Sudoku

An application of message passing

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数独

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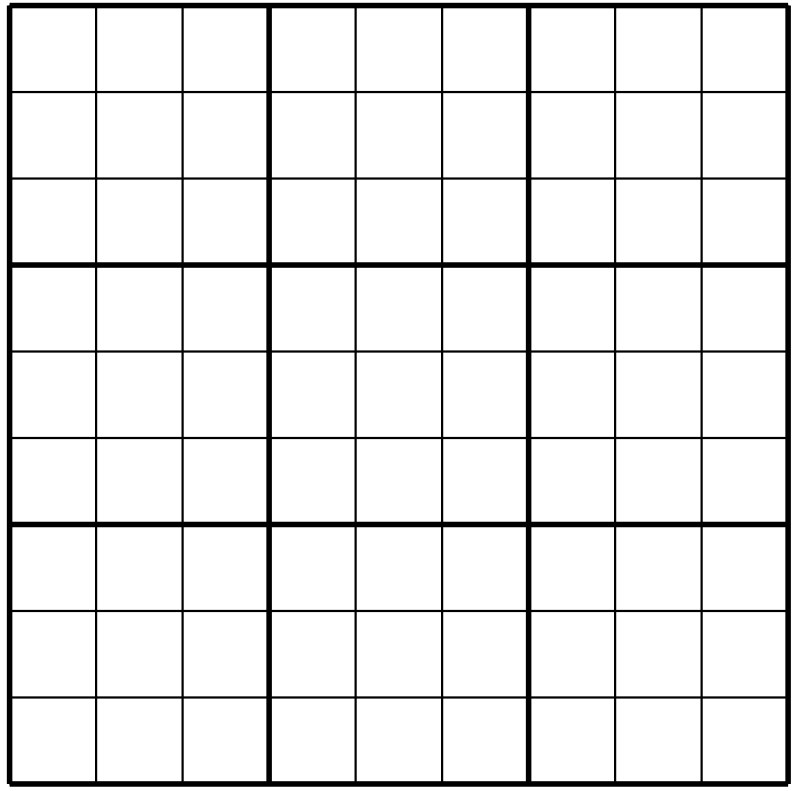

