Tri-bimaximal lepton mixing: models, deviations and alternatives

Max-Planck-Institut
für Kernphysik

Werner Rodejohann
(MPIK, Heidelberg)
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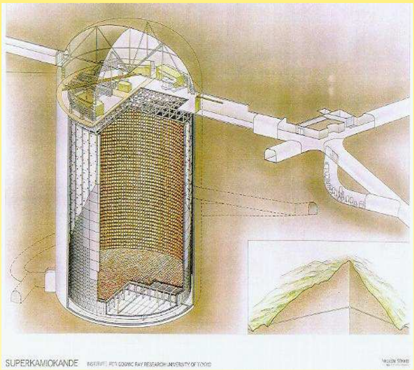
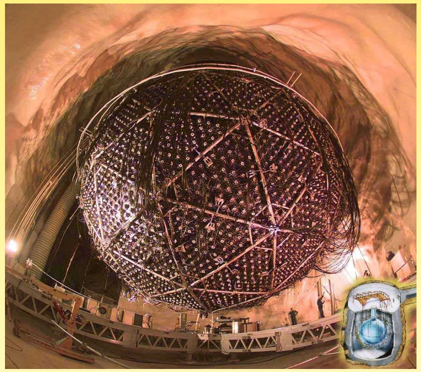
Outline

- Current Status of PMNS
- Tri-bimaximal Mixing (TBM):
  - how to model
  - deviating from TBM: how to get $\theta_{13} \simeq 0.1$
- alternatives to TBM
- sterile neutrinos and flavor symmetries

$$U = \begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, e^{i\delta} \\ -s_{12}\, c_{23} - c_{12}\, s_{23}\, s_{13}\, e^{-i\delta} & c_{12}\, c_{23} - s_{12}\, s_{23}\, s_{13}\, e^{-i\delta} & s_{23}\, c_{13} \\ s_{12}\, s_{23} - c_{12}\, c_{23}\, s_{13}\, e^{-i\delta} & -c_{12}\, s_{23} - s_{12}\, c_{23}\, s_{13}\, e^{-i\delta} & c_{23}\, c_{13} \end{pmatrix} =$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\text{atmospheric and LBL accelerator}} \underbrace{\begin{pmatrix} c_{13} & 0 & s_{13}\, e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}\, e^{i\delta} & 0 & c_{13} \end{pmatrix}}_{\text{SBL reactor}} \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{solar and LBL reactor}}$$





$$U = \begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, e^{i\delta} \\ -s_{12}\, c_{23} - c_{12}\, s_{23}\, s_{13}\, e^{-i\delta} & c_{12}\, c_{23} - s_{12}\, s_{23}\, s_{13}\, e^{-i\delta} & s_{23}\, c_{13} \\ s_{12}\, s_{23} - c_{12}\, c_{23}\, s_{13}\, e^{-i\delta} & -c_{12}\, s_{23} - s_{12}\, c_{23}\, s_{13}\, e^{-i\delta} & c_{23}\, c_{13} \end{pmatrix} =$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\sin^2\theta_{23} = \tfrac{1}{2}) \qquad (\sin^2\theta_{13} = 0) \qquad (\sin^2\theta_{12} = \tfrac{1}{3})$$

$$\Delta m^2_{\mathrm{A}} \qquad \Delta m^2_{\mathrm{A}} \qquad \Delta m^2_{\odot}$$

$$U = \begin{pmatrix} c_{12}\,c_{13} & s_{12}\,c_{13} & s_{13}\,e^{i\delta} \\ -s_{12}\,c_{23} - c_{12}\,s_{23}\,s_{13}\,e^{-i\delta} & c_{12}\,c_{23} - s_{12}\,s_{23}\,s_{13}\,e^{-i\delta} & s_{23}\,c_{13} \\ s_{12}\,s_{23} - c_{12}\,c_{23}\,s_{13}\,e^{-i\delta} & -c_{12}\,s_{23} - s_{12}\,c_{23}\,s_{13}\,e^{-i\delta} & c_{23}\,c_{13} \end{pmatrix} =$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\sin^2\theta_{23} = \tfrac{1}{2}) \qquad (\sin^2\theta_{13} = \epsilon^2) \qquad (\sin^2\theta_{12} = \tfrac{1}{3})$$

$$\Delta m^2_{\mathrm{A}} \qquad \Delta m^2_{\mathrm{A}} \qquad \Delta m^2_{\odot}$$

Tri-bimaximal Mixing

$$U_{\rm TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}$$

Harrison, Perkins, Scott (2002)

with mass matrix

$$(m_\nu)_{\rm TBM} = U^*_{\rm TBM}\, m_\nu^{\rm diag}\, U^\dagger_{\rm TBM} = \begin{pmatrix} A & B & B \\ \cdot & \frac{1}{2}(A+B+D) & \frac{1}{2}(A+B-D) \\ \cdot & \cdot & \frac{1}{2}(A+B+D) \end{pmatrix}$$

$$A = \frac{1}{3}\left(2\, m_1 + m_2\, e^{-2i\alpha}\right) , \quad B = \frac{1}{3}\left(m_2\, e^{-2i\alpha} - m_1\right) , \quad D = m_3\, e^{-2i\beta}$$

$\Rightarrow$ Flavor symmetries...

Mass Matrix

Special case of μ–τ symmetry

$$(m_\nu)_{\rm TBM} = U^*_{\rm TBM}\, m_\nu^{\rm diag}\, U^\dagger_{\rm TBM} = \begin{pmatrix} A & B & B \\ \cdot & \frac{1}{2}(A+B+D) & \frac{1}{2}(A+B-D) \\ \cdot & \cdot & \frac{1}{2}(A+B+D) \end{pmatrix}$$

$$A = \frac{1}{3}\left(2\, m_1 + m_2\, e^{-2i\alpha}\right)\,, \quad B = \frac{1}{3}\left(m_2\, e^{-2i\alpha} - m_1\right)\,, \quad D = m_3\, e^{-2i\beta}$$

- $m_{e\mu} = m_{e\tau}$ and $m_{\mu\mu} = m_{\tau\tau}$
- $m_{ee} + m_{e\mu} + m_{e\tau} = m_{\mu e} + m_{\mu\mu} + m_{\mu\tau} = m_{\tau e} + m_{\tau\mu} + m_{\tau\tau}$
- masses independent on mixing (i.e., not $V_{us} = \sqrt{m_d/m_s}$)

Correlations between mass matrix elements $\leftrightarrow$ flavor symmetries

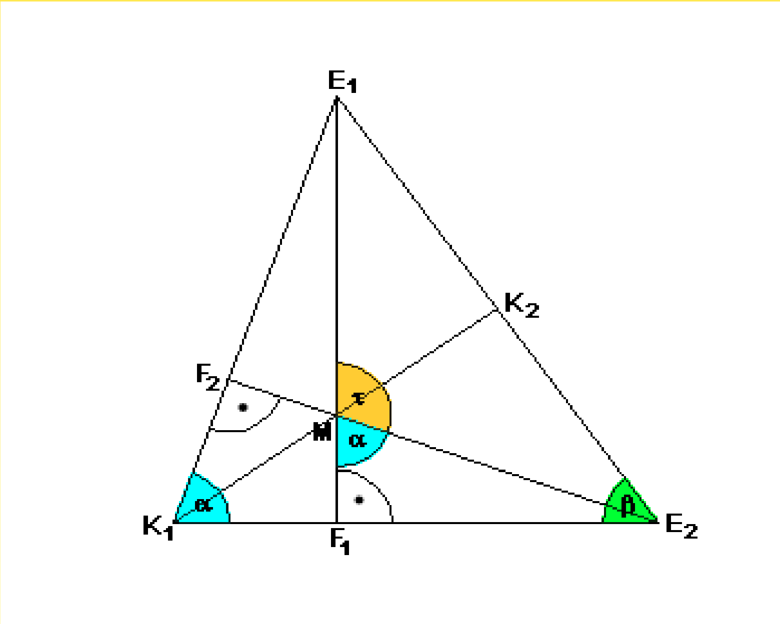
How to choose the group

Group	d	Irr. Repr.'s	Presentation
$D_3 \sim S_3$	6	1, $1'$, 2	$A^3 = B^2 = (AB)^2 = 1$
D_4	8	$1_1, \ldots 1_4, 2$	$A^4 = B^2 = (AB)^2 = 1$
D_5	10	1, $1'$, 2, 2'	$A^5 = B^2 = (AB)^2 = 1$
D_6	12	$1_1, \ldots 1_4$, 2, 2'	$A^6 = B^2 = (AB)^2 = 1$
D_7	14	1, $1'$, 2, $2'$, $2''$	$A^7 = B^2 = (AB)^2 = 1$
A_4	12	1, $1'$, $1''$, 3	$A^3 = B^2 = (AB)^3 = 1$
$A_5 \sim PSL_2(5)$	60	1, 3, $3'$, 4, 5	$A^3 = B^2 = (BA)^5 = 1$
T'	24	1, $1'$, $1''$, 2, $2'$, $2''$, 3	$A^3 = (AB)^3 = R^2 = 1,\ B^2 = R$
S_4	24	1, $1'$, 2, 3, $3'$	$BM : A^4 = B^2 = (AB)^3 = 1$ $TB : A^3 = B^4 = (BA^2)^2 = 1$
$\Delta(27) \sim Z_3 \rtimes Z_3$	27	$1_1, \ldots 1_9, 3, \overline{3}$	
$PSL_2(7)$	168	$1, 3, \overline{3}, 6, 7, 8$	$A^3 = B^2 = (BA)^7 = (B^{-1}A^{-1}BA)^4 = 1$
$T_7 \sim Z_7 \rtimes Z_3$	21	$1, 1', \overline{1'}, 3, \overline{3}$	$A^7 = B^3 = 1,\ AB = BA^4$

Altarelli, Feruglio, 1002.0211

How to choose the group: A_4

- minimality
  - smallest group with 3 irrep
  - has 3 one-dimensional irreps 1, $1'$, $1''$
- geometry



angle between two faces: $\alpha = 2\,\theta_{\rm TBM}$, where $\sin^2\theta_{\rm TBM} = \frac{1}{3}$

A role model (Altarelli, Feruglio)

	l	e^c	μ^c	τ^c	ν^c	$h_{u,d}$	θ	φ_T	φ_S	ξ	φ_0^T	φ_0^S	ξ_0
A_4	3	1	1"	1'	3	1	1	3	3	1	3	3	1
Z_3	ω	ω^2	ω^2	ω^2	ω^2	1	1	1	ω^2	ω^2	1	ω^2	ω^2
$U(1)_{\mathrm{FN}}$	0	4	2	0	0	0	-1	0	0	0	0	0	0

- Z_3 to separate charged leptons and neutrinos
- Froggatt-Nielsen to get charged lepton hierarchy
- φ_T and φ_S acquire vevs and break the symmetry ("vev alignment")
- φ_0^T, φ_0^S and ξ_0 live to make the vevs look the way they do ("driving fields")

