with Lorentz factor

$$\gamma := \frac{1}{(1 - v^2/c^2)^{1/2}} \geq 1$$

Inverse transformation:

$$\begin{aligned} x &= \gamma(x' + vt')\\ y &= y'\\ z &= z'\\ ct &= \gamma\left(ct' + \frac{v}{c}x'\right) \end{aligned}$$

(interchange prime and unprimed quantities, replace $v \to -v$).

4 Contraction & Dilation

- **Length contraction**: Stick of length $dx' = x'_1 - x'_2$ carried in K'. Length of this stick in K (measured at same [!] time t in K):

$$dx' = x'_1 - x'_2 = \gamma(x_1 - vt) - \gamma(x_2 - vt) = \gamma(x_1 - x_2) = \gamma dx$$

$$\Rightarrow \boxed{dx = \frac{dx'}{\gamma}}$$

Length of a moving object measured along (!) its direction of motion is shorter than length as measured in proper frame of object.

- **Time dilation**: Clock at rest in moving frame K': time interval $dt' = t'_1 - t'_2$ (x' remains constant, $dx' = 0$). Time interval as measured in K:

$$dt = t_1 - t_2 = \gamma(t'_1 + \frac{v}{c^2}x') - \gamma(t'_2 + \frac{v}{c^2}x') = \gamma(t'_1 - t'_2) = \gamma dt'$$

$$\Rightarrow \boxed{dt = \gamma dt'}$$

Time interval in K has increased, moving clock appears to have slowed down.

5 Transformation of Velocities

Differentials perpendicular to motion do not change, $dy = dy'$, $dz = dz'$.

$$u_x := \frac{dx}{dt} = \frac{\gamma(dx' + v dt')}{\gamma\left(dt' + \frac{v}{c^2} dx'\right)} = \frac{u'_x + v}{1 + \frac{v}{c^2}\, u'_x} \tag{5}$$

$$u_y := \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{v}{c^2} dx'\right)} = \frac{u'_y}{\gamma(1 + \frac{v}{c^2}\, u'_x)} \tag{6}$$

$$u_z := \frac{dz}{dt} = \frac{dz'}{\gamma\left(dt' + \frac{v}{c^2} dx'\right)} = \frac{u'_z}{\gamma(1 + \frac{v}{c^2}\, u'_x)} \tag{7}$$

Generalizing to components of $\vec{u}$ perpendicular and parallel to $\vec{v}$:

$$u_\parallel = \frac{u'_\parallel + v}{1 + \frac{v}{c^2}\, u'_\parallel} \tag{8}$$

$$u_\perp = \frac{u'_\perp}{\gamma(1 + \frac{v}{c^2}\, u'_\parallel)} \tag{9}$$

6 Aberration & Beaming

Direction of velocities in K and K' are related by aberration formula:

$$\tan\theta = \frac{u_\perp}{u_\parallel} = \frac{u'_\perp}{\gamma(u'_\parallel + v)} = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)}$$

with $u' = |\vec{u}'|$. Azimuthal angle remains unchanged!

For light, $u' = c$, aberration formula becomes

$$\begin{aligned}
\tan\theta &= \frac{\sin\theta'}{\gamma(\cos\theta' + v/c)} \\
\cos\theta &= \frac{u_\parallel}{u} = \frac{1}{c}\frac{u'_\parallel + v}{1 + \frac{v}{c^2}u'_\parallel} = \frac{1}{c}\frac{c\cos\theta' + v}{1 + \frac{v}{c^2}c\cos\theta'} = \frac{\cos\theta' + v/c}{1 + (v/c)\cos\theta'} \\
\sin\theta &= \frac{u_\perp}{c} = \frac{\sin\theta'}{\gamma(1 + \frac{v}{c}\cos\theta')}
\end{aligned}$$

Consider case $\theta' = \pi/2$ (photon emitted at right angles to $\vec{v}$ in K'):

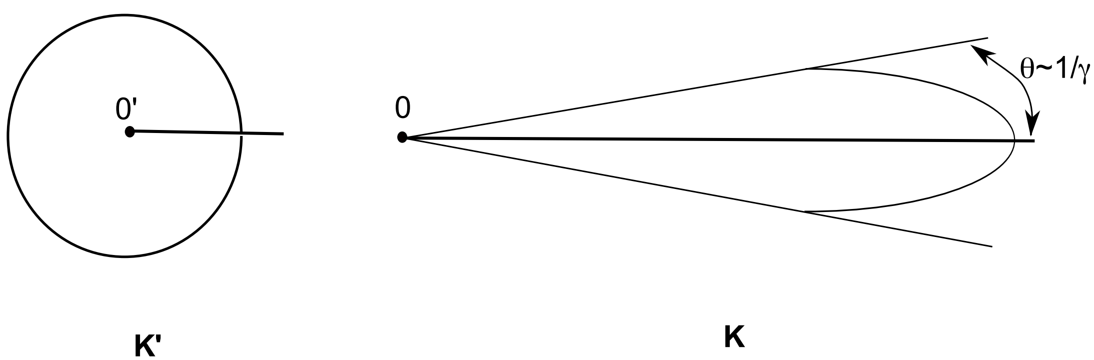
$$\tan\theta = \frac{c}{\gamma v} \quad \text{and} \quad \cos\theta = \frac{v}{c} \tag{10}$$

$$\sin\theta = \sqrt{1 - \cos\theta^2} = \frac{1}{\gamma} \tag{11}$$

$$\tag{12}$$

For highly relativistic speeds, $\gamma \gg 1$, and $\theta \sim 1/\gamma$.