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	7		5		8		3	
		2				9		
	1		6		4		5	
5								6
	2		1		5		9	
		4				7		
	8		7		2		4	
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8	6	2	3	4	7	9	1	5
9	1	3	6	8	4	2	5	7
5	4	7	2	9	3	1	8	6
6	2	8	1	7	5	3	9	4
2	3	4	8	5	9	7	6	1
1	8	6	7	3	2	5	4	9
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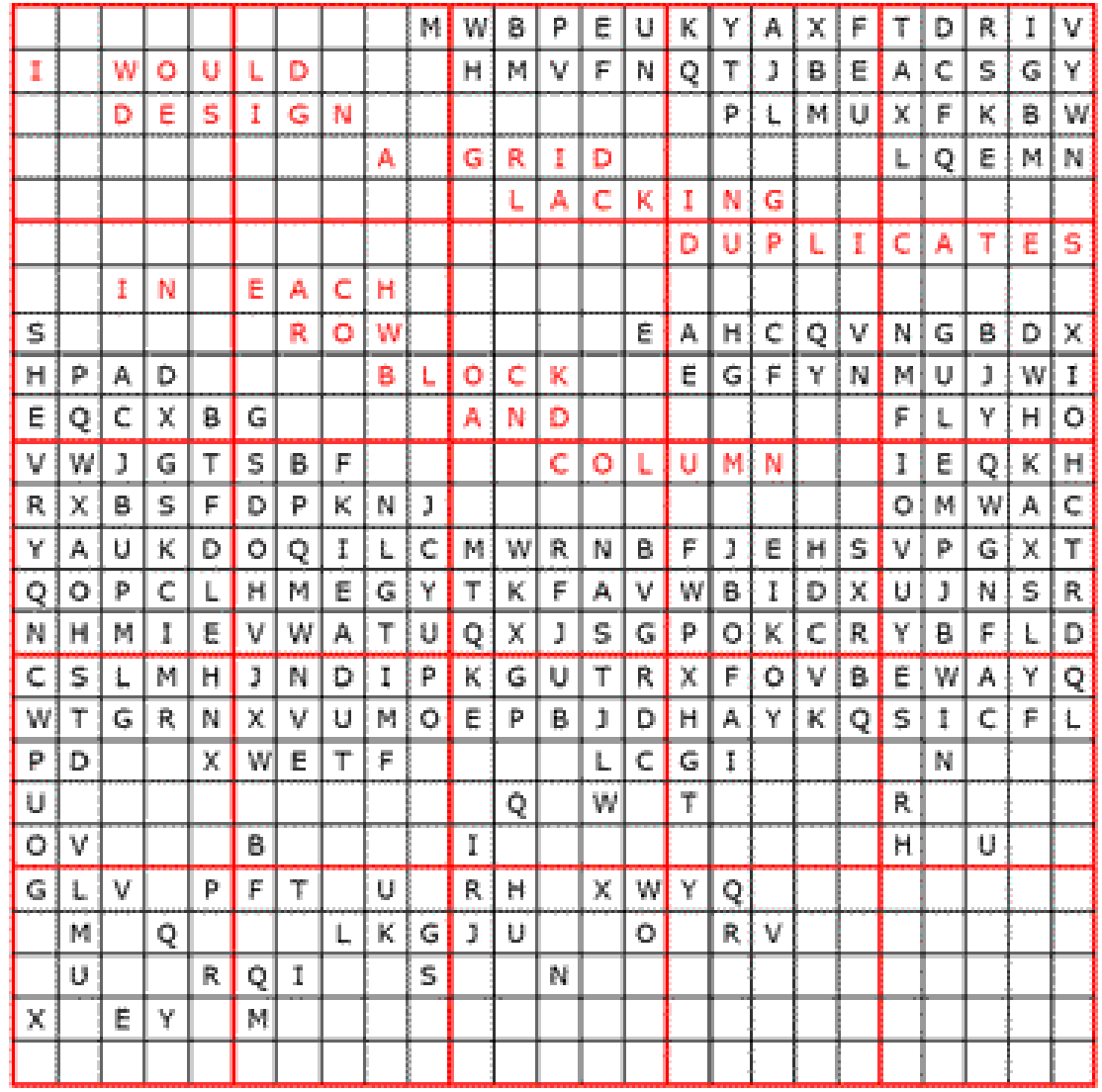
- a well formed Sudoku has a unique solution