A role model

$$w_l = y_e e^c(\varphi_T l) + y_\mu \mu^c(\varphi_T l)' + y_\tau \tau^c(\varphi_T l)''$$
$$+y(\nu^c l) + x_A \xi(\nu^c \nu^c) + x_B(\varphi_S \nu^c \nu^c) + h.c. + \dots$$

where $y_e e^c(\varphi_T l)$ stands for $y_e e^c(\varphi_T l) h_d \theta^4/\Lambda^5$

leads to TBM if $\langle\varphi_T\rangle = v_T(1,1,1)$ and $\langle\varphi_S\rangle = v_S(1,0,0)$:

$$m_\nu^D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} y v_u\,, \quad M = \begin{pmatrix} A+2B/3 & -B/3 & -B/3 \\ -B/3 & 2B/3 & A-B/3 \\ -B/3 & A-B/3 & 2B/3 \end{pmatrix}$$

with $A = 2\,x_A\langle\xi\rangle$, $B = 2\,x_B\,v_S/v_u$

"NLO" terms are higher order terms, suppressed by powers of Λ, which will generate corrections to TBM

The Zoo (of A_4 models)

Type	L_i	ℓ_i^c	ν_i^c	Δ	References
A1				-	[1–14] [15]#
A2	$\underline{3}$	$\underline{1}, \underline{1}', \underline{1}''$	-	$\underline{1}, \underline{1}', \underline{1}'', \underline{3}$	[16–18]
A3				$\underline{1}, \underline{3}$	[19]
B1	$\underline{3}$	$\underline{1}, \underline{1}', \underline{1}''$	$\underline{3}$	-	[4, 20–27]# [28–30]* [31–45]
B2				$\underline{1}, \underline{3}$	[46]#
C1				-	[2, 47, 48]
C2	$\underline{3}$	$\underline{3}$	-	$\underline{1}$	[49, 50] [51]#
C3				$\underline{1}, \underline{3}$	[52]
C4				$\underline{1}, \underline{1}', \underline{1}'', \underline{3}$	[53]
D1				-	[54, 55]# [56, 57]* [58]
D2	$\underline{3}$	$\underline{3}$	$\underline{3}$	$\underline{1}$	[59] [60]*
D3				$\underline{1}'$	[61]*
D4				$\underline{1}', \underline{3}$	[62]*
E	$\underline{3}$	$\underline{3}$	$\underline{1}, \underline{1}', \underline{1}''$	-	[63, 64]
F	$\underline{1}, \underline{1}', \underline{1}''$	$\underline{3}$	$\underline{3}$	$\underline{1}$ or $\underline{1}'$	[65]
G	$\underline{3}$	$\underline{1}, \underline{1}', \underline{1}''$	$\underline{1}, \underline{1}', \underline{1}''$	-	[66]
H	$\underline{3}$	$\underline{1}, \underline{1}, \underline{1}$	-	-	[67]
I	$\underline{3}$	$\underline{1}, \underline{1}, \underline{1}$	$\underline{1}, \underline{1}, \underline{1}$	-	[68]*
J	$\underline{3}$	$\underline{1}, \underline{1}, \underline{1}$	$\underline{3}$	-	[12, 39, 69, 70]
K	$\underline{3}$	$\underline{1}, \underline{1}, \underline{1}$	$\underline{1}, \underline{1}$	$\underline{1}$	[71]*
L	$\underline{3}$	$\underline{1}, \underline{1}, \underline{1}$	$\underline{1}$	-	[72]*
M	$\underline{1}, \underline{1}', \underline{1}''$	$\underline{1}, \underline{1}'', \underline{1}'$	$\underline{3}, \underline{1}$	-	[73, 74]
N	$\underline{1}, \underline{1}', \underline{1}''$	$\underline{1}, \underline{1}'', \underline{1}'$	$\underline{3}, \underline{1}', \underline{1}''$	-	[75]

Barry, W.R., PRD **81**, 093002 (2010), updated regularly on
http://www.mpi-hd.mpg.de/personalhomes/jamesb/Table_A4.pdf

How to distinguish?

- LFV
- low scale scalars: Higgs, LFV
- compatible with GUTs?
- leptogenesis possible?
- neutrino mass observables
  - sum-rules, such as $2m_2 + m_3 = m_1$
  - correlation with oscillation parameters

	Bari	GM-I	STV	TBM
$\sin\theta_{13}$	$0.145^{+0.022}_{-0.031}$	$0.097^{+0.053}_{-0.047}$	$0.130^{+0.025}_{-0.041}$	0
$\sin^2\theta_{23}$	$0.42^{+0.08}_{-0.03}$	$0.462^{+0.082}_{-0.050}$	$0.51^{+0.06}_{-0.06}$	0.5
$\sin^2\theta_{12}$	$0.306^{+0.018}_{-0.015}$	$0.319^{+0.016}_{-0.016}$	$0.316^{+0.016}_{-0.016}$	0.333

2012 (DC + T2K + MINOS) at 3σ:

$$\sin\theta_{13} = 0.146^{+0.084}_{-0.119}$$

all groups find deviations from one or more TBM values, typically

$$\delta\theta_{13} > \delta\theta_{12} \gtrsim \delta\theta_{23}$$

Taking Bari results as example:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}\, e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}\, e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.762 & -0.684 \\ 0 & -0.684 & 0.762 \end{pmatrix} \begin{pmatrix} 0.989 & 0 & 0.145\, e^{-i\delta} \\ 0 & 1 & 0 \\ -0.145\, e^{i\delta} & 0 & 0.989 \end{pmatrix} \begin{pmatrix} 0.833 & 0.553 & 0 \\ -0.553 & 0.883 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.824 & 0.547 & 0.145\, e^{-i\delta} \\ -0.421 - 0.078\, e^{i\delta} & 0.634 - 0.052\, e^{i\delta} & 0.641 \\ 0.359 - 0.092\, e^{i\delta} & -0.540 - 0.061\, e^{i\delta} & 0.754 \end{pmatrix}$$

`Abbas, Smirnov, PRD` **`82`**`, 013008 (2010)`:

deviations from $m_\nu^{\rm TBM}$ possible: "TBM accidental?"

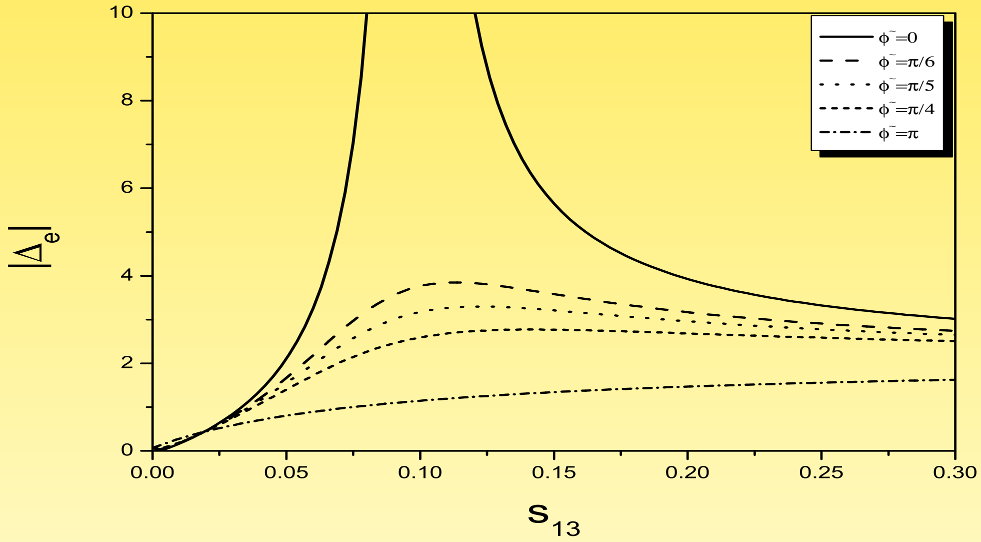
Recall the three TBM conditions

$$m_{e\mu} = m_{e\tau}$$

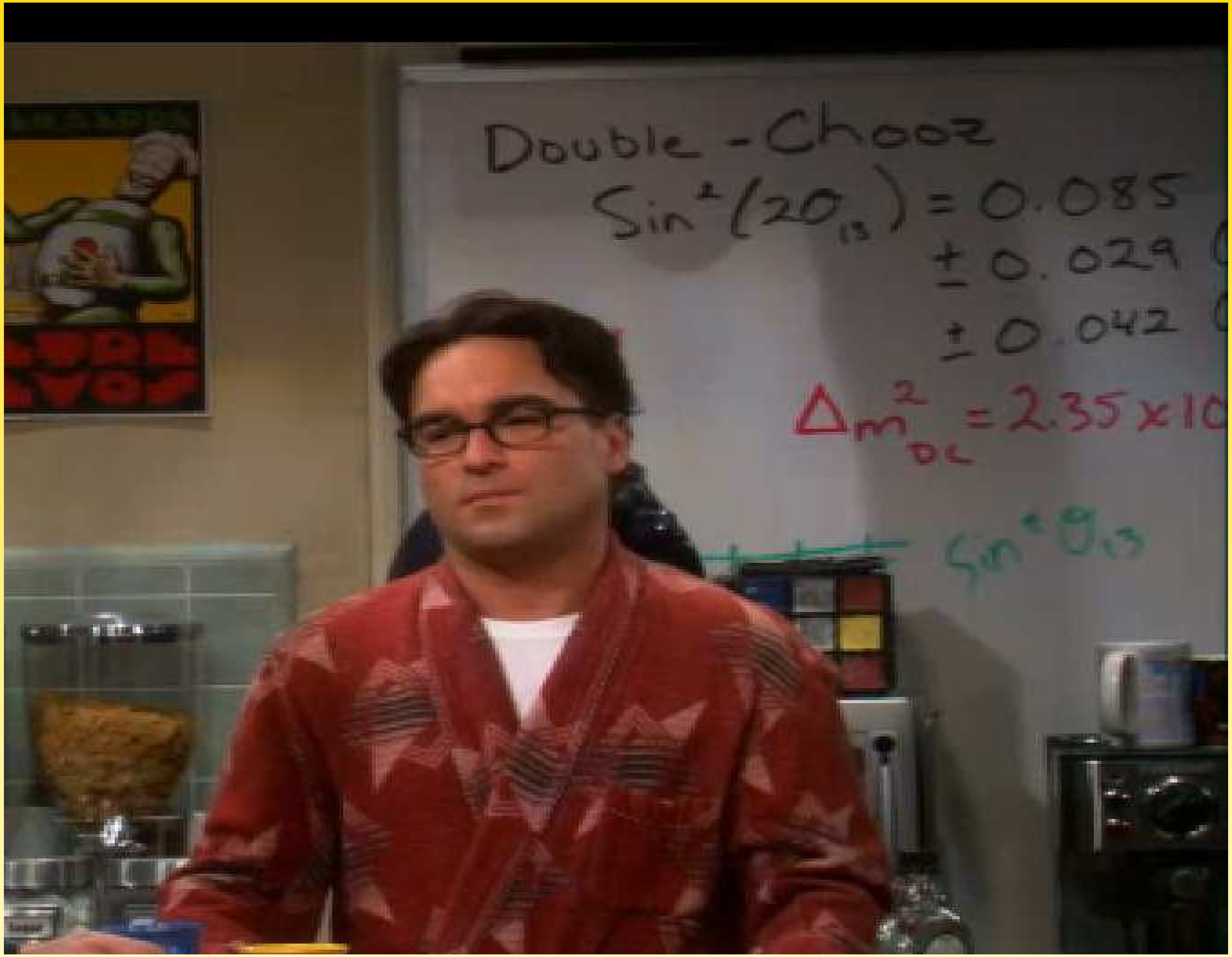
$$m_{\mu\mu} = m_{\tau\tau}$$

$$m_{ee} + m_{e\mu} + m_{e\tau} = m_{\mu e} + m_{\mu\mu} + m_{\mu\tau} = m_{\tau e} + m_{\tau\mu} + m_{\tau\tau}$$

thus, define $\Delta_e = \dfrac{(m_\nu)_{e\mu} - (m_\nu)_{e\tau}}{(m_\nu)_{e\mu}}$, $\Delta_\mu = \dfrac{(m_\nu)_{\mu\mu} - (m_\nu)_{\tau\tau}}{(m_\nu)_{\tau\tau}}$, Δ_Σ