$\Rightarrow$ in K photons are concentrated in cone of half angle $1/\gamma$ = beaming effect.



7 Example: Brightness Increase due to Relativistic Beaming

Light from source emitting isotropically is equally distributed over sphere, so observed flux F [erg/s/cm^2] at distance r:

$$F = \frac{L}{4\pi r^2}$$

with L luminosity of source [erg/s]. If this power is concentrated into solid angle $\Delta\Omega$ instead, observed flux will be:

$$F_{foc} = \frac{L}{\Delta\Omega r^2}$$

Thus, brightness increase due to focusing by factor:

$$b := \frac{F_{foc}}{F} = \frac{4\pi}{\Delta\Omega}$$

Solid angle subtended by cone with opening angle α:

$$\Delta\Omega = \int_0^{2\pi} d\phi \int_0^{\alpha/2} \sin\theta d\theta = 2\pi[-\cos\theta]_0^{\alpha/2} = 2\pi(1 - \cos[\alpha/2])$$

For small x, $\cos x \simeq 1 - x^2/2$, so $\Delta\Omega \simeq \pi\frac{\alpha^2}{4}$. For beaming $\alpha/2 \simeq 1/\gamma$, $\Delta\Omega \simeq \pi/\gamma^2$, so **brightness increase** $b \simeq 4\gamma^2$ $\Rightarrow$ Naively taking $4\pi\times$ (measured flux) as proxy for real source luminosity strongly over-estimates required energetics.

8 Transformation of Acceleration

Have $da := \frac{du}{dt}$. Consider x-component (motion along x-axis), $u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$, use chain rule:

$$\begin{aligned} du_x &= \frac{du'_x(1 + \frac{vu'_x}{c^2}) - (u'_x + v)(\frac{vdu'_x}{c^2})}{(1 + \frac{vu'_x}{c^2})^2} \\ dt &= \gamma\left(dt' + \frac{v}{c^2}dx'\right) = \gamma dt'\left(1 + \frac{vu'_x}{c^2}\right) \end{aligned}$$

So:

$$\begin{aligned} a_x = \frac{du_x}{dt} &= \frac{du'_x(1 + \frac{vu'_x}{c^2}) - (u'_x + v)(\frac{vdu'_x}{c^2})}{\gamma dt'\left(1 + \frac{v}{c^2}u'_x\right)^3} = \frac{du'_x[1 + \frac{vu'_x}{c^2} - \frac{u'_x v}{c^2} - \frac{v^2}{c^2}]}{\gamma dt'(1 + \frac{vu'_x}{c^2})^3} \\ &= \frac{\frac{du'_x}{dt'}[1 - \frac{v^2}{c^2}]}{\gamma(1 + \frac{vu'_x}{c^2})^3} = \frac{a'_x}{\gamma^3(1 + \frac{vu'_x}{c^2})^3} \end{aligned}$$

Do similarly for a_y and a_z. In instantaneous rest frame of particle K' ($\vec{u}' = 0$):

$$\boxed{a'_\parallel = \gamma^3 a_\parallel} \tag{13}$$

$$\boxed{a'_\perp = \gamma^2 a_\perp} \tag{14}$$

9 Relativistic Invariants

Invariants = Quantities that do not change under Lorentz trafo, i.e., stay the same in all inertial frames.

- **Total emitted power:** $dE/dt = dE'/dt'$ is Lorentz invariant.
  1. Energy dE is zero component of momentum four vector $[dE/c, d\vec{p}]$
     (where $E := \gamma mc^2$, $\vec{p} := \gamma m\vec{v}_p$; $\vec{v}_p$ = velocity of particle)
  2. cdt is zero-component of displacement four vector $[cdt, d\vec{r}]$.

  $\Rightarrow$ both components transform in same way between inertial frames
  $\Rightarrow$ ratio dE/dt invariant.

  More detailed: In rest frame of accelerated particle, total energy loss dE' has dipole symmetry (Larmor's formula), thus zero net momentum $d\vec{p}' = 0$.
  Energy trafo $dE = \gamma(dE' + udp'_u) = \gamma dE'$.
  Time trafo $dt = \gamma dt'$ (dt' proper time).

$$\Rightarrow \quad \boxed{P := \frac{dE}{dt} = \frac{dE'}{dt'} =: P'}$$

- **Phase space volume:** $dV_{ps} = dV'_{ps}$ is Lorentz invariant

  where $dV_{ps} := d^3\mathbf{x}d^3\mathbf{p}$, $d^3\mathbf{x} := dxdydz$, and $d^3\mathbf{p} := dp_x dp_y dp_z$.