Sudoku variants

- larger Sudokus



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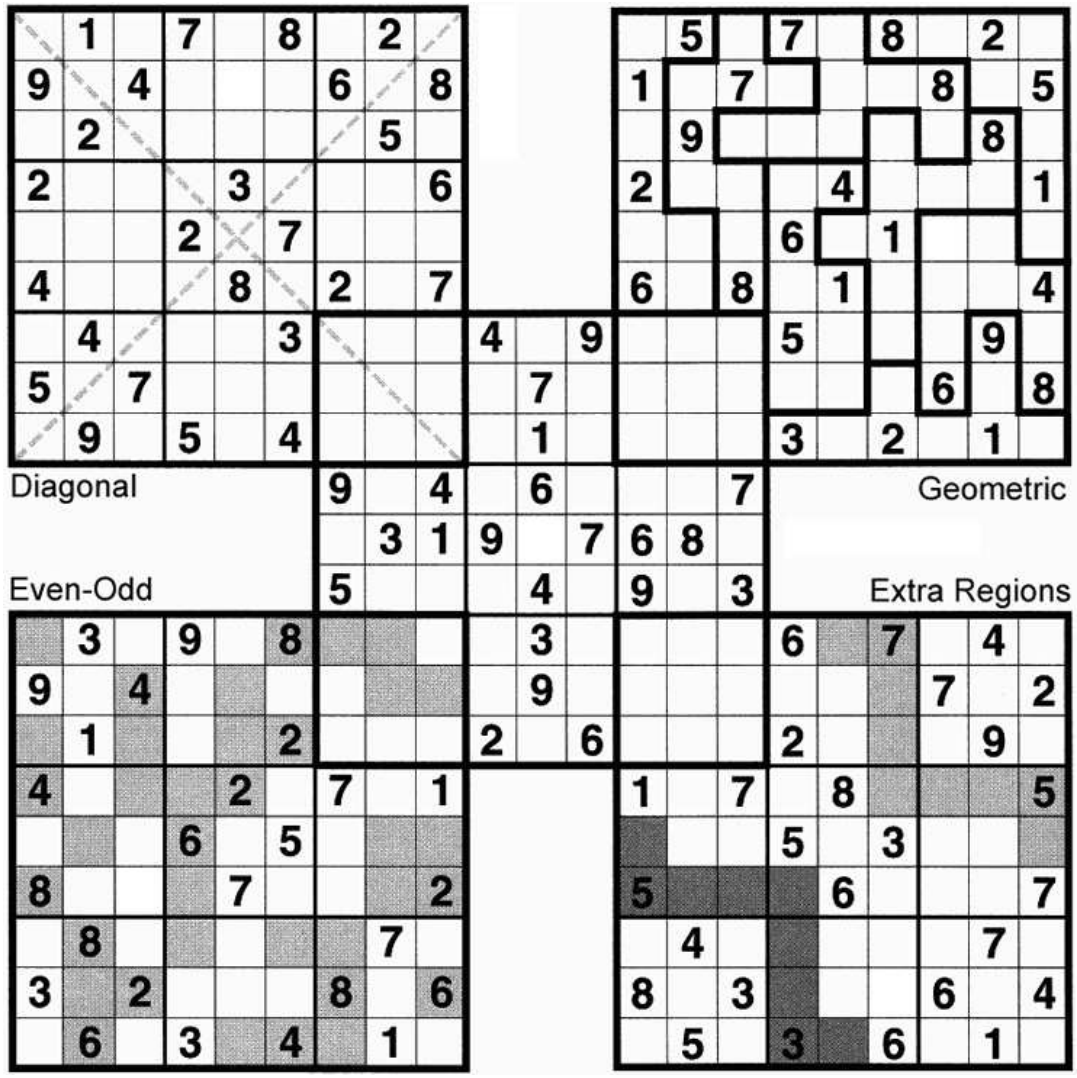