Non-zero θ_{13}



Possibilities

- study deviations from TBM
- construct alternative scenarios

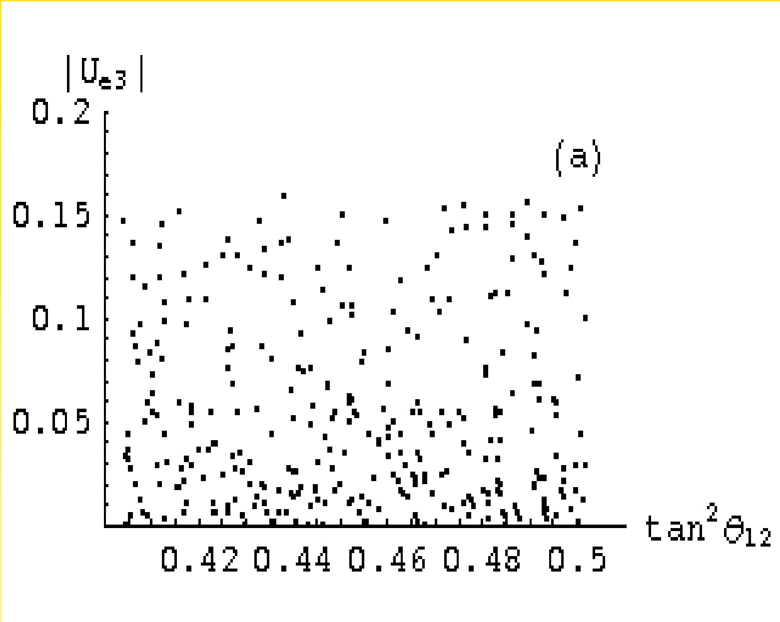
How to perturb a mixing scenario/model

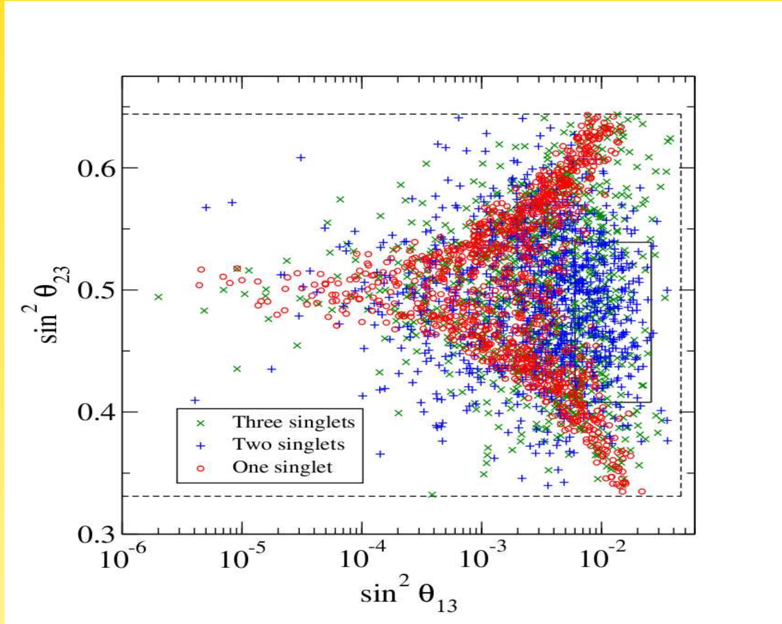
- VEV misalignment, NLO terms
- explicit naive breaking
- renormalization
- charged leptons

VEV misalignment, NLO terms

- **"naive misalignment"**:

  if $\langle \text{flavon} \rangle = (1, 1, 1)^T$, perturb it to $\langle \text{flavon} \rangle = (1, 1 + \epsilon_1, 1 + \epsilon_2)^T$





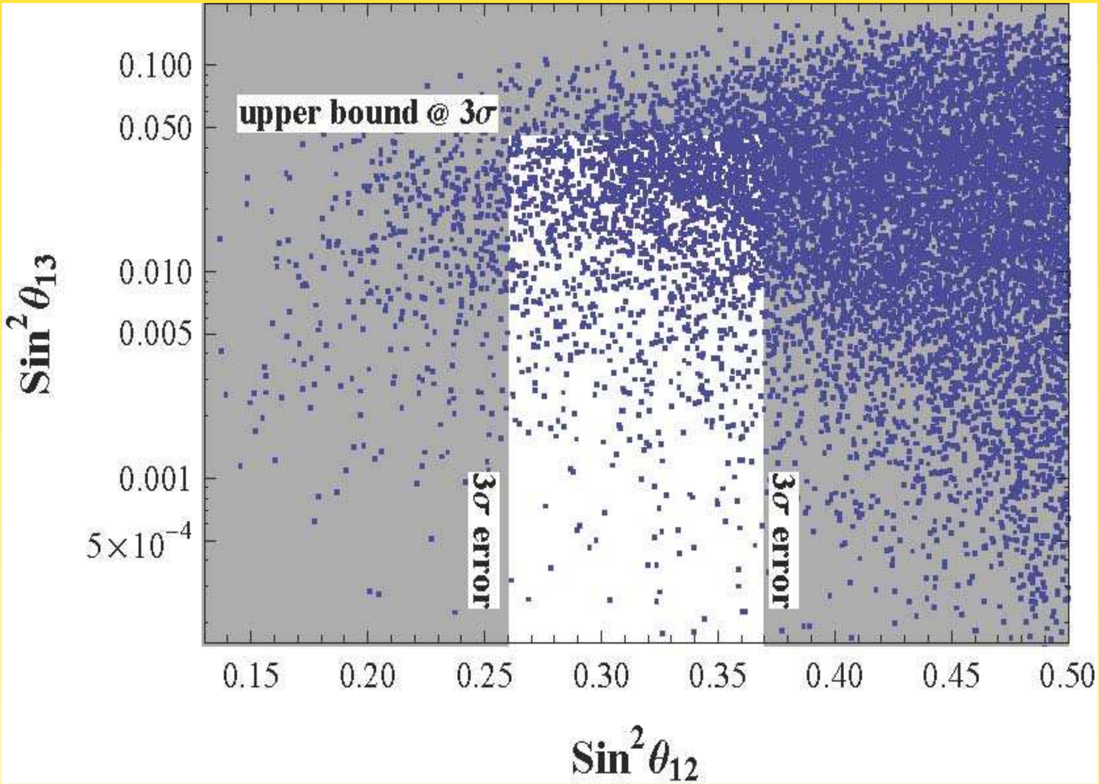
Honda, Tanimoto

Barry, W.R.

- of order $\langle \text{flavon} \rangle / \Lambda$ or $\langle \text{flavon} \rangle / M_R$, typically $\mathcal{O}(0.1)$ or $\mathcal{O}(\lambda_C)$ or $\mathcal{O}(0.01)$
- typically of the same order for θ_{23} and $|U_{e3}|$
- solar neutrino mixing angle receives slightly larger corrections

VEV misalignment, NLO terms

- NLO terms, VEV misalignment due to terms allowed by the symmetry
  $\Rightarrow$ model-dependent!
  - Altarelli, Feruglio, Merlo, JHEP 0905:



$\delta \sin^2 \theta_{12} \simeq \delta |U_{e3}| = \mathcal{O}(\lambda)$ and $\delta \sin^2 \theta_{23} = \mathcal{O}(\lambda^2)$

- Altarelli, Feruglio, Hagedorn, JHEP 0803:
  corrections $\mathcal{O}(\lambda^2)$ to all mixing angles
- Lin, NPB 824:
  $\delta|U_{e3}| = \mathcal{O}(\lambda)$ and $\delta \sin^2\theta_{12} \simeq \delta \sin^2\theta_{23} = \mathcal{O}(\lambda^2)$
- Hagedorn, Ziegler, 1007.1888:
  $\delta|U_{e3}|^2 = \mathcal{O}(\lambda^2)$ and $\delta \sin^2\theta_{12} = \mathcal{O}(\lambda)$
- Ishimori *et al.*, 1004.5004:
  $\delta|U_{e3}|^2 = \mathcal{O}(\lambda^2)$ and $\delta \sin^2\theta_{12} = \mathcal{O}(\lambda)$ and $\delta \sin^2\theta_{23} = \mathcal{O}(\lambda^2)$
- etc.:
  etc.