  Consider particles with small spread in position and momentum (but not energy, $dE' = 0$) in K':

  1. In K, $d^3\mathbf{x} = \gamma^{-1} d^3\mathbf{x}'$ due to length contraction in x-direction.
  2. Momentum transforms as four vector ($dp_y dp_z = dp'_y dp'_z$) with

  $$dp_x = \gamma(dp'_x + v dE'/c^2) = \gamma dp'_x$$

  $$\Rightarrow \quad \boxed{dV_{ps} := d^3\mathbf{x}d^3\mathbf{p} = \frac{d^3\mathbf{x}'}{\gamma}\gamma d^3\mathbf{p}' =: dV'_{ps}}$$

- **Phase space distribution:** $f := \frac{dN}{dV_{ps}}$ is Lorentz invariant.

  Number of particles within phase volume element, dN is countable quantity (conserved), thus $dN = dN'$.
  Phase space element $dV_{ps} = dV'_{ps}$ (just shown)

$$\Rightarrow \boxed{f = \frac{dN}{dV_{ps}} = \frac{dN'}{dV'_{ps}} = f'}$$

- **Intensity**: I_ν/ν^3 is Lorentz invariant.

  Remember (lecture 3):

$$I_\nu = \frac{2h\nu^3}{c^2} f$$

  But f is invariant, so

$$\boxed{\frac{I_\nu}{\nu^3} = \frac{I'_{\nu'}}{\nu'^3}}$$

  Note: Same holds for source function $S_\nu = j_\nu/\alpha_\nu$
  (cf. transfer eq.: $dI_\nu/d\tau_\nu = S_\nu - I_\nu$).

10 Transformation Properties of Radiation Quantities

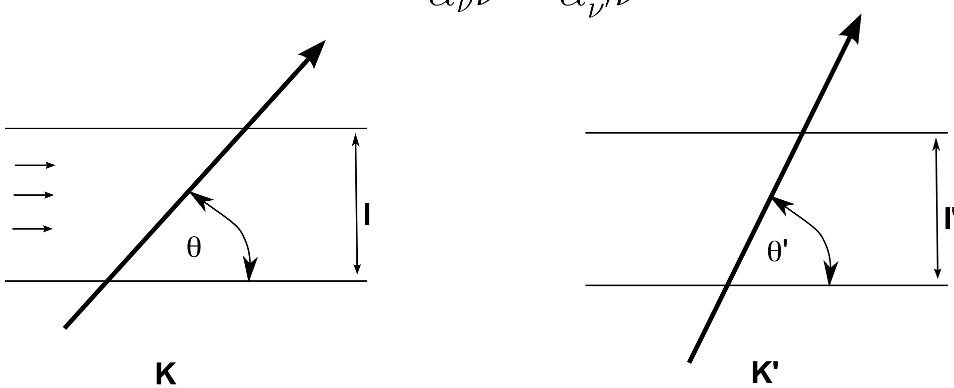
- **Optical depth:** $\tau = \tau'$=invariant, since $e^{-\tau}$ gives fraction of photons passing through material ("counting" - conserved).

- **Absorption coefficient:** $\nu\alpha_\nu$ is invariant. Have $\tau_\nu := \int \alpha_\nu ds$, so

$$\tau = \frac{l\alpha_\nu}{\sin\theta} = \frac{l}{\nu\sin\theta}(\nu\alpha_\nu) \stackrel{!}{=} \tau'$$

But for perpendicular components $l = l'$ and photon 4-momentum $k^\nu = (\omega/c, \vec{k})$, so component $k_y \propto \nu\sin\theta$ with $k_y = k'_y$, thus

$$\alpha_\nu \nu = \alpha'_{\nu'}\nu'$$



- **Emission coefficient:** j_ν/ν^2 is invariant.
  Have $j_\nu = S_\nu \alpha_\nu$. Hence

$$\frac{j_\nu}{\nu^2} = \frac{S_\nu \alpha_\nu}{\nu^2} = \frac{S'_{\nu'} \left(\frac{\nu}{\nu'}\right)^3 (\alpha'_{\nu'} \frac{\nu'}{\nu})}{\nu^2} = \frac{S'_{\nu'} \alpha'_{\nu'}}{\nu'^2} = \frac{j'_{\nu'}}{\nu'^2}$$

  as source function S_ν transforms like intensity $I_\nu = (\nu/\nu')^3 \, I'_{\nu'}$.

- **Number density:** $n = \gamma n'$.

$$n = \frac{dN}{dV} = \frac{dN}{d^3\mathbf{x}} = \frac{dN'}{\gamma^{-1} d^3\mathbf{x}'} = \gamma n'$$

11 Larmor Formula for a Relativistically Moving Particle

Known: $P = \frac{dE}{dt}$=invariant, and in instantaneous rest frame $P' = \frac{2}{3}\frac{q^2}{c^3}|\dot{\vec{v}}\,'|^2$, so

$$\left(\frac{dE}{dt}\right)_K = \left(\frac{dE'}{dt'}\right)_{K'} = \frac{2}{3}\frac{q^2}{c^3}|\dot{\vec{v}}\,'|^2$$

with $\dot{\vec{v}}\,' = \vec{a}'$ and $\vec{a}' \cdot \vec{a}' = a'^2_\perp + a'^2_\parallel$.

