RADIATION TRANSFER (Basics)

1 Overview

- Knowledge in HEA essentially depends on proper understanding of radiation received from distant objects.
- This lecture introduces/recaptures some of the basic radiation properties, definitions etc.
- It closes with black body radiation and discusses an application in the context of AGN.
2 Flux Density $F$

Energy flux density $F :=$ Energy $dE$ passing through area $dA$ in time interval $dt$

\[
F := \frac{dE}{dA \, dt} \quad (1)
\]

Units $[F] = \text{erg s}^{-1} \, \text{cm}^{-2}$.

Note: $F$ depends on orientation of $dA$ and can also depend on frequency $\nu$, i.e. $F_\nu := dE/[dA \, dt \, d\nu]$. 
3 Inverse-Square Law for Flux Density

Consider: Flux from an isotropic radiation source = source emitting equal amounts of energy in all directions.

Energy conservation ($dE = dE_1$) implies that flux through two spherical surfaces is identical:

$$4\pi r^2 F(r) = 4\pi r_1^2 F(r_1)$$

so that

$$F(r) = \frac{F(r_1)r_1^2}{r^2} = \frac{\text{const}}{r} = \frac{L}{4\pi r^2}$$

where $L$ is called luminosity.

\[\text{Note: Spherically symmetric stars are isotropic emitters, other objects like AGN are generally not (apparent isotropic luminosity).}\]
4 Specific Intensity $I_\nu$

- *Ray Optics Approximation:* Energy carried along individual rays (straight lines).

- *Problem:* Rays are infinitely thin, single ray carries essentially no energy.

$\Rightarrow$ Consider set of rays: construct area $dA$ normal to a given ray and look at all rays passing through area element within solid angle $d\Omega$ of the given ray.
• **Specific Intensity**, $I_\nu$, in frequency band $\nu, ..., \nu + d\nu$ is defined via

$$dE =: I_\nu dA dt d\Omega d\nu$$

• Units $[I_\nu] = \text{erg s}^{-1} \text{ cm}^{-2} \text{ ster}^{-1} \text{ Hz}^{-1}$.

• **Note:**

1. $I_\nu(\nu, \Omega)$ depends on location, direction and frequency. *For isotropic radiation field, $I_\nu = constant for all directions (independent of angle).*

2. Total integrated intensity $I := \int I_\nu d\nu \ [\text{erg s}^{-1} \text{ cm}^{-2} \text{ ster}^{-1}]$.

3. *Specific Intensity* is sometimes called *spectral intensity* or *spectral brightness*, or loosely just *brightness*. 
5 Net Flux $F_{\nu}$ and Specific Intensity $I_{\nu}$

For area element $dA$ at some arbitrary orientation $\vec{n}$ with respect to ray. Contribution $dF_{\nu}$ to flux in direction $\vec{n}$ from flux in direction of $d\Omega$:

$$dF_{\nu} := I_{\nu} \cos \theta d\Omega$$

or

$$dE = I_{\nu} \cos \theta dAdtd\Omega d\nu$$

Integrate over all angles to obtain net (spectral) flux

$$F_{\nu} = \int I_{\nu} \cos \theta d\Omega = \int_0^{\pi} \int_0^{2\pi} I_{\nu}(\theta, \phi) \cos \theta \sin \theta d\theta d\phi$$

Note:

1. For isotropic radiation $I_{\nu}$ (independent of angle) the net flux is zero; there is as much energy crossing $dA$ in the $\vec{n}$ direction as in the $-\vec{n}$ direction.