Sign and size of RG correction

model	mass ordering	θ_{12}	θ_{23}
SM	$\Delta m^2_{31} > 0$	$\searrow$	$\searrow$
	$\Delta m^2_{31} < 0$	$\searrow$	$\nearrow$
MSSM	$\Delta m^2_{31} > 0$	$\nearrow$	$\nearrow$
	$\Delta m^2_{31} < 0$	$\nearrow$	$\searrow$

angle	NH	IH	QD
$\delta\theta_{12}$	1	$\Delta m^2_{\rm A}/\Delta m^2_{\odot}$	$m_0^2/\Delta m^2_{\odot}$
$\delta\theta_{13}$	1	1	$m_0^2/\Delta m^2_{\rm A}$
$\delta\theta_{23}$	1	1	$m_0^2/\Delta m^2_{\rm A}$

Note: potentially huge effect for θ_{12} unless (Majorana) phase suppression

Large $|U_{e3}|$ and RG

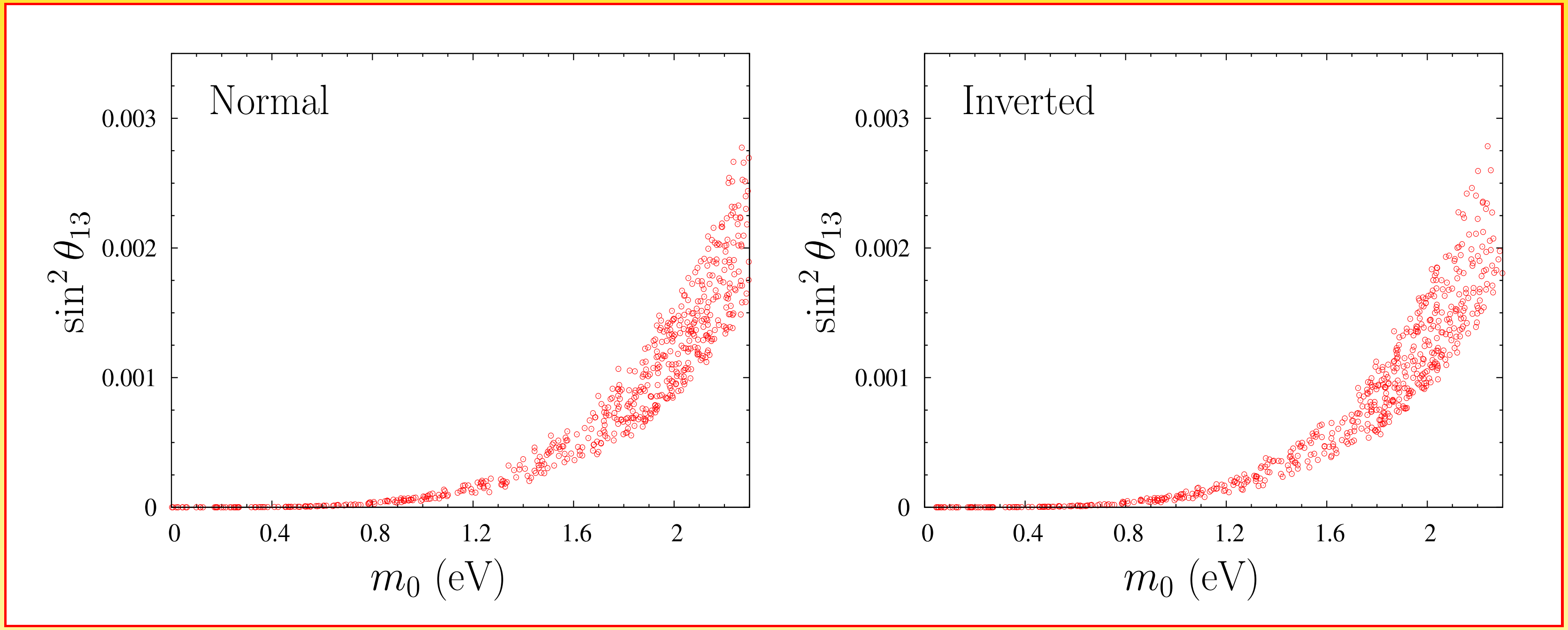
aim: get $|U_{e3}| = 0.1$ from TBM

- constraint: keep $\sin^2\theta_{12}$ close to TBM value
- what is $\sin^2\theta_{23}$?

Goswami, Petcov, Ray, W.R., PRD **80** (2009) 053013

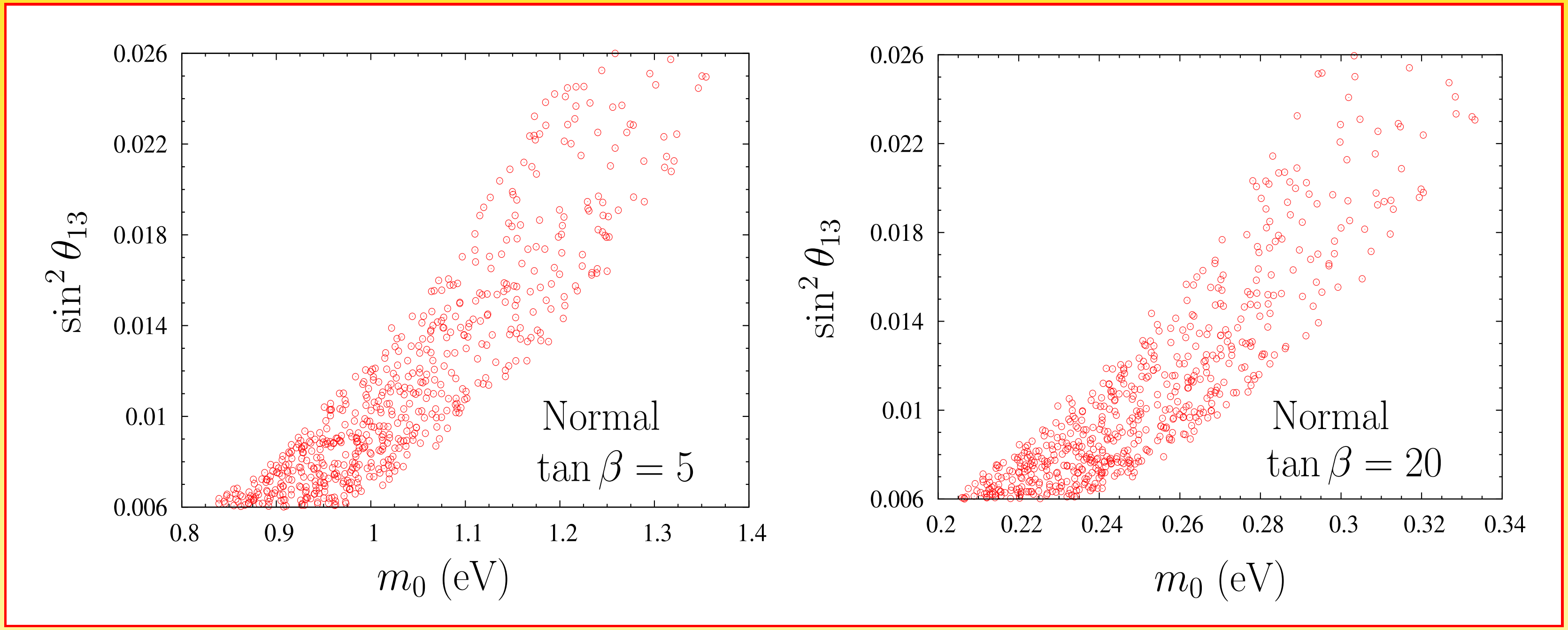
- we took "Bari hint": $0.077 \leq |U_{e3}| \leq 0.161$

Renormalization and $|U_{e3}| \simeq 0.1$

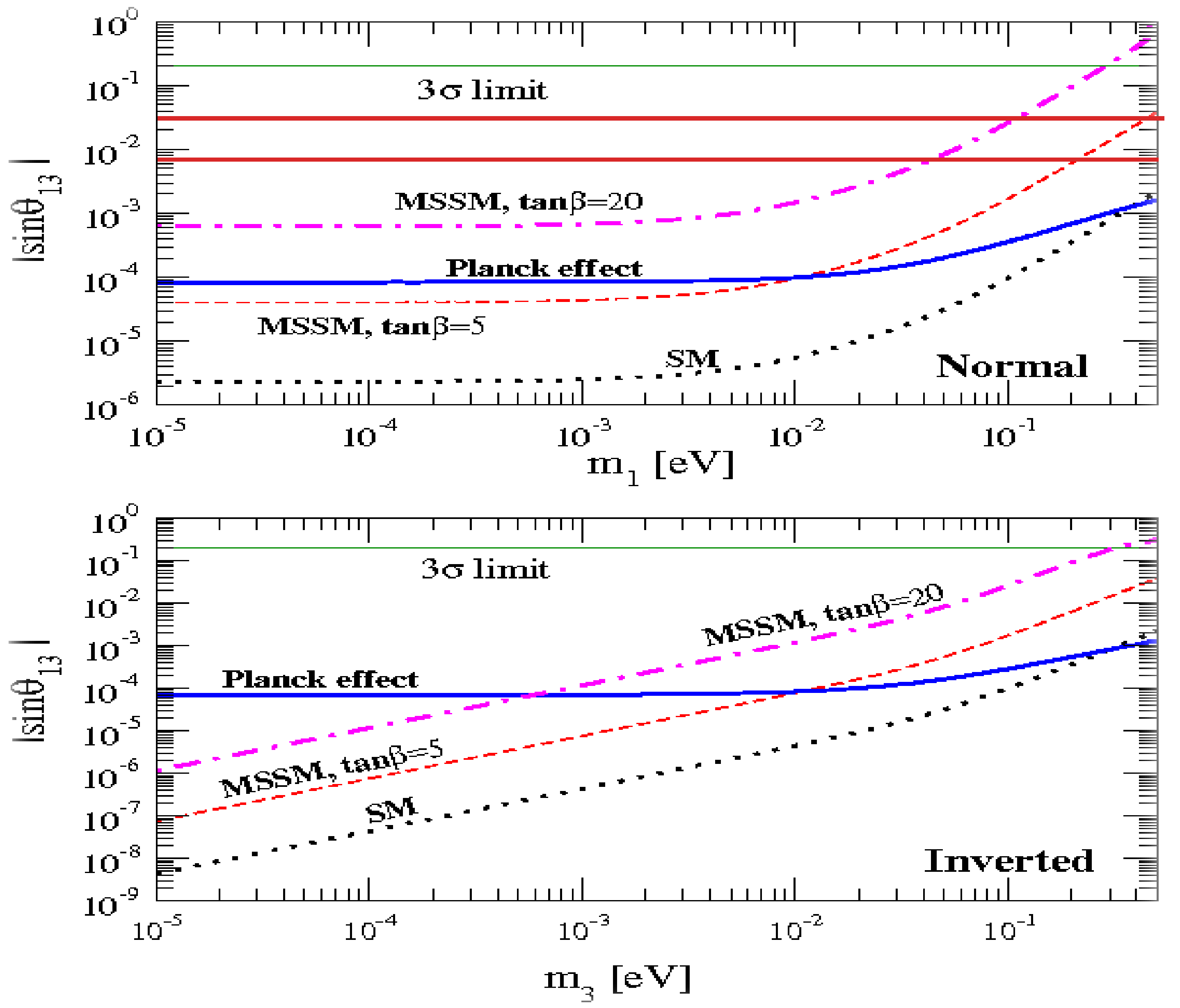


- SM: doesn't work

Renormalization and $|U_{e3}| \simeq 0.1$



- MSSM: quasi-degenerate neutrinos and $4 \lesssim (m_0/\mathrm{eV})\tan\beta \lesssim 7$


3σ limit
MSSM, tanβ=20
Planck effect
MSSM, tanβ=5
SM
Normal
$|\sin\theta_{13}|$
m$_1$ [eV]
3σ limit
MSSM, tanβ=20
Planck effect
MSSM, tanβ=5
SM
Inverted
$|\sin\theta_{13}|$
m$_3$ [eV]