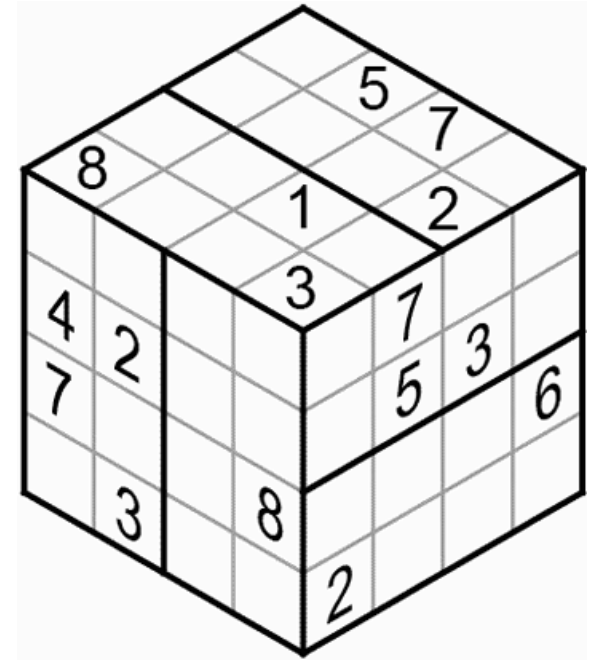
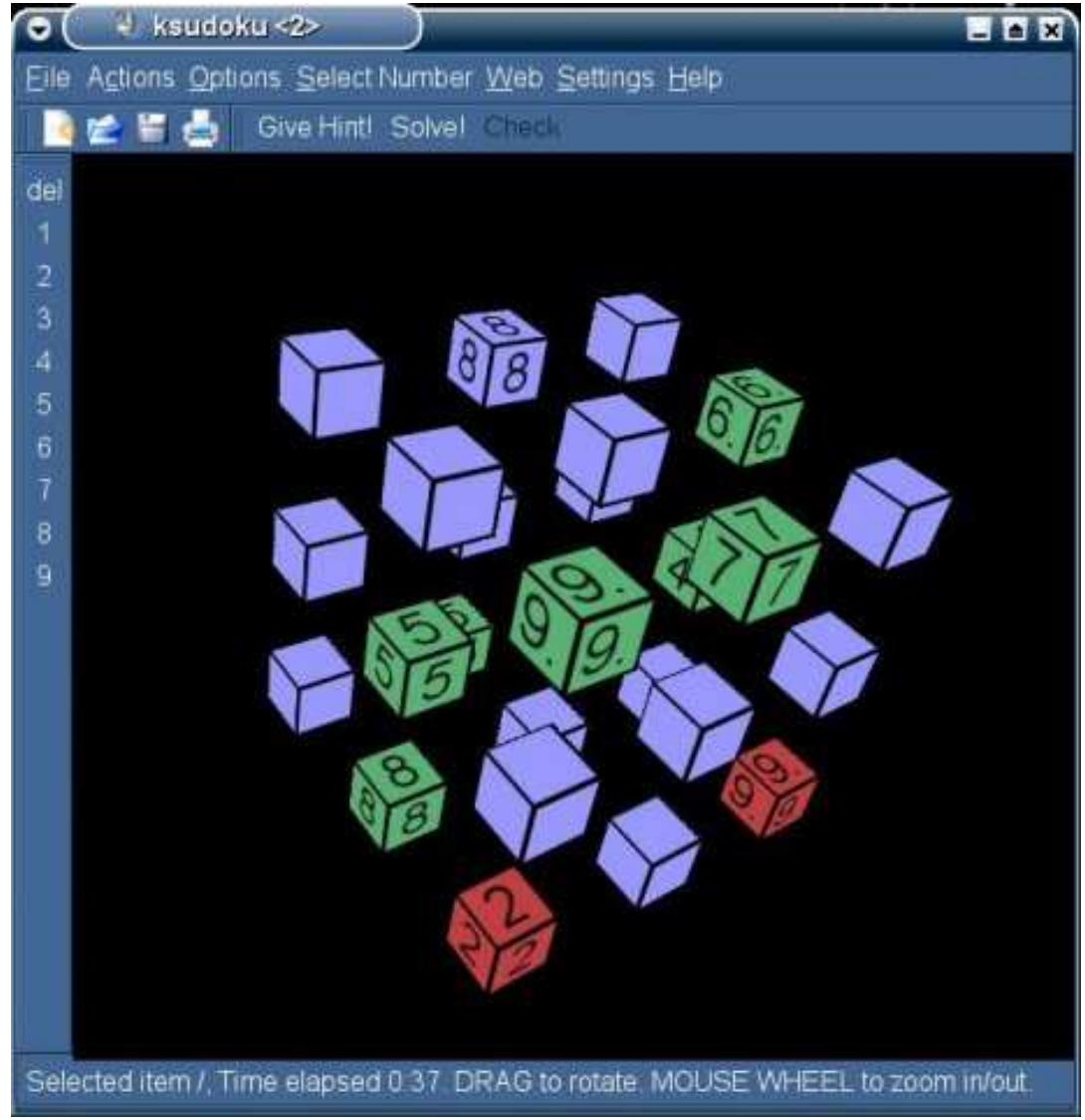
- other Sudokus variants