Transformation properties for components of acceleration (instantaneous rest frame K') eq. (13f)

$$\begin{aligned} a'_\parallel &= \gamma^3 a_\parallel \\ a'_\perp &= \gamma^2 a_\perp \end{aligned} \tag{15}$$

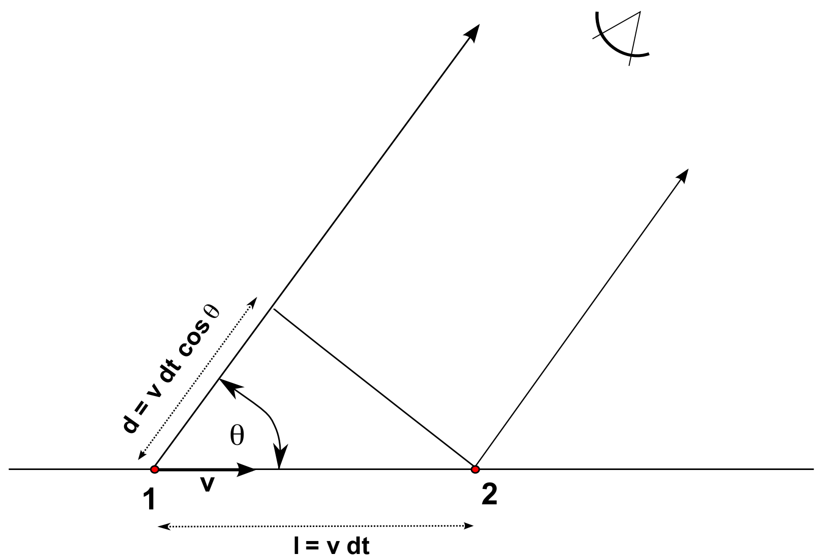
Hence

$$\boxed{\left(\frac{dE}{dt}\right)_K = \frac{2}{3}\frac{q^2\gamma^4}{c^3}(a_\perp^2 + \gamma^2 a_\parallel^2)} \tag{16}$$

(Note: In instantaneous rest frame K' particle has zero velocity at a certain time; it will not remain at rest in this frame, since it can accelerate, but for infinitesimally neighbouring times it will move non-relativistically.)

12 Relativistic Doppler Effect

=frequency relation between by observer **received** and in K' **emitted** radiation (taking travel time into account)



In rest frame K of observer, emitting source moves from 1 to 2 at velocity v. Photons emitted in interval dt'_{em} in moving frame K' of source are separated in K by (time-dilation)

$$dt = \gamma dt'_{em}$$

But in this time, source has moved in K distance $l = vdt$ along axis, and $d = vdt\cos\theta$ towards observer.
Difference in **arrival times** as seen by observer for radiation emitted at 1 and 2

$$\Delta t_A = dt - \frac{d}{c} = dt - \left(\frac{v}{c}dt\cos\theta\right) = dt\left(1 - \frac{v}{c}\cos\theta\right) = \gamma\left(1 - \frac{v}{c}\cos\theta\right) dt'_{em}$$

Using frequencies $\nu_{obs} := \frac{1}{\Delta t_A}$, $\nu' := \frac{1}{dt'_{em}}$, and $\beta := \frac{v}{c}$, then

$$\boxed{\nu_{obs} = \frac{1}{\gamma(1-\beta\cos\theta)}\nu' =: D\nu'} \tag{17}$$

with D = Doppler factor.

Note:
Doppler factor depends on angle between observer and direction of motion and can be very large (for $v \to c$), e.g., for head-on motion ($\theta = 0$) and large speeds

$$D = \frac{1}{\gamma(1-\beta)} = \frac{(1+\beta)}{\gamma(1-\beta)(1+\beta)} = \frac{(1+\beta)}{\gamma\frac{1}{\gamma^2}} \simeq 2\gamma$$