Effect on θ_{12}

$$\theta_{12} \to \theta_{12}^0 + \epsilon_{\rm RG}\, k_{12}$$

with $\epsilon_{\rm RG} = C/(16\pi^2)\, m_\tau^2/v^2 \log \lambda/\Lambda$ and $C = -3/2$ or $C = (1 + \tan^2\beta)$

for solar mixing angle:

$$k_{12} = \frac{\sqrt{2}}{3}\,\frac{\left|m_1 + m_2\, e^{i\alpha_2}\right|^2}{\Delta m_\odot^2} \propto \begin{cases} 1 & \text{NH} \\ \dfrac{\Delta m_{\rm A}^2}{\Delta m_\odot^2}\,(1 + e^{i\alpha_2}) & \text{IH} \\ \dfrac{m_0^2}{\Delta m_\odot^2}\,(1 + e^{i\alpha_2}) & \text{QD} \end{cases}$$

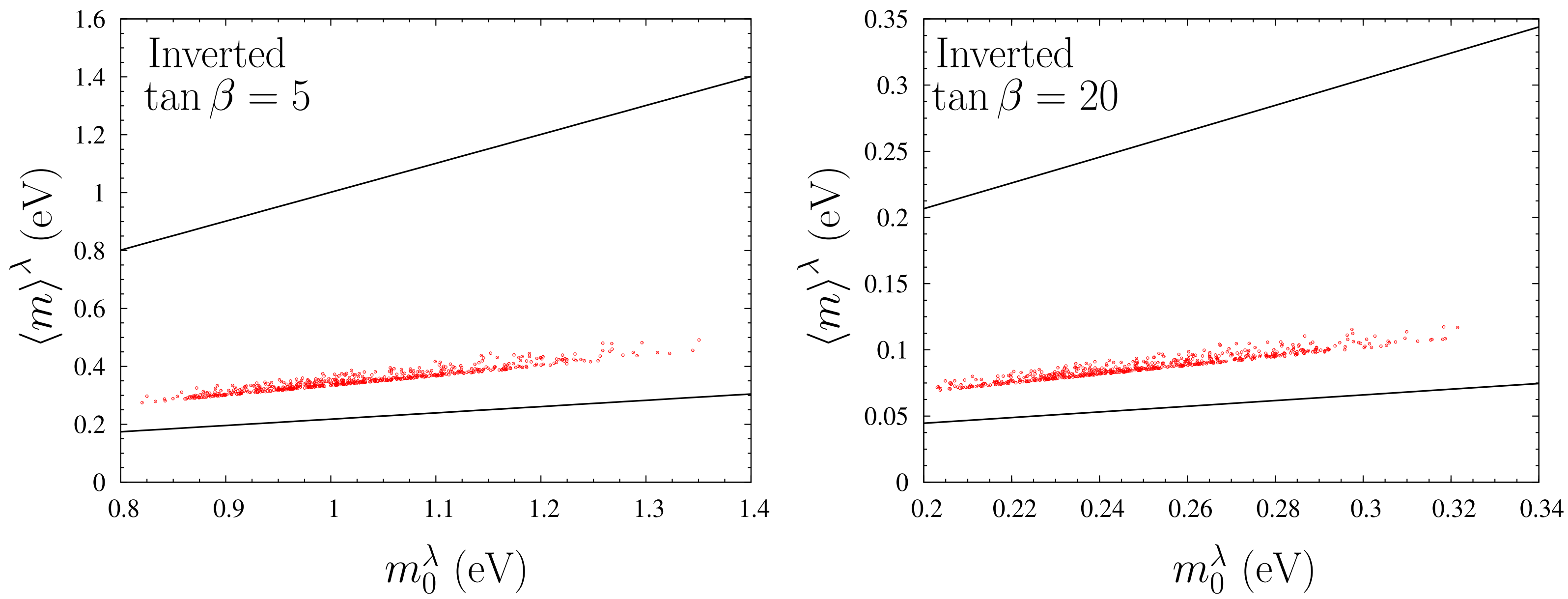
$\Rightarrow$ strong effect for IH and QD

$\Rightarrow$ suppress with $\alpha_2 = \pi$

$$|m_{ee}| \simeq m_0\,\sqrt{1 - \sin^2 2\theta_{12}\,\sin^2\alpha_2/2} \stackrel{\alpha_2=\pi}{\to} m_0\,\cos 2\theta_{12}$$

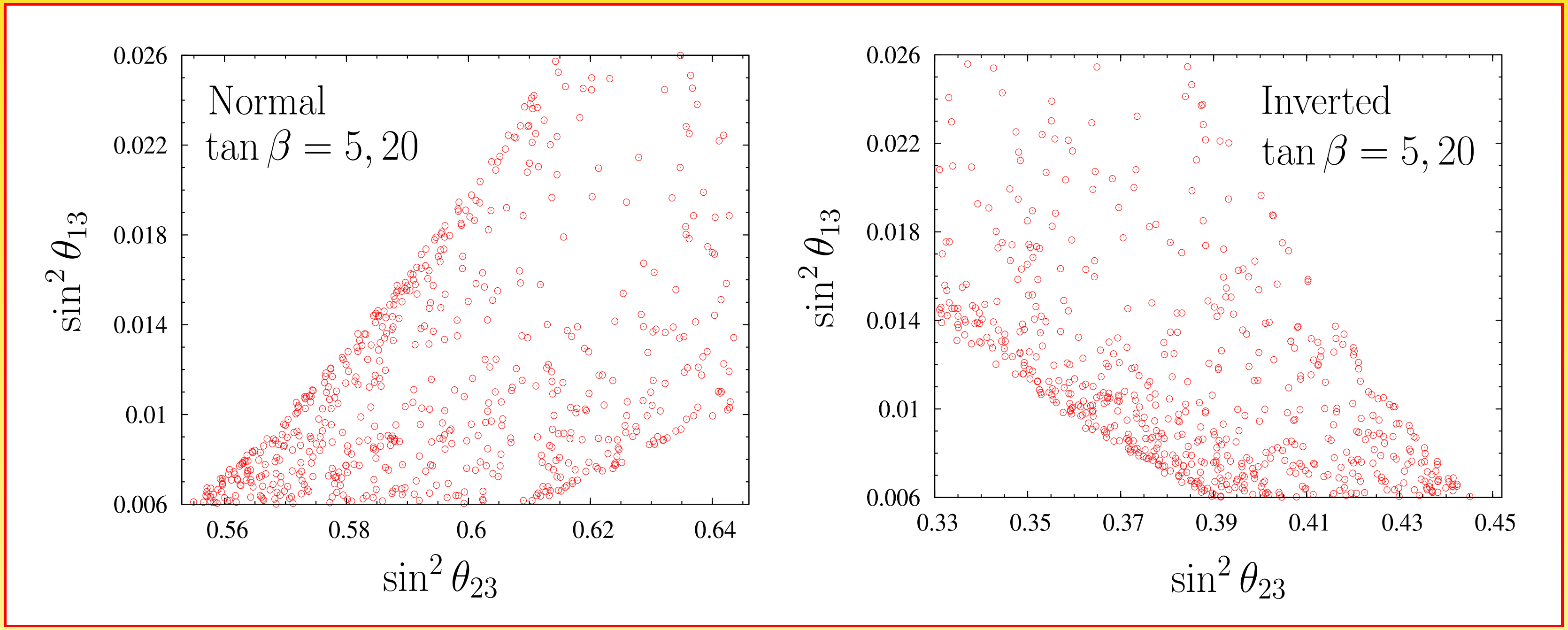
large cancellations in $0\nu\beta\beta$!

Renormalization and $|U_{e3}| \simeq 0.1$



- $|m_{ee}| \simeq c_{13}^2\, m_0\, |c_{12}^2 + s_{12}^2\, e^{i\alpha_2}|$
- $\tan\beta = 5$: $|m_{ee}|$ takes values between 0.26 and 0.50 eV; general upper and lower limits: 0.2 eV and 1.4 eV
- $\tan\beta = 20$: $|m_{ee}|$ takes values between 0.07 and 0.11 eV; general upper and lower limits: 0.05 eV and 0.34 eV

Renormalization and $|U_{e3}| \simeq 0.1$



- $|\theta_{23} - \pi/4| = \mathcal{O}(|U_{e3}|)$
- can NOT be maximal

Alternatives to TBM

- μ–τ symmetry $(Z_2, D_4, \ldots)$:

$$m_\nu = \begin{pmatrix} a & b & b \\ \cdot & d & e \\ \cdot & \cdot & d \end{pmatrix} \Rightarrow U_{e3} = 0,\ \theta_{23} = \pi/4$$

solar neutrino mixing unconstrained $(\theta_{12} = \mathcal{O}(1))$

`countless papers`

Alternatives to TBM

- Golden Ratio φ_1 (A_5)

$$\boxed{\cot\theta_{12} = \varphi} \quad \Rightarrow \sin^2\theta_{12} = \frac{1}{1+\varphi^2} = \frac{2}{5+\sqrt{5}} \quad \simeq 0.276$$

(Datta, Ling, Ramond; Kajiyama, Raidal, Strumia; Everett, Stuart)

- Golden Ratio φ_2 (D_5)

$$\boxed{\cos\theta_{12} = \frac{\varphi}{2}} \quad \Rightarrow \sin^2\theta_{12} = \frac{1}{4}(3-\varphi) = \frac{5-\sqrt{5}}{8} \quad \simeq 0.345$$

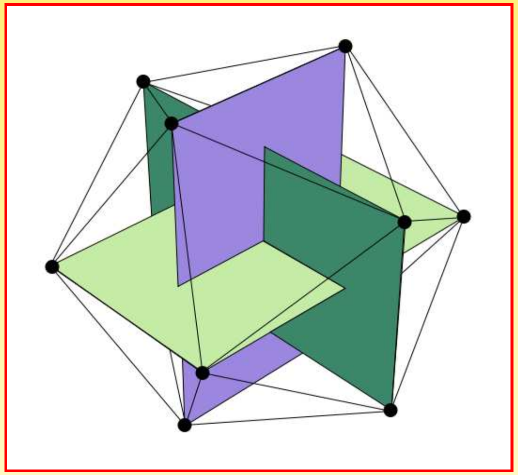
(W.R.; Adulpravitchai, Blum, W.R.)

Golden Ratio Prediction φ_1

$\boxed{\cot\theta_{12} = \varphi}$ or: $\tan 2\theta_{12} = 2$

can be generated by $m_\nu = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow Z_2 : \;\; S = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$

Model based on A_5 (isomorphic to rotational icosahedral symmetry group)?