Sudoku variants

- three-dimensional Sudoku



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- uniform probability measure over set of solutions

$$p(x_1, x_2, \ldots, x_{81}) = \frac{1}{Z} \prod_{i \in \text{cols, rows, squares}} f(x_{i_1}, x_{i_2}, \ldots, x_{i_9})$$

$$f(y_1, y_2, \ldots, y_9) = \mathbb{1}\left[(y_1, y_2, \ldots, y_9) \text{ is a perm. of } (1, 2, \ldots, 9)\right]$$

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$$\mu_i(x_i) = \sum_{x_j, j \in \{1,2,\ldots,81\} \setminus \{i\}} p(x_1, x_2, \ldots, x_{81}) = \mathbb{1}\left[x_i = v_i\right]$$

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- probability distribution can be expressed in terms of a factor graph

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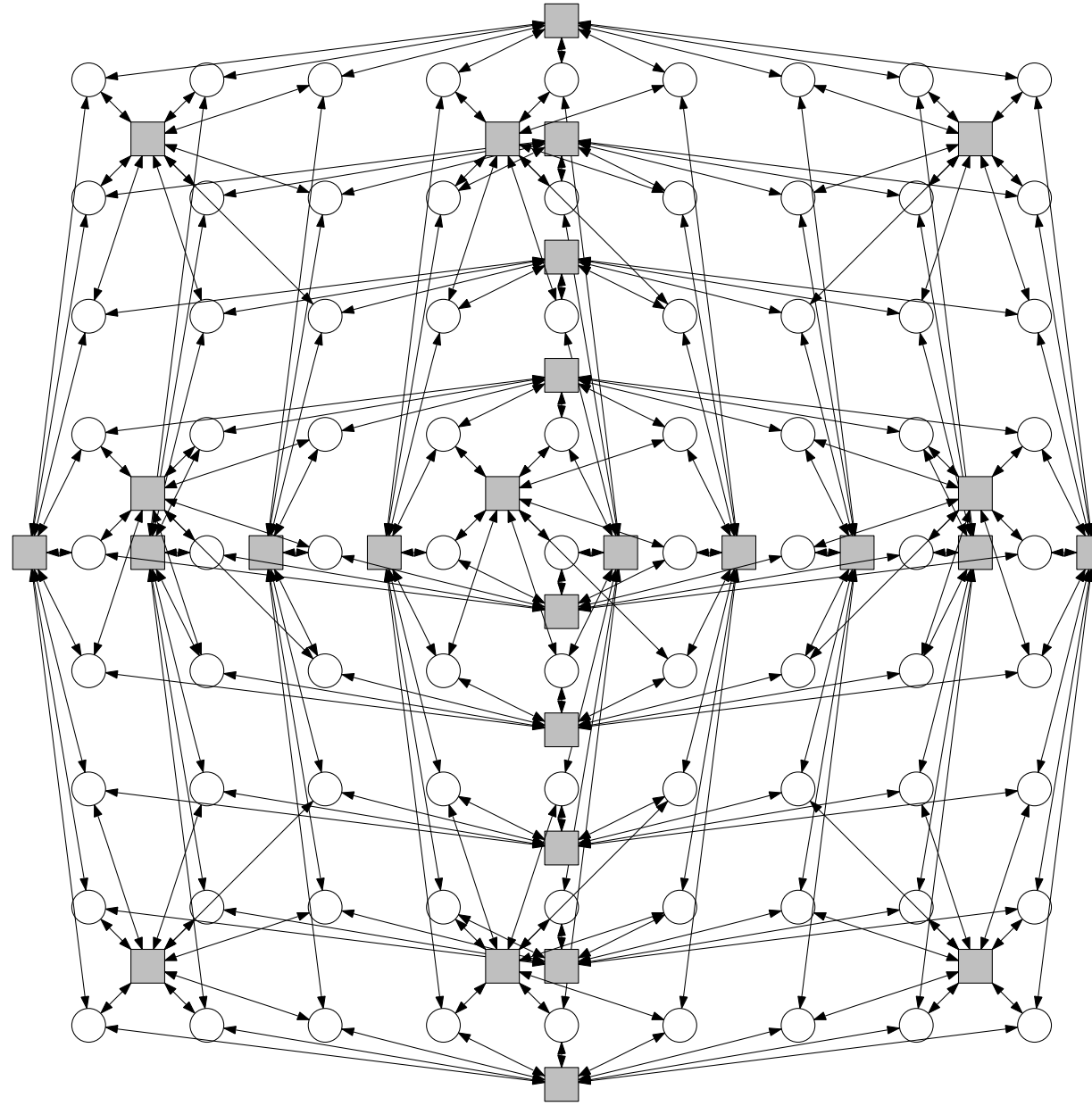