13 Superluminal Motion in AGN

Apparent velocity measured in many AGN jets are $v_{app} > c$

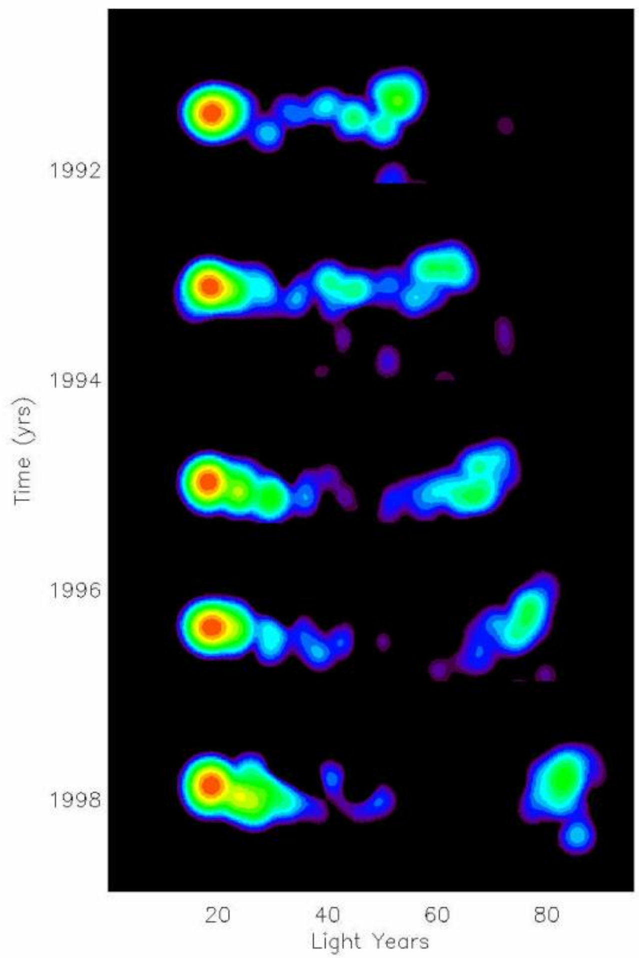
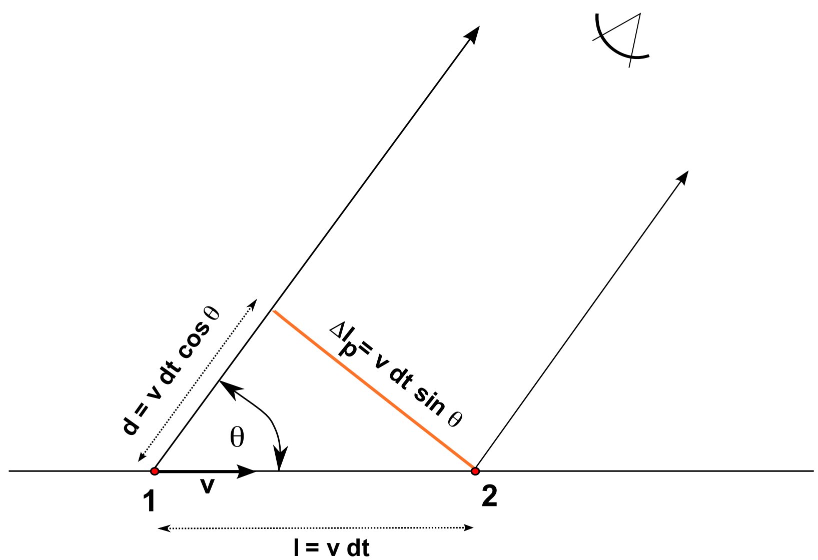


Figure 3: Apparent superluminal motion in the jet of the AGN 3C279 ($z = 0.536$): Rightmost (blue-green) radio (22GHz) blob moved about 25 light years from 1991 to 1998, translating into an apparent speed of $\sim 25/7 = 3.5$ c [Credits: NRAO/AUI].

Explanation: Motion with speed close to c at small viewing angles (in K).



- Consider blob moving with speed v at angle θ to line of sight, emitting light signals at t_1 and $t_2 = t_1 + dt$.
- Light travel time: Observers sees signal separated by:

$$\Delta t_A = dt - \frac{d}{c} = dt \left(1 - \frac{v}{c}\cos\theta\right)$$

- Observed distance traveled in plane of sky:

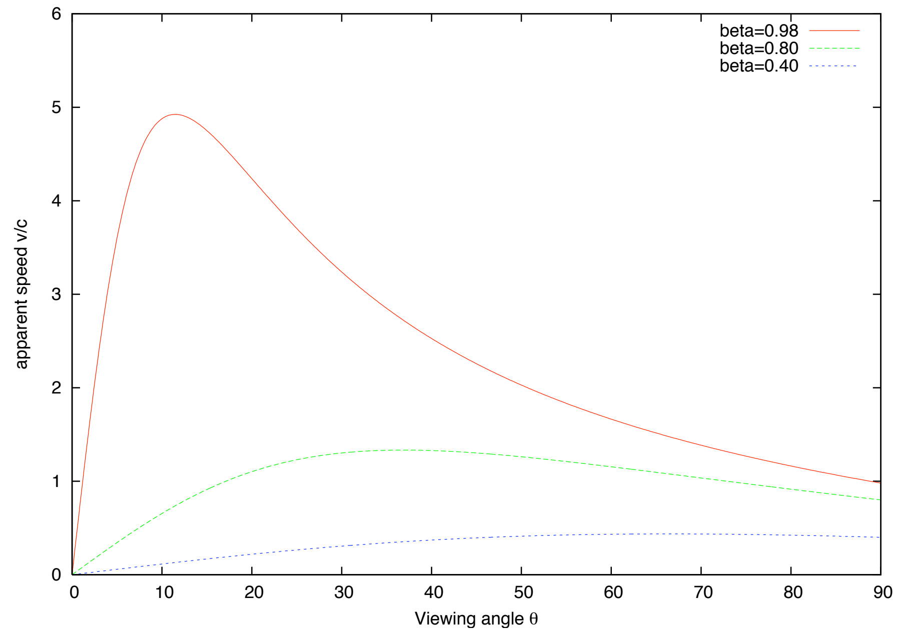
$$\Delta l_p = v dt \sin\theta$$

- Apparent velocity inferred from observations:

$$\boxed{v_{app} = \frac{\Delta l_p}{\Delta t_A} = \frac{v dt \sin\theta}{dt\left(1 - \frac{v}{c}\cos\theta\right)} = \frac{v\sin\theta}{\left(1 - \frac{v}{c}\cos\theta\right)}}$$

$\Rightarrow$ For v/c large and θ small: $v_{app} > c$

- Maximum: $dv_{app}/d\theta = 0$ at $\cos\theta = \frac{v}{c}$, with apparent velocity $v_{app}^{max} = \gamma v$.



14 Solid Angle Trafo for Radiation Emitted within $d\Omega'$ in K'

- Solid angle element $d\Omega' := \sin\theta' d\theta' d\phi' = -d\cos\theta'\, d\phi$
- Azimuthal angle not affected, so: $d\phi' = d\phi$.
- Aberration formula: $\cos\theta' = \frac{\cos\theta - v/c}{1-(v/c)\cos\theta}$.
- Differentiating this gives

$$d\cos\theta' = \frac{d\cos\theta(1-\beta\cos\theta) - (\cos\theta - \beta)[-\beta\ d\cos\theta]}{(1-\beta\cos\theta)^2} = \frac{d\cos\theta\ (1-\beta^2)}{(1-\beta\cos\theta)^2}$$

- Hence

$$\boxed{d\Omega' = -\frac{d\cos\theta}{\gamma^2(1-\beta\cos\theta)^2} d\phi' = \frac{d\Omega}{\gamma^2(1-\beta\cos\theta)^2} = D^2 d\Omega}$$

- Brightness increase due to focusing $b := F_{iso}/F_{foc} = 4\pi/\Delta\Omega$ and $\Delta\Omega = 4\pi/D^2$, so $b = D^2$. For head-on motion ($\theta = 0$) $D \simeq 2\gamma$, so $b \simeq 4\gamma^2$ as before.

15 Luminosity boosting I

Remember: I_ν/ν^3 was invariant under Lorentz transformation ($I_\nu \equiv I(\nu)$ the specific intensity).

- Thus, observed intensity of a moving blob:

$$\frac{I(\nu_{obs})}{\nu_{obs}^3} = \frac{I(\nu')'}{\nu'^3}$$

so

$$I(\nu_{obs}) = I(\nu')' \left(\frac{\nu_{obs}}{\nu'}\right)^3 = D^3 I(\nu')'$$

- If we are interested in energy flux: $\nu_{obs} I(\nu_{obs}) = D^4 \nu' I(\nu')'$
- For a blob with power law $I(\nu')' = A\ \nu'^{-\alpha} = A\ \left(\frac{\nu_{obs}}{D}\right)^{-\alpha} = D^\alpha\ A\nu_{obs}^{-\alpha}$:

$$\boxed{I(\nu_{obs}) = D^{3+\alpha} I(\nu_{obs})'}$$

- **Consequence:** For a relativistic flow with $\beta \simeq 0.97$ ($\gamma = 4$) flux in forward direction can be boosted by a factor ~ 1000, and de-boosted by the same amount in the backwards direction (noting $\theta_b \to \pi + \theta$).

16 Luminosity boosting II

Remember: $dF_\nu := \frac{dE}{dAdtd\nu}$ and $dE = I_\nu \cos\theta dAdtd\Omega d\nu$.

$\Rightarrow$ observed flux density for an optically-thin source $F_\nu := \int_{\text{source}} I_\nu(\theta,\phi)\cos\theta d\Omega$

"integrated over the solid angle subtended by the source".

- For small angular source sizes, $\theta \ll 1$ rad, $\cos\theta \simeq 1$, thus

$$F_\nu \equiv F(\nu) \simeq \int_{\text{source}} I(\nu) d\Omega$$

- Since $d\Omega = dA/d_L^2$, d_L = distance, $F_\nu \propto d_L^{-2}$ (inverse-square law).
  Spectral luminosity $L_\nu := 4\pi d_L^2 F_\nu$ intrinsic property of the source!

- With transformation properties for specific intensity $I(\nu)$:

$$F(\nu) = \int_{source} I(\nu)\, d\Omega = D^3 \int_{source} I(\nu')' \,\frac{dA}{d_L^2} = D^3 \int_{source} j(\nu')' \,\frac{dx' dA}{d_L^2}$$

  with intensity evaluated at transformed frequency $\nu' = \nu/D$.