Cartesian coordinates of its 12 vertices:

$$(0,\, \pm 1,\, \pm\varphi)$$

$$(\pm 1,\, \pm\varphi,\, 0)$$

$$(\pm\varphi,\, 0,\, \pm 1)$$

Golden Ratio Prediction φ_1

A_5 has irreps $\mathbf{1}$, $\mathbf{3}$, $\mathbf{3'}$, $\mathbf{4}$, $\mathbf{5}$

e.g., generators for triplet representation $\mathbf{3}$

$$S_3 = \frac{1}{2}\begin{pmatrix} -1 & \varphi & 1/\varphi \\ \varphi & 1/\varphi & 1 \\ 1/\varphi & 1 & -\varphi \end{pmatrix} \text{ and } T_3 = \frac{1}{2}\begin{pmatrix} 1 & \varphi & 1/\varphi \\ -\varphi & 1/\varphi & 1 \\ 1/\varphi & -1 & \varphi \end{pmatrix}$$

Everett, Stuart, PRD **79**, 085005 (2009)

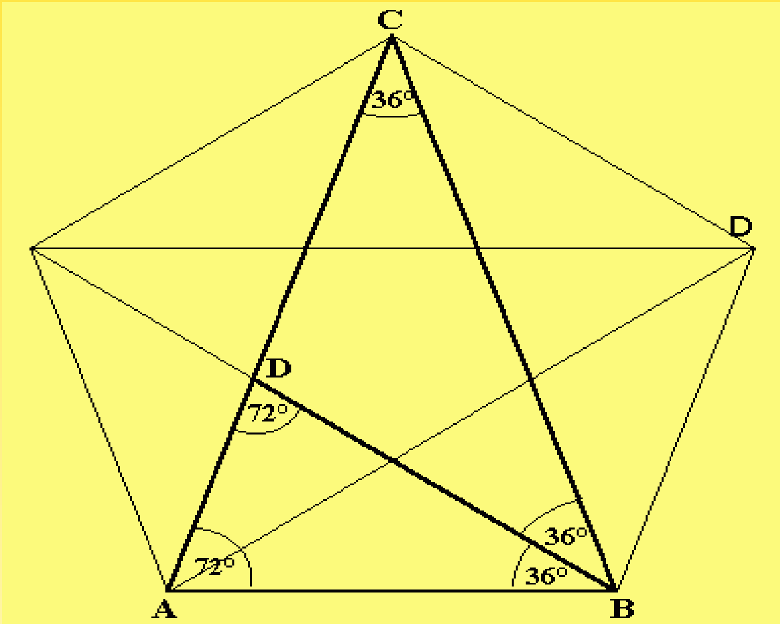
4th generation model Chen, Kephart, Yuan, 1011.3199

Golden Ratio Prediction φ_2

$$\boxed{\cos\theta_{12} = \varphi/2} \quad \text{or: } \theta_{12} = \frac{\pi}{5}$$

$$\sin^2\theta_{12} = \sin^2\pi/5 = \tfrac{5-\sqrt{5}}{8} \simeq 0.345$$

(W.R., PLB **671**, 267 (2009))

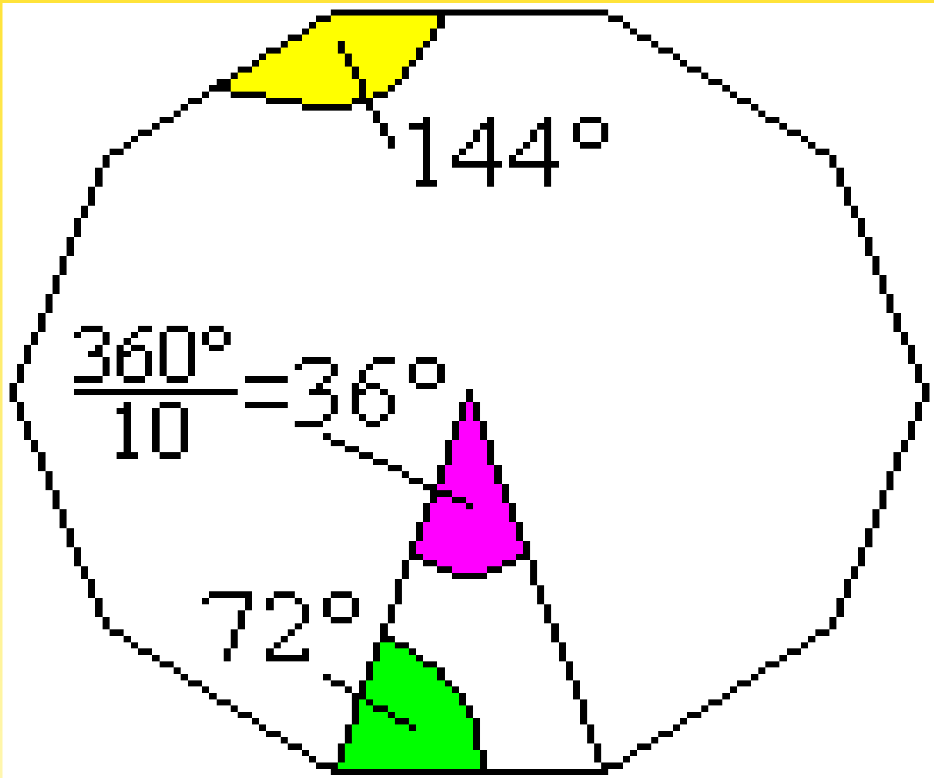


$$\overline{\mathrm{AD}} = \varphi\,\overline{\mathrm{AB}}$$

symmetry group of pentagon: D_5

Golden Ratio Prediction φ_2

symmetry group of decagon: D_{10}



A Model based on D_{10}

Adulpravitchai, Blum, W.R., New J. Phys. **11**, 063026 (2009)

Field	$l_{1,2}$	l_3	$e^c_{1,2}$	e^c_3	$h_{u,d}$	σ^e	$\chi^e_{1,2}$	$\xi^e_{1,2}$	$\rho^e_{1,2}$	σ^ν	$\varphi^\nu_{1,2}$	$\chi^\nu_{1,2}$	$\xi^\nu_{1,2}$
D_{10}	$\underline{\mathbf{2}}_4$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{2}}_2$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{2}}_2$	$\underline{\mathbf{2}}_3$	$\underline{\mathbf{2}}_4$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{2}}_1$	$\underline{\mathbf{2}}_2$	$\underline{\mathbf{2}}_3$
Z_5	ω	ω	ω^2	ω^2	1	ω^2	ω^2	ω^2	ω^2	ω^3	ω^3	ω^3	ω^3

Dihedral Groups

Blum, Hagedorn, Lindner, Hohenegger, PRD **77**, 076004 (2008):

D_n has several $\mathbf{2_j}$, generated by

$$A = \begin{pmatrix} e^{2\pi i \frac{j}{n}} & 0 \\ 0 & e^{-2\pi i \frac{j}{n}} \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and Z_2 is generated by

$$B\,A^k = \begin{pmatrix} 0 & e^{-2\pi i \frac{j}{n}\,k} \\ e^{2\pi i \frac{j}{n}\,k} & 0 \end{pmatrix}$$

Thus, break D_n such that m_ν invariant under $B\,A^{k_\nu}$ and m_ℓ under $B\,A^{k_\ell}$:

$$|U_{e1}|^2 = \left|\cos \pi \frac{j}{n} \left(k_\nu - k_\ell\right)\right|^2$$

Again, D_5 or D_{10} to obtain $\pi/5$

- hexagonal mixing (D_6)

$$\theta_{12} = \pi/6 \Rightarrow \sin^2\theta_{12} = \frac{1}{4}$$

(Albright, Dueck, W.R.; Kim, Seo)

Kim, Seo, 1005.4684: D_{12} model

introduces 13 Higgs doublets(...), but achieved (QLC) $\theta_{12} = \pi/6$, $\theta_C = \pi/12 = 15^0 \simeq \theta_C + 1.8^0$ and called it dodecal

Systematic Search for Mixing Patterns

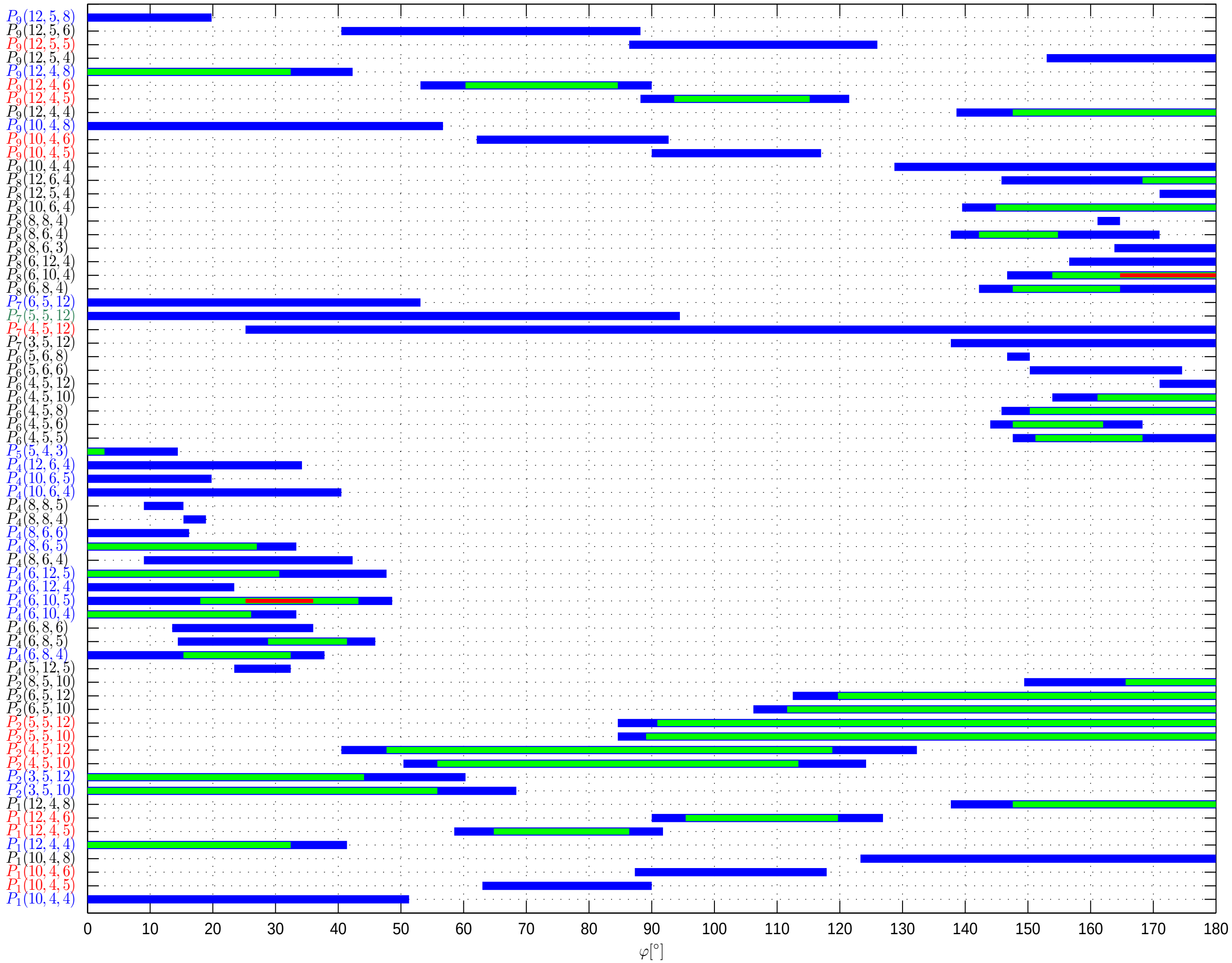
$$
\begin{aligned}
P_1: \quad U &= R_{12}(\vartheta_1) R_{23}(\vartheta_2, \varphi) R_{12}^{-1}(\vartheta_3) \\
P_2: \quad U &= R_{23}(\vartheta_1) R_{12}(\vartheta_2, \varphi) R_{23}^{-1}(\vartheta_3) \\
P_3: \quad U &= R_{23}(\vartheta_1) R_{13}(\vartheta_2, \varphi) R_{12}(\vartheta_3) \\
P_4: \quad U &= R_{12}(\vartheta_1) R_{13}(\vartheta_2, \varphi) R_{23}^{-1}(\vartheta_3) \\
P_5: \quad U &= R_{13}(\vartheta_1) R_{12}(\vartheta_2, \varphi) R_{13}^{-1}(\vartheta_3) \\
P_6: \quad U &= R_{12}(\vartheta_1) R_{23}(\vartheta_2, \varphi) R_{13}(\vartheta_3) \\
P_7: \quad U &= R_{23}(\vartheta_1) R_{12}(\vartheta_2, \varphi) R_{13}^{-1}(\vartheta_3) \\
P_8: \quad U &= R_{13}(\vartheta_1) R_{12}(\vartheta_2, \varphi) R_{23}(\vartheta_3) \\
P_9: \quad U &= R_{13}(\vartheta_1) R_{23}(\vartheta_2, \varphi) R_{12}^{-1}(\vartheta_3)
\end{aligned}
$$