2. Total integrated flux $F := \int F_{\nu} d\nu$ [erg s$^{-1}$ cm$^{-2}$].
6 Constancy of Specific Intensity along Line of Sight

Consider any ray and two points along it: Energy carried through $dA_1$ and $dA_2$:

\[
\begin{align*}
    dE_1 &= I_{\nu_1} dA_1 dt d\Omega_1 d\nu_1 \\
    dE_2 &= I_{\nu_2} dA_2 dt d\Omega_2 d\nu_2
\end{align*}
\]

with $d\nu_1 = d\nu_2$, and $d\Omega_1$ solid angle subtended by $dA_2$ at $dA_1$, and vice versa:

\[
\begin{align*}
    d\Omega_1 &= dA_2 / R^2 \\
    d\Omega_2 &= dA_1 / R^2
\end{align*}
\]
Energy conservation implies $dE_1 = dE_2$, i.e.

$$ I_{\nu_1} dA_1 dt \frac{dA_2}{R^2} d\nu = I_{\nu_2} dA_2 dt \frac{dA_1}{R^2} d\nu $$

$\Rightarrow I_{\nu_1} = I_{\nu_2} = \text{const}$, or $dI_{\nu}/ds = 0$ where $ds$ length element along ray.

**Implication:** Spectral intensity is the same at the source and at the detector. Can think of $I_{\nu}$ in terms of energy flowing out of source or as energy flowing into detector.
7 Constancy of Specific Intensity and Inverse Square Law?

Intensity = const. along ray does not contradict inverse square law!
Consider sphere of uniform brightness $B$. Flux measured at point $P = \text{flux from all visible points of sphere:}$

$$F = \int I \cos \theta d\Omega = B \int_0^{2\pi} d\phi \int_0^{\theta_c} \sin \theta \cos \theta d\theta$$

with $d\Omega = \sin \theta d\theta d\phi$ (spherical coordinates), and $\sin \theta_c = R/r$. Thus

$$F = 2\pi B \int_0^{\arcsin(R/r)} \sin \theta \cos \theta d\theta = 2\pi B \frac{1}{2} \left( \frac{R}{r} \right)^2 = \pi B \left( \frac{R}{r} \right)^2 \propto \frac{1}{r^2}$$

using $\int_0^{\alpha} \sin x \cos x dx = \frac{1}{2} \sin^2 \alpha$.

$\Rightarrow$ Inverse-Square Law is consequence of decreasing solid angle of object!
8 Energy Density $u_\nu$ and Mean Intensity $J_\nu$

Radiative (Specific) Energy Density $u_\nu :=$ energy per unit volume per unit frequency range.

Radiative energy density $u_\nu(\Omega)$ per unit solid angle is defined via

$$dE =: u_\nu(\Omega)dVd\Omega d\nu$$

For light, volume element $dV = cdt \cdot dA$, so

$$dE = cu_\nu(\Omega)dAdtd\Omega d\nu$$

Comparison with definition of intensity

$$dE =: I_\nu dAdtd\Omega d\nu$$

implies

$$u_\nu(\Omega) = I_\nu/c$$
The specific energy density \( u_\nu \) then is

\[
  u_\nu = \int u_\nu(\Omega)d\Omega = \frac{1}{c} \int I_\nu(\Omega)d\Omega =: \frac{4\pi}{c} J_\nu
\]

where the mean intensity is defined by

\[
  J_\nu := \frac{1}{4\pi} \int I_\nu(\Omega)d\Omega
\]

Note that for an isotropic radiation field, \( I_\nu(\Omega) = I_\nu \), \( I_\nu = J_\nu \).

Total radiation energy density [erg cm\(^{-3}\)] by integration

\[
  u = \int u_\nu d\nu = \frac{4\pi}{c} \int J_\nu d\nu
\]
9  Photons and Intensity

• Total number of photons (states) in phase space:

\[ N := nV = \int f \frac{g}{h^3} d^3p \, d^3x \]

with \( n = N/V \) photon density, \( f=\)distribution function aka phase space occupation number, \( g=\)degeneracy factor measuring internal degree of freedom.