- **Note:** Assuming beaming, integration over "solid angle subtended by source" utilises ($d\Omega := \sin\theta d\theta d\phi = -d\cos\theta d\phi$ and $d\Omega' = D^2 d\Omega$)

$$\begin{aligned}
\int_{source} d\Omega &= \int_{\theta=0}^{\theta=1/\gamma} d\Omega = -2\pi \int_{\theta=0}^{\theta=1/\gamma} d\cos\theta = -2\pi[\cos\theta]_0^{1/\gamma} \simeq \pi/\gamma^2 \\
&= \int_{\theta'=0}^{\theta'=\pi/2} \frac{1}{D^2} d\Omega' = -2\pi \int_{\theta'=0}^{\theta'=\pi/2} \frac{d\cos\theta'}{\gamma^2(1+\beta\cos\theta')^2} \simeq \pi/\gamma^2
\end{aligned}$$

with $\cos x \simeq 1 - x^2/2$ and $-\int_1^0 dx/(1+x)^2 = [1/(1+x)]_1^0$, and noting that

$$\begin{aligned}
D &= \frac{1}{\gamma(1-\beta\cos\theta)} = \frac{1}{\gamma\left(1-\beta\frac{\cos\theta'+\beta}{1+\beta\cos\theta'}\right)} \\
&= \frac{1+\beta\cos\theta'}{\gamma(1+\beta\cos\theta'-\beta\cos\theta'-\beta^2)} = \gamma(1+\beta\cos\theta')
\end{aligned}$$

17 Advancing and Receding Jets

For jet-like feature=series of unresolved, uniformly-spaced blobs. Need to consider emission per unit length l in the observer frame. Number of blobs observed per unit length $\propto 1/D$, so have one D less in the boosting formula compared with single blob.

Alternatively:

- Observed flux is given by integration over solid angle subtended by the source
$$F_\nu = \int I_\nu d\Omega = \int j_\nu \frac{dl dA}{d_L^2} = \frac{D^2}{d_L^2} \int j'_{\nu'} dV = \frac{D^{2+\alpha}}{d_L^2} \int j'_\nu dV$$
Noting $j_\nu = D^2 j'_{\nu'}$, and with volume integration being performed in observer's frame. Last expression valid for power-law $j'_{\nu'} \propto \nu'^{-\alpha} \propto (\nu/D)^{-\alpha}$ with $\nu = D\nu'$, where $D = 1/[\gamma(1 - \beta \cos\theta)]$.

- For receding jet, $\theta_{rec} = \pi + \theta$, so $\cos\theta_{rec} = -\cos\theta$, $D_{rec} = 1/[\gamma(1 + \beta\cos\theta)] \leq 1/\gamma$, so strong de-boosting $\Rightarrow$ one-sidedness of jet!

Application: Ratio of fluxes measured from (identical) advancing and receding jets:

$$R := \frac{F_\nu^{adv}}{F_\nu^{rec}} = \frac{D_{adv}^{2+\alpha}}{D_{rec}^{2+\alpha}} = \left(\frac{1+\beta\cos\theta}{1-\beta\cos\theta}\right)^{2+\alpha}$$

So

$$R^{1/[2+\alpha]} = \frac{1+\beta\cos\theta}{1-\beta\cos\theta}$$

i.e.

$$\beta\cos\theta + \beta\cos\theta R^{1/[2+\alpha]} = R^{1/[2+\alpha]} - 1$$

thus

$$\boxed{\beta\cos\theta = \frac{R^{1/[2+\alpha]} - 1}{R^{1/[2+\alpha]} + 1}}$$

$\Rightarrow$ Can constrain angle θ from measured flux ratio (upper limit for $\beta = 1$).