and choose π/n for the ϑ_i with $n = 1, 2, 3, 4, 5, 6, 8, 10, 12$

$\Rightarrow 9^4 = 6561$ possibilities

Systematic Search for Mixing Patterns

$\vartheta = \frac{\pi}{n}$	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$\frac{\pi}{6}$	$\frac{\pi}{8}$	$\frac{\pi}{10}$	$\frac{\pi}{12}$
$\sin^2\vartheta$	0	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{5-\sqrt{5}}{8}$	$\frac{1}{4}$	$\frac{2-\sqrt{2}}{4}$	$\frac{3-\sqrt{5}}{8}$	$\frac{2-\sqrt{3}}{4}$
$\cos^2\vartheta$	1	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3+\sqrt{5}}{8}$	$\frac{3}{4}$	$\frac{2+\sqrt{2}}{4}$	$\frac{5+\sqrt{5}}{8}$	$\frac{2+\sqrt{3}}{4}$



W.R., Zhang, Zhou, NPB **855**

Alternatives to TBM

Bi-maximal

$$U_{\rm BM} = \begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \sqrt{\frac{1}{2}} \\ \frac{1}{2} & -\frac{1}{2} & \sqrt{\frac{1}{2}} \end{pmatrix}$$

S_4: Altarelli, Feruglio, Merlo, JHEP **0905**, 020 (2009) (needs large NLO corrections)

CKM(-like) charged lepton corrections may also resurrect it:

- $\mathrm{QLC}_0 : \theta_{12} = \frac{\pi}{4} - \theta_C \Rightarrow \sin^2\theta_{12} \simeq 0.280$
- $\mathrm{QLC}_1 : U = V^\dagger\, U_{\rm BM} \Rightarrow \sin^2\theta_{12} \simeq \frac{1}{2} - \lambda/\sqrt{2}\,\cos\phi \simeq 0.331\ldots 0.670$
- $\mathrm{QLC}_2 : U = U_{\rm BM}\, V^\dagger \Rightarrow \sin^2\theta_{12} \simeq \frac{1}{2} - \lambda\,\cos\phi' \simeq 0.276\ldots 0.762$

"Quark-Lepton Complementarity"

Alternatives to TBM

Tri-maximal Mixing(s)

- TM$_2$ $(S_{3,4}, \Delta(27))$

$$\begin{pmatrix} |U_{e2}|^2 \\ |U_{\mu 2}|^2 \\ |U_{\tau 2}|^2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

(Lam; Grimus, Lavoura)

- TM$_1$, TM$_3$, TM^1, TM^2, TM^3, e.g.,

$$\text{TM}^1: \qquad \left(|U_{e1}|^2, \; |U_{e2}|^2, \; |U_{e3}|^2\right) = \left(\tfrac{2}{3}, \; \tfrac{1}{3}, \; 0\right)$$

$$\text{TM}_1: \qquad \begin{pmatrix} |U_{e1}|^2 \\ |U_{\mu 1}|^2 \\ |U_{\tau 1}|^2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/6 \\ 1/6 \end{pmatrix}$$

(Lam; Albright, W.R.; Friedberg, Lee; He, Zee)

Minimal Modification of TBM

want non-zero θ_{13} and $\sin^2\theta_{12} \leq \frac{1}{3}$:

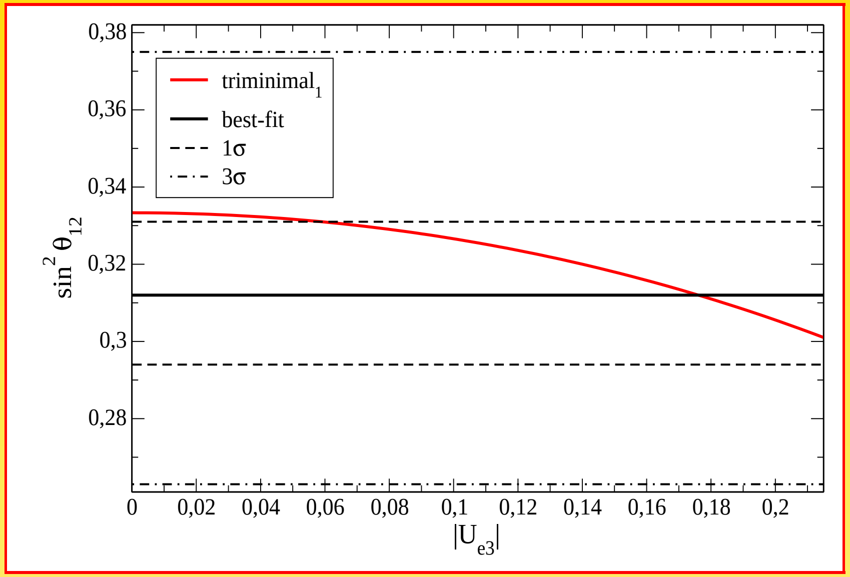
$$\mathrm{TM}_1 : \quad \begin{pmatrix} |U_{e1}|^2 \\ |U_{\mu 1}|^2 \\ |U_{\tau 1}|^2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/6 \\ 1/6 \end{pmatrix}$$

gives observables

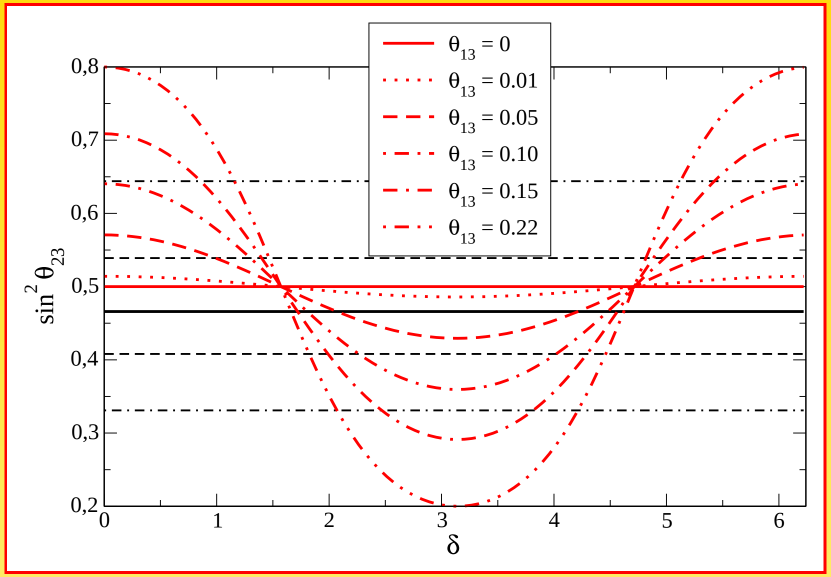
$$\sin^2\theta_{12} = \frac{1}{3}\,\frac{1 - 3\,|U_{e3}|^2}{1 - |U_{e3}|^2} \simeq \frac{1}{3}\,\left(1 - 2\,|U_{e3}|^2\right) \leq \frac{1}{3}$$

$$\cos\delta\,\tan 2\theta_{23} = -\frac{1 - 5\,|U_{e3}|^2}{2\sqrt{2}\,|U_{e3}|\,\sqrt{1 - 3\,|U_{e3}|^2}} \simeq \frac{-1}{2\sqrt{2}\,|U_{e3}|} + \frac{7}{4\sqrt{2}}\,|U_{e3}|$$

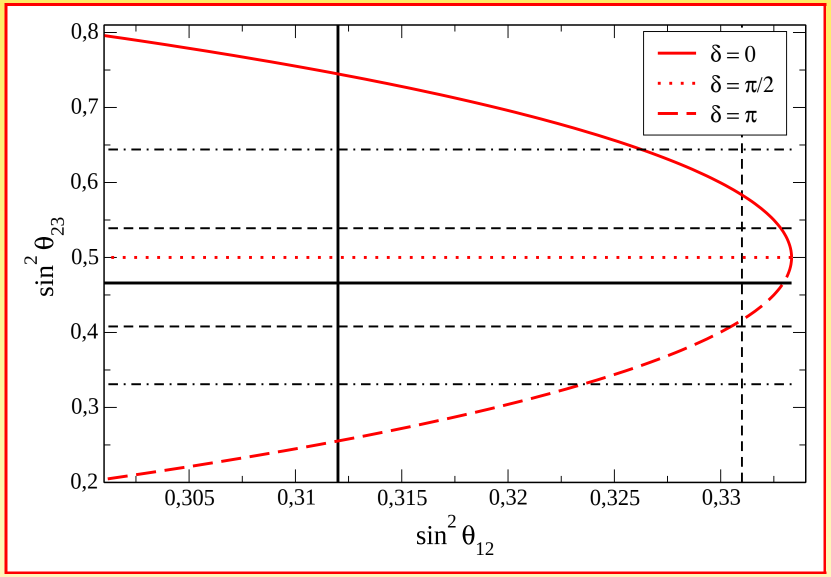
`Albright, W.R., EPJC`**62**



triminimal1
best-fit
1σ
3σ
sin2 θ12
|Ue3|
0,38
0,36
0,34
0,32
0,3
0,28
0
0,02
0,04
0,06
0,08
0,1
0,12
0,14
0,16
0,18
0,2