(Heisenberg uncertainty principle \( \Delta x \Delta p \sim h \); one particle mode occupies a volume \( h^3 \); \( g = 2 \) for photon with spin \( \pm 1 \) [left-/right-handed circularly polarized]). In phase space element

\[ dN = 2f d^3x \frac{d^3p}{h^3} \]

• Energy flow through volume element \( dV = d^3x = dA \, (cdt) \):

\[ dE = h\nu dN = 2f h\nu dA (cdt) \frac{d^3p}{h^3} \]

• For photons \( |p| = \frac{h\nu}{c} \), and using spherical coordinates

\[ d^3p = p^2 dp d\Omega = \left( \frac{h}{c} \right)^3 \nu^2 d\nu d\Omega \]
so that

\[ dE = \frac{2h\nu^3}{c^2} f dA dt d\Omega d\nu \]

- Compare with definition of specific intensity

\[ dE =: I_\nu dA dt d\Omega d\nu \]

then

\[ I_\nu = \frac{2h\nu^3}{c^2} f \]

- **Note:** Since \( I_\nu /\nu^3 \propto f \), \( I_\nu /\nu^3 \) is Lorentz invariant because phase space distribution \( f \propto dN/d^3x d^3p \) is invariant (since phase space volume element \( dV_{ps} := d^3x d^3p \) is itself invariant).
10 Radiation Transfer

- Transport of radiation ("radiation transfer") through matter is affected by emission and absorption. Specific intensity $I_\nu$ will in general not remain constant.

Figure 1: Interaction of photons with two energy states of matter ($E_1$ and $E_2$, $E_2 > E_1$). (A) stimulated absorption, (B) spontaneous emission, and (C) stimulated emission. Often "absorption" is taken to include both, true absorption (A) and stimulated emission (C), as both are proportional to the intensity of the incoming beam. The "net absorption" could thus be positive or negative, depending on which process dominates.
**Emission:** Radiation can be emitted, adding energy to beam

\[ dE =: j_\nu dV d\Omega dt d\nu \]

where \( j_\nu \) [erg cm\(^{-3}\) s\(^{-1}\) ster\(^{-1}\) Hz\(^{-1}\)] is (specific) coefficient for spontaneous emission = energy added per unit volume, unit solid angle, unit time and unit frequency. Note that \( j_\nu \) depends on direction.

Comparison with specific intensity \( dE =: I_\nu dAdt d\Omega d\nu \) gives changes in intensity, using \( dV = dAds \), \( ds \) distance along ray

\[
\boxed{dI_\nu = j_\nu ds}
\]

If emission is isotropic

\[ j_\nu =: \frac{1}{4\pi} P_\nu \]

with \( P_\nu \) radiated power per unit volume and frequency.
• **Absorption**: Radiation can also be absorbed, taking energy away.

Consider medium with particle number density $n$ [cm$^{-3}$], each having effective absorbing area as cross-section $\sigma_\nu$ [cm$^2$]:

- Number of absorbers: $ndA_{ds}$
- Total absorbing area: $n\sigma_\nu dA_{ds}$

$\Rightarrow$ Energy absorbed out of beam:

$$-dI_\nu dA dt d\Omega d\nu = I_\nu dA_{\text{absorb}} d\Omega dt d\nu = I_\nu (n\sigma_\nu dA_{ds}) d\Omega dt d\nu$$

Thus change in intensity:

$$dI_\nu = -n\sigma_\nu I_\nu ds =: -\alpha_\nu I_\nu ds$$

where $\alpha_\nu = n\sigma_\nu$ is absorption coefficient [cm$^{-1}$].
• **Equation of Radiation Transfer:** Combining emission and absorption

\[
\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu
\]

• **Example I:** Pure emission, \(\alpha_\nu = 0\), so

\[
\frac{dI_\nu}{ds} = j_\nu
\]

Separation of variables and assuming path starts at \(s = 0\) gives

\[
I_\nu(s) = I_\nu(0) + \int_0^s j_\nu(s')ds'
\]

*Brightness increase is equal to integrated emission along line of sight.*

• **Example II:** Pure absorption, \(j_\nu = 0\), so

\[
dI_\nu = -\alpha_\nu I_\nu ds
\]

Assuming path starts at \(s = 0\) gives

\[
I_\nu(s) = I_\nu(0) \exp \left[ - \int_0^s \alpha_\nu(s')ds' \right]
\]

*Brightness decreases exponentially with absorption along line of sight.*
• **Optical Depth:** The optical depth $\tau_\nu$ is defined by

$$\tau_\nu(s) := \int_0^s \alpha_\nu(s')ds' = \int_0^s n(s')\sigma_\nu ds' = n\sigma_\nu s$$

where last step is valid for a homogeneous medium.