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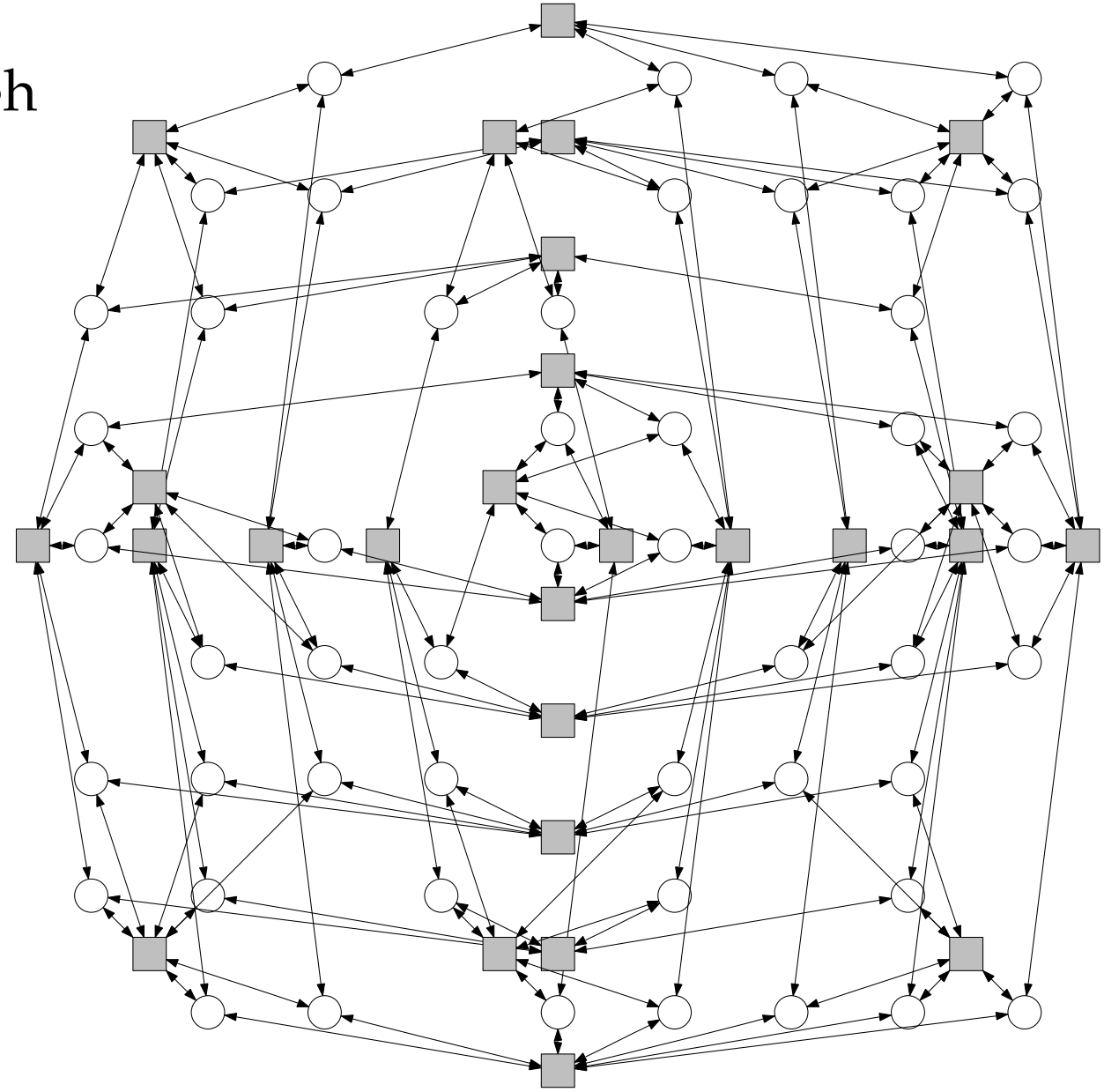