18 Example: Relativistic Effects in GRBs

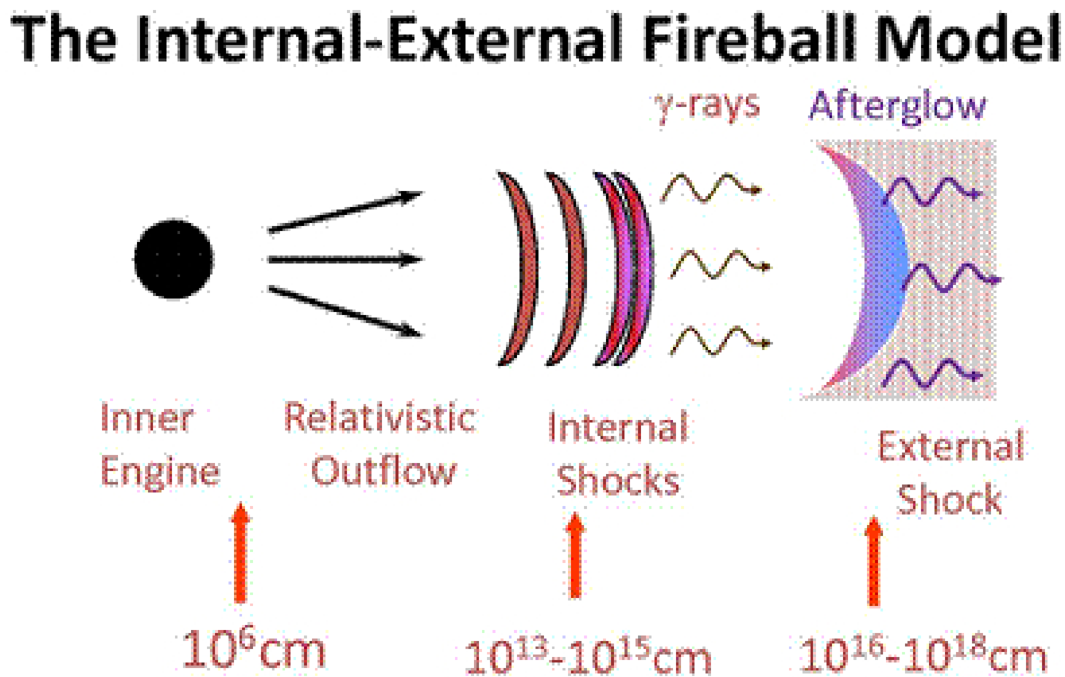


Figure 4: Sketch of GRB model where HE prompt emission is related to internal shocks, and afterglow emission to external shock in ambient medium. The prompt emission (between a few millisec to tens of min) in mostly confined to gamma-rays, while afterglow emission (from weeks to months) is seen in X-rays, optical and radio [Credit: T. Piran].

Prompt HE emission associated with internal shocks = colliding shells

Assume: Central engine ejects two shells separated by time Δt_{var} (observer's frame) with velocity $v_2 > v_1$, (with $v_2 \simeq c$, $\gamma \equiv \gamma(v_1)$).

Collision for $r_1 = r_2$ where

$$r_1 = v_1(t + \Delta t_{var}) \quad \text{and} \quad r_2 = v_2 t$$

viz. at $t = \frac{v_1}{(v_2 - v_1)}\Delta t_{var}$, so radius at which collision occurs:

$$\begin{aligned} r = r_2 &= \frac{v_1 v_2}{v_2 - v_1}\Delta t_{var} = \frac{v_1}{1 - v_1/v_2}\Delta t_{var} = \frac{v_1(1 + v_1/v_2)}{1 - v_1^2/v_2^2}\Delta t_{var} \\ &\simeq \frac{2v_1}{1 - v_1^2/c^2}\Delta t_{var} \\ &\simeq 2c\gamma^2 \Delta t_{var} \end{aligned} \tag{18}$$

Consider **photon-photon pair production** in GRBs ($\gamma + \gamma \to e^+ + e^-$):

Optical depth (pure number) τ is an invariant. Calculate it in flow rest frame

$$\tau_{\gamma\gamma} = \tau'_{\gamma\gamma} \simeq \sigma_{\gamma\gamma} n'_\gamma \Delta r'$$

- Doppler formula $\nu = D\nu' \Rightarrow D\Delta t_{var} = \Delta t'_{var}$. Causality implies

  $$\Delta r' = c\Delta t'_{var} = cD\Delta t_{var} \simeq 2\gamma c\Delta t_{var}$$

  with head-on motion approximation $D \simeq 2\gamma$.

- Photon number density given we know source luminosity

  $$n'_\gamma = \frac{L'_\gamma}{4\pi r'^2 ch\nu'} \simeq \frac{L_\gamma/[2\gamma]^{3+\alpha}}{4\pi r^2 ch\nu/[2\gamma]}$$

  using luminosity boosting (jet scaling, $L \sim F_\nu \nu$) and flux spectrum $F_\nu \propto \nu^{-\alpha}$.

- Using r from above, optical depth becomes

$$\tau_{\gamma\gamma} = \tau'_{\gamma\gamma} \simeq \sigma_{\gamma\gamma} \frac{L_\gamma/[2\gamma]^{3+\alpha}}{4\pi r^2 ch\nu/[2\gamma]} 2\gamma c\Delta t_{var} \simeq \frac{2^{-(1+\alpha)}\sigma_{\gamma\gamma}L_\gamma\Delta t_{var}}{4\pi(2c\gamma^2\Delta t_{var})^2 h\nu\gamma^{1+\alpha}}$$

$$\simeq \left(\frac{\sigma_{\gamma\gamma}L_\gamma}{4\pi c^2\Delta t_{var}h\nu}\right)\frac{2^{-3+\alpha}}{\gamma^{5+\alpha}}$$

- With typical numbers $L_\gamma \simeq 10^{51}$ erg/s, $\Delta t_{var} \simeq 5$ ms, $h\nu = 0.5MeV = 8 \times 10^{-7}$ erg, $\alpha \simeq 4/3$, $\sigma_{\gamma\gamma} \simeq \sigma_T/5$:

$$\tau_{\gamma\gamma} \sim \frac{2 \times 10^{11}}{\gamma^{5+\alpha}}$$

- **Transparency requires $\tau < 1$ so need $\gamma \gtrsim 100$**, i.e. ultra-relativistic speeds of GRB outflows.