With this definition, for pure absorption ($j_\nu = 0$):

$$I_\nu(s) = I_\nu(0)e^{-\tau_\nu}$$

Medium is said to be optically thick or opaque when $\tau_\nu > 1$, and optically thin or transparent when $\tau_\nu < 1$. 
**Mean Optical Depth and Mean Free Path of Radiation:**

"Mean free path" as equivalent concept; have exponential absorption law:

\[ \Rightarrow \text{Probability of photon to travel at least an optical depth } \tau_\nu: \exp(-\tau_\nu) \]

\[ \Rightarrow \text{Mean optical depth traveled} \]

\[ <\tau_\nu> := \int_0^\infty \tau_\nu \exp(-\tau_\nu) d\tau_\nu = \left[ -e^{-\tau_\nu}(\tau_\nu + 1) \right]_0^\infty = 1 \]

*Average distance a photon can travel without being absorbed = mean free path*, \(<\ell_\nu>\), determined by

\[ <\tau_\nu> = n\sigma_\nu <\ell_\nu> \]

Thus

\[ <\ell_\nu> = \frac{1}{n\sigma_\nu} \]

**Example:** Sun \((R_s = 7 \times 10^{10} \text{ cm}, \ M_\odot = 2 \times 10^{33} \text{ g})\), average density \(\rho \sim M_\odot/V \simeq 1.4 \text{ g/cm}^3\). Assume Sun is made up predominately of ionized hydrogen, mean number density of electrons (same as for protons) \(n \sim \rho/m_p \sim 10^{24} \text{ particles/cm}^3\). A photon in the sun scattering on free electrons (Thomson cross-section \(\sigma_T = 6.65 \times 10^{-25} \text{ cm}^2\)), thus travels on average only a distance \(<\ell> \sim 1/n\sigma_T \sim 1 \text{ cm}\) before being scattered...
11 Formal Solution to Radiation Transfer Equation

Have

\[
\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu
\]

With \( \tau_\nu(s) := \int_0^s \alpha_\nu(s') ds' \) viz. \( d\tau_\nu = \alpha_\nu ds \):

\[
\alpha_\nu \frac{dI_\nu}{d\tau_\nu} = j_\nu - \alpha_\nu I_\nu
\]

and thus

\[
\frac{dI_\nu}{d\tau_\nu} = \frac{j_\nu}{\alpha_\nu} - I_\nu =: S_\nu - I_\nu
\]

with \( S_\nu := j_\nu / \alpha_\nu \) called the source function.

Multiplying with \( \exp(\tau_\nu) \) gives

\[
\exp(\tau_\nu) \frac{dI_\nu}{d\tau_\nu} = \exp(\tau_\nu) S_\nu - \exp(\tau_\nu) I_\nu
\]

viz.

\[
e^{\tau_\nu} \frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu
\]

21
so
\[ \frac{d(e^{\tau \nu} I_{\nu})}{d\tau_{\nu}} = e^{\tau \nu} S_{\nu} \]

with solution
\[ e^{\tau \nu} I_{\nu}(\tau_{\nu}) - I_{\nu}(0) = \int_{0}^{\tau_{\nu}} e^{\tau' \nu} S_{\nu}(\tau'_{\nu}) d\tau'_{\nu} \]
or
\[ I_{\nu}(\tau_{\nu}) = I_{\nu}(0)e^{-\tau_{\nu}} + \int_{0}^{\tau_{\nu}} e^{-(\tau_{\nu} - \tau'_{\nu})} S_{\nu}(\tau'_{\nu}) d\tau'_{\nu} \]

**Interpretation:**
Intensity emerging from an absorbing medium equals the initial intensity diminished by absorption, plus the integrated source contribution diminished by absorption.