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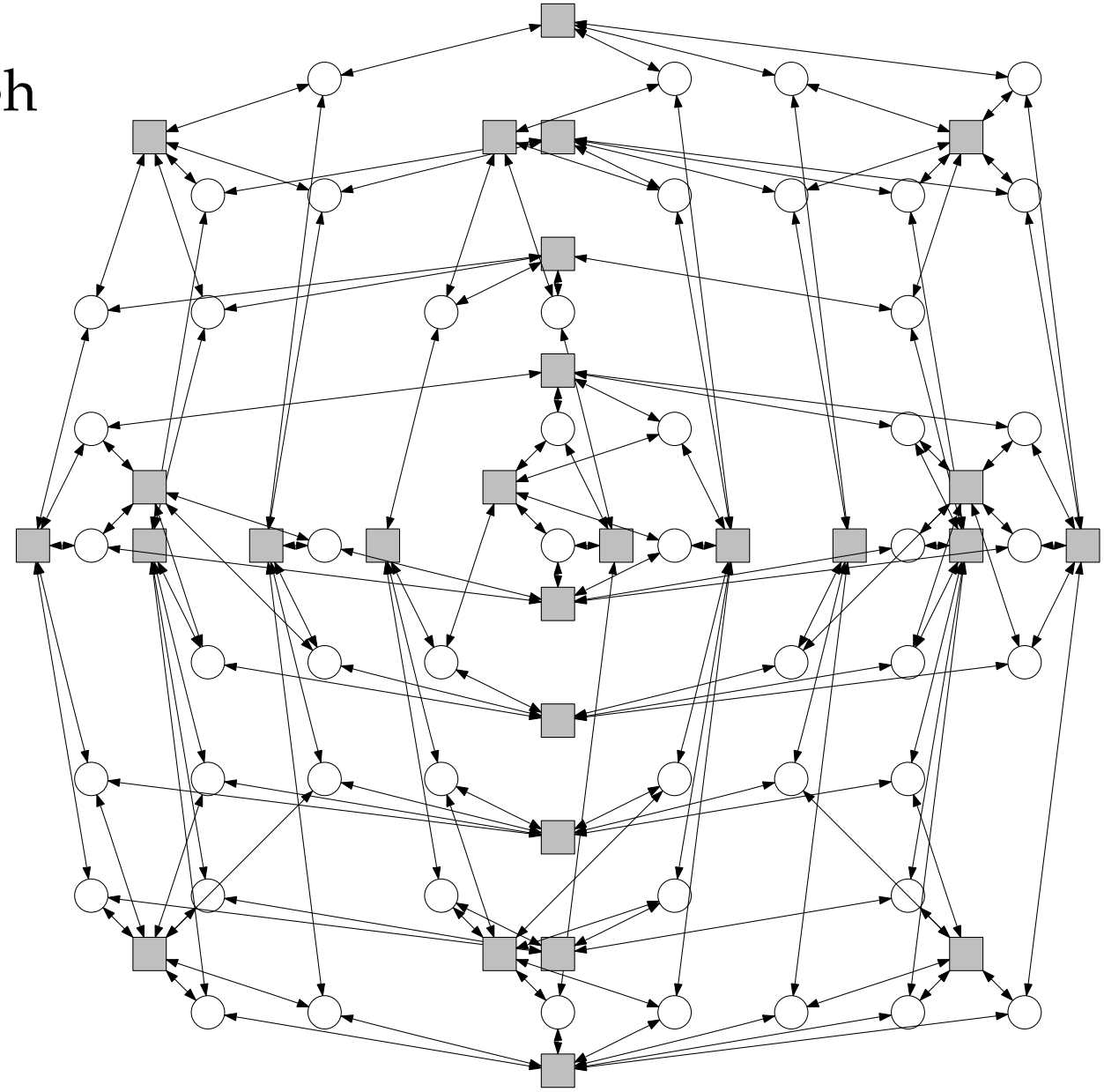

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- Will message passing work?