• **Example:** Constant source function \( S_{\nu} \)

\[ I_{\nu}(\tau_{\nu}) = I_{\nu}(0)e^{-\tau_{\nu}} + S_{\nu}(1 - e^{-\tau_{\nu}}) = S_{\nu} + e^{-\tau_{\nu}}(I_{\nu}(0) - S_{\nu}) \]

For \( \tau_{\nu} \to \infty \), one has \( I_{\nu} \to S_{\nu} \).
• If radiation is in thermodynamic equilibrium, nothing is allowed to change 

\[
d\frac{I_\nu}{d\tau_\nu} = S_\nu - I_\nu = 0
\]

so that \( S_\nu = I_\nu =: B_\nu(T) \), where \( B_\nu \) is the Planck function/spectrum

\[
B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1}
\]

• Remember, we had:

\[
I_\nu = \frac{2h\nu^3}{c^2} f
\]

But photon occupation number in thermodynamical equilibrium (i.e., Bose-Einstein distribution with chemical potential \( \mu = 0 \))

\[
f(E) = \frac{1}{\exp([E - \mu]/kT) - 1}
\]
(Note: In radiative equilibrium we have: absorption rate $R_{abs} = R_{em}$ emission rate. With $N_G$ and $N_E$ number of atoms in ground and excited states, respectively, $n$ photon distribution, and $Q$ probability for transition between states: $R_{abs} = QN_Gn$ and $R_{em} = QN_En + QN_E$ (stimulated + spontaneous emission). Solving for $n$ gives $n = \frac{1}{(N_G/N_E) - 1}$. Particle states in thermodynamical equilibrium (Boltzmann statistics), $N_E/N_G = \exp(-\Delta E/kT)$. Then $n = \frac{1}{\exp(\Delta E/kT) - 1}$ etc.)
12 Black Body (BB) Spectrum

Frequency space:

\[
B_\nu(T) = \frac{2\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1}
\]

Wavelength space: using \( B_\nu d\nu = -B_\lambda d\lambda \), i.e. \( B_\lambda = -B_\nu d\nu/d\lambda, \nu = c/\lambda \):

(Minus sign as increment of frequency corresponds to decrement of wavelength)

\[
B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp(hc/\lambda kT) - 1}
\]

\( B_\lambda \): Energy emitted per second, unit area, steradian and wavelength interval

- \( h = 6.63 \times 10^{-27} \) erg s = Planck constant
- \( k = 1.38 \times 10^{-16} \) erg/K = Boltzmann constant
Figure 2: Planck spectrum of black body radiation as a function of frequency. Note logarithmic scales [Credits: J. Wilms].
13 Limits

• **Rayleigh-Jeans Law**: For $h\nu \ll kT$, i.e. $\nu \ll 2 \times 10^{10} \left( \frac{T}{1K} \right)$

$$\exp \left( \frac{h\nu}{kT} \right) = 1 + \frac{h\nu}{kT} + \ldots.$$ 

so that

$$B_{\nu}(T) \simeq \frac{2\nu^2}{c^2} kT \quad \text{or} \quad B_{\lambda}(T) \simeq \frac{2c}{\lambda^4} kT$$

**Note**: In radio astronomy this is often used to define the ”brightness temperatures” $T_B := I_{\nu} c^2 / (2k\nu^2)$ as a measure of intensity.

• **Wien Spectrum**: For $h\nu \gg kT$,

$$\exp(h\nu/kT) - 1 \simeq \exp(h\nu/kT)$$

so that

$$B_{\nu}(T) \simeq \frac{2h\nu^3}{c^2} \exp \left( -\frac{h\nu}{kT} \right)$$
• **Wien Displacement law:** Frequency of maximum intensity $\nu_{\text{max}}$ is obtained by solving

$$\frac{\partial B_\nu}{\partial \nu} = 0$$

This is equivalent to $x = 3 \ (1 - \exp(-x))$, where $x := h \nu_{\text{max}} / kT$. Numerically, $x = 2.82$, so

$$h \nu_{\text{max}} = 2.82 \ kT$$

Likewise for $B_\lambda$, $\lambda_{\text{max}} T = 0.28989 \ \text{cm K.}$

**Note:**

1. $\lambda_{\text{max}} \nu_{\text{max}} \neq c!$
2. ”Non-thermal” $\gg$ thermal energies $\epsilon \simeq 2.5 \times 10^4 \ (T/10^8 K) \text{ eV}$
Figure 3: Limiting cases: Rayleigh-Jeans $B_\nu \propto \nu^2$ and Wien law $B_\nu \propto \nu^3 \exp(-h\nu/kT)$. Note logarithmic scales [Credits: J. Wilms].
14 Stefan Boltzmann (SB) Law

Total energy density of BB radiation (using $\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$ and $x = h\nu/kT$):

$$u_{BB}(T) = \int_0^\infty \frac{4\pi}{c} B_{\nu} d\nu$$

$$= \frac{8\pi^5}{15} \left(\frac{kT}{hc}\right)^3 kT$$

$$=: a T^4 \quad (2)$$

with radiation density constant

$$a := \frac{8\pi^5 k^4}{15 c^3 h^3} = 7.566 \times 10^{-15} \text{ erg cm}^{-3} \text{K}^{-4}$$

If we are interested in flux diffusing out through surface, $F = dE/dAdt$, then

$$F = \frac{c}{4} u_{BB} = \frac{ac}{4} T^4 =: \sigma_{SB} T^4$$

with $\sigma_{SB} := \frac{2\pi^5 k^4}{15 c^2 h^3} = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{s}^{-1} \text{K}^{-4} = \text{Stefan Boltzmann constant.}$
Note: The SB law gives the power emitted per unit area of the emitting body

\[ F = \int \int_{\nu=0}^{\infty} B_{\nu} \cos \theta d\nu d\Omega = \frac{c u_{BB}}{4 \pi} \int_{0}^{2\pi} d\phi \int_{0}^{\pi/2} \cos \theta \sin \theta d\theta = \frac{c u_{BB}}{2} \left[ -\frac{1}{2} \cos^2(\theta) \right]_{0}^{\pi/2} = \frac{c u_{BB}}{4} \]

using \( d\Omega = \sin \theta d\theta d\phi \) (\( d\theta \) polar angle).
15 Example: Big Blue Bump (BBB) in AGN

Accretion disk in AGN: Gravitational potential energy released as radiation from optically-thick disk of area $\pi r^2$ (two sides)

$$L = \frac{GM_{BH}\dot{M}}{2r} = 2\pi r^2 \sigma_{SB} T^4$$

gives

$$T = \left( \frac{GM_{BH}\dot{M}}{4\pi \sigma_{SB} r^3} \right)^{1/4}$$

In terms of Eddington accretion rate (lecture 1): $\dot{M}_{Edd} = \frac{L_{Edd}}{\eta c^2} \simeq 2.2 \left( \frac{M_{BH}}{10^8 M_\odot} \right) M_\odot / \text{yr},$

$$T(r) \simeq 6.3 \times 10^5 \left( \frac{\dot{M}}{\dot{M}_{Edd}} \right)^{1/4} \left( \frac{10^8 M_\odot}{M_{BH}} \right)^{1/4} \left( \frac{r_s}{r} \right)^{3/4} \text{ K}$$

using $r_s := 2GM_{BH}/c^2$. For Eddington emitting source, emission maximized at

$$\nu_{max} = 2.82 kT / h = 3.6 \times 10^{16} \left( \frac{T}{6.3 \times 10^5 \text{ K}} \right) \text{ Hz}$$
Thermal emission expected to be prominent in UV $[10^{15} \text{ Hz}, 3 \times 10^{16}] = \text{BBB}$. 

Figure 4: AGN (Seyfert) template used to infer BH masses from the location of the Big blue bump (BBB) peak in recent work by Calderone et al. 2013 (MNRAS 431, 210). The BBB is related to thermal radiation from the inner parts of the accretion disk, while the IR bump is thermal radiation emitted from further out (e.g., dust torus at $\sim 1 \text{ pc}$ from black hole).