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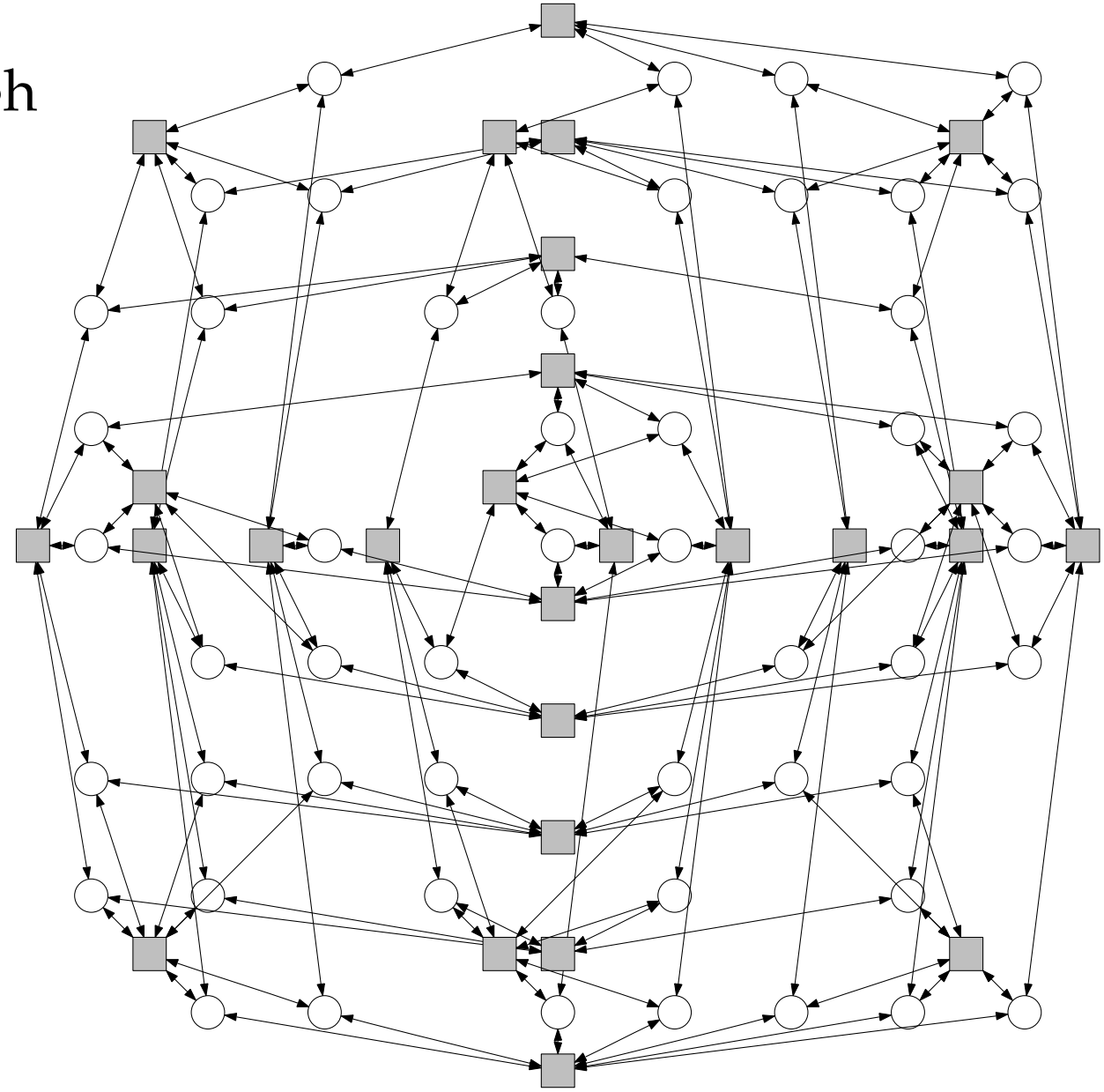

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- function node processing: marginalisation with respect to x

$$\mu_{i\to k}(x) \sim \sum_{\substack{x_l\in\{1,2,\ldots,9\},\\ l\in\{\text{n.n. of } i\}\setminus\{k\}}} f(x,\ldots,x_l,\ldots) \prod_{j\in\{\text{n.n. of } i\}\setminus\{k\}} \mu_{j\to i}(x_j)$$

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  - random update: converges to some solution

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$\mathbb{K}$	"$(+,0)$"	"$(\cdot,1)$"	algorithm
$\mathbb{R}_{\geq 0}$	$(+,0)$	$(\cdot,1)$	sum product
$\mathbb{R}_{\geq 0} \cup \{\infty\}$	$(\min,\infty)$	$(\cdot,1)$	min product
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$\mathbb{R} \cup \{\infty\}$	$(\min,\infty)$	$(+,0)$	min sum
$\mathbb{R} \cup \{-\infty\}$	$(\max,-\infty)$	$(+,0)$	max sum

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- no numerical instabilities, outperforms sum product algorithm slightly (solves 71 % of Sudokus with 17 givens)

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- What about other message passing schemes, e. g. formulations with binary variables?