θ13 = 0
θ13 = 0.01
θ13 = 0.05
θ13 = 0.10
θ13 = 0.15
θ13 = 0.22
sin2 θ23
δ
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0
1
2
3
4
5
6



δ = 0
δ = π/2
δ = π
sin2 θ23
sin2 θ12
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,305
0,31
0,315
0,32
0,325
0,33

Alternatives to TBM

- tetra-maximal (Xing; Zhang, Zhou)

$$U = \text{diag}(1, 1, i)\, \tilde{R}_{23}(\pi/4; \pi/2)\, \tilde{R}_{13}(\pi/4; 0)\, \tilde{R}_{12}(\pi/4; 0)\, \tilde{R}_{13}(\pi/4; \pi)$$

- symmetric mixing $U = U^T$ (Joshipura, Smirnov; Hochmuth, W.R.)

$$|U_{e3}| = \frac{\sin\theta_{12}\, \sin\theta_{23}}{\sqrt{1 - \sin^2\delta\, \cos^2\theta_{12}\, \cos^2\theta_{23}} + \cos\delta\, \cos\theta_{12}\, \cos\theta_{23}}$$

Alternatives to TBM

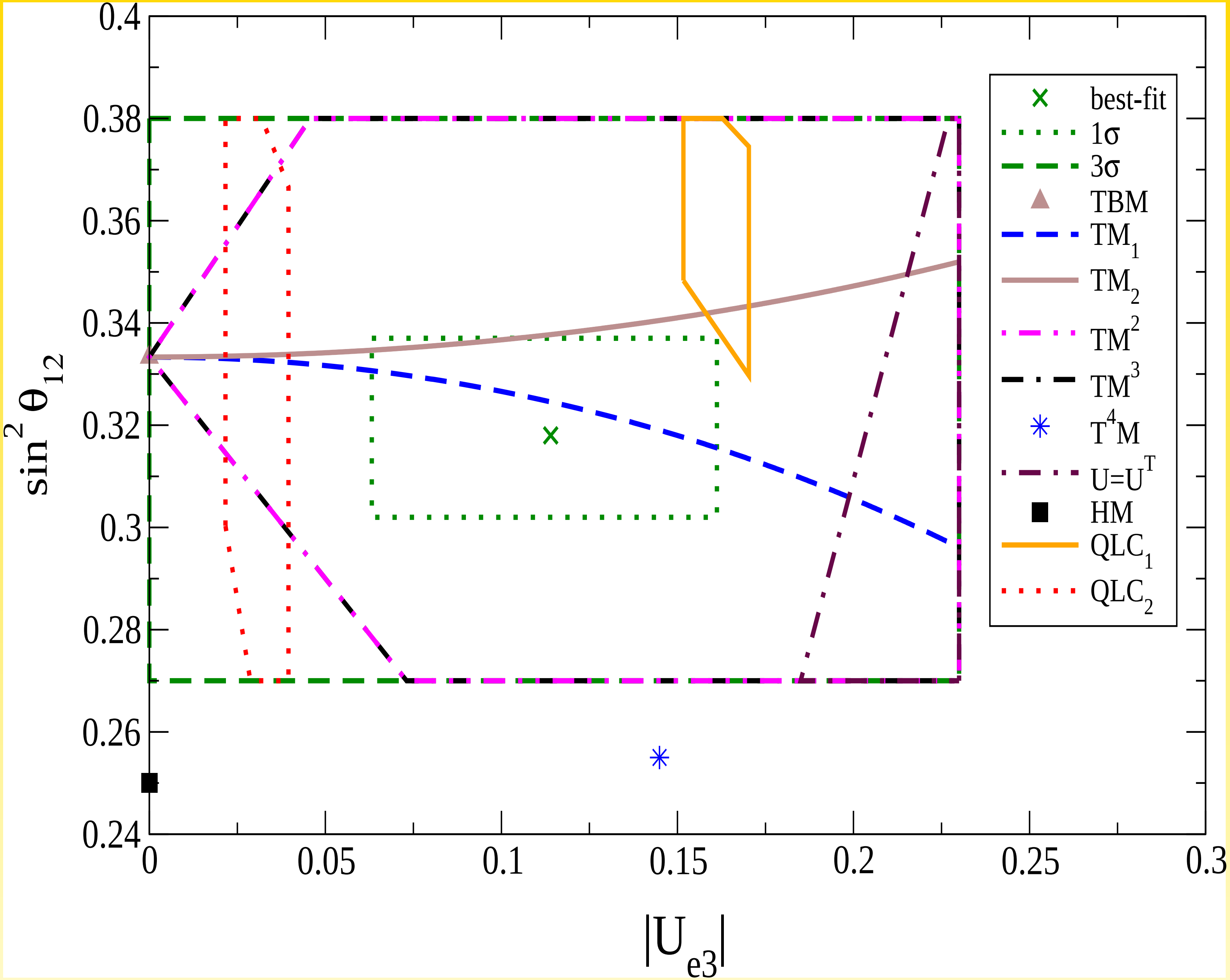
- various other proposals after T2K and DC: Xing, 1106.3244; Qui, Ma, 1106.3284; He, Zee, 1106.4359; Zheng, Ma, 1106.4040; Zhou, 1106.4808; Araki, 1106.5211; Haba, Takahashi, 1106.5926; Morisi, Patel, Peinado, 1107.0696, Chao, Zheng, 1107.0738; Zhang, Zhou, 1107.1097; Chu, Dhen, Hambye, 1107.1589; Toorop, Feruglio, Hagedorn, 1107.3486; Antusch, Maurer, 1107.3728; Rodejohann, Zhang, Zhou, 1107.3970; Ahn, Cheng, Oh, 1107.4549; Marzocca, Petcov, Romanino, Spinrath, 1108.0614; Ge, Dicus, Repko, 1108.0964; Riazuddin, 1108.1469; Ludl, Morisi, Peinado, 1109.3393; Verma, 1109.4228; Meloni, 1110.5210; Kitabayashi, Yasue, 1110.5162; He, Majee, 1111.2293; Rashed, 1111.3072; Buchmuller, Domcke, Schmitz, 1111.3872; King, Luhn, 1112.1959; Eby, Frampton, 1112.2675; Gupta, Joshipura, Patel, 1112.6113,...

Once you start playing with numbers...

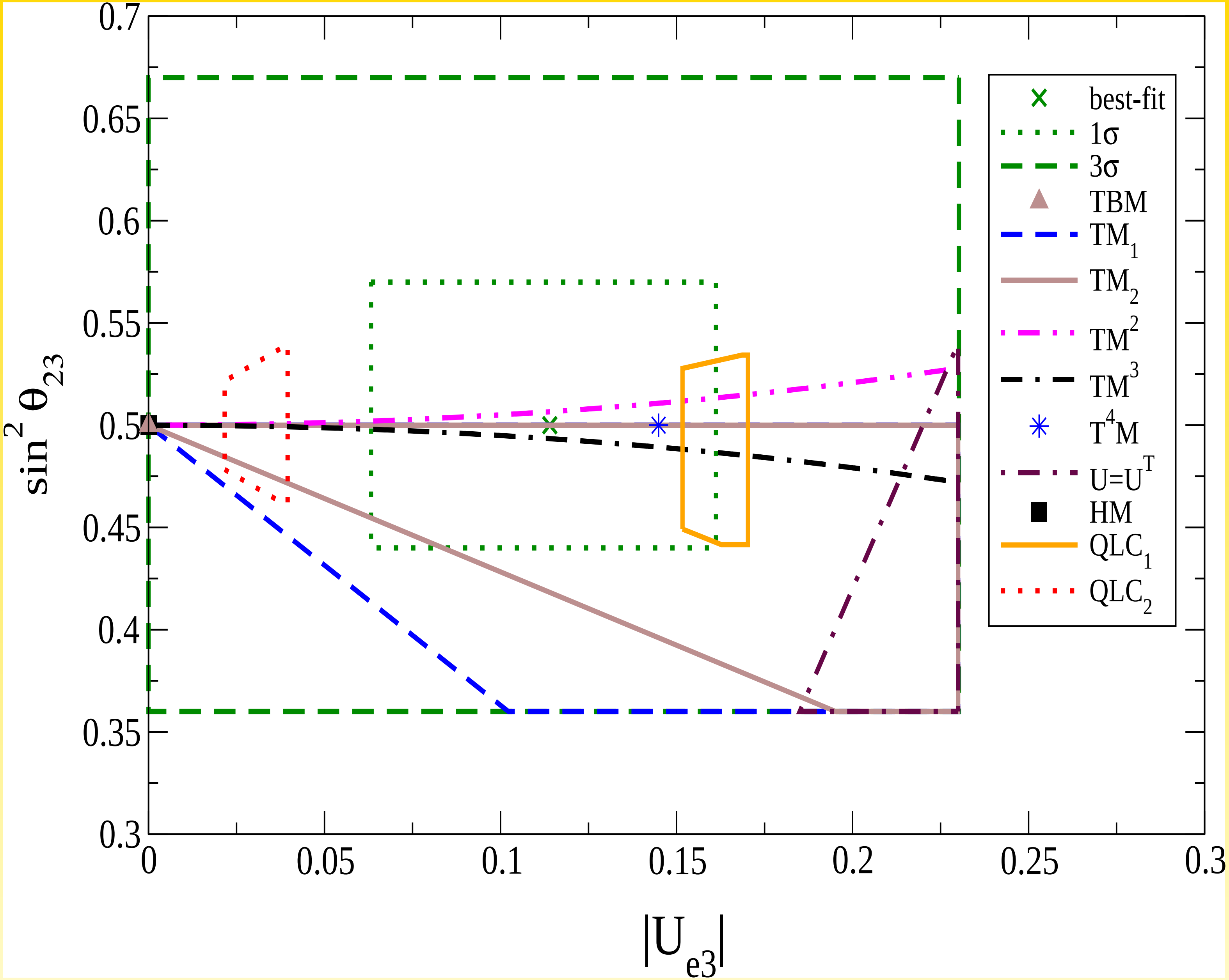
- "transcendental mixing":
  - $\sin\theta_{13} = \theta_{13} \Rightarrow \theta_{13} = 0$
    is the fixed point of $\sin(x)$
  - $\cos\theta_{23} = \theta_{23} = 0.739085133\ldots \Rightarrow \sin^2\theta_{23} = 0.454$
    is the fixed point of $\cos(x)$
    "Dottie's number" $d = \lim\limits_{n\to\infty} \cos_n(x)$ is irrational like π, e, $\sqrt{2}$
- Euler-Mascheroni constant:
  - $\theta_{12} = \gamma = 0.577215664\ldots \Rightarrow \sin^2\theta_{12} = 0.298$
  - $|U_{e2}| = \gamma \Rightarrow \sin^2\theta_{12} = 0.298\ldots0.314$
- Euler's number:
  - $\tan 2\theta_{12} = e = 2.718281828\ldots \Rightarrow \sin^2\theta_{12} = 0.327$
- etc   :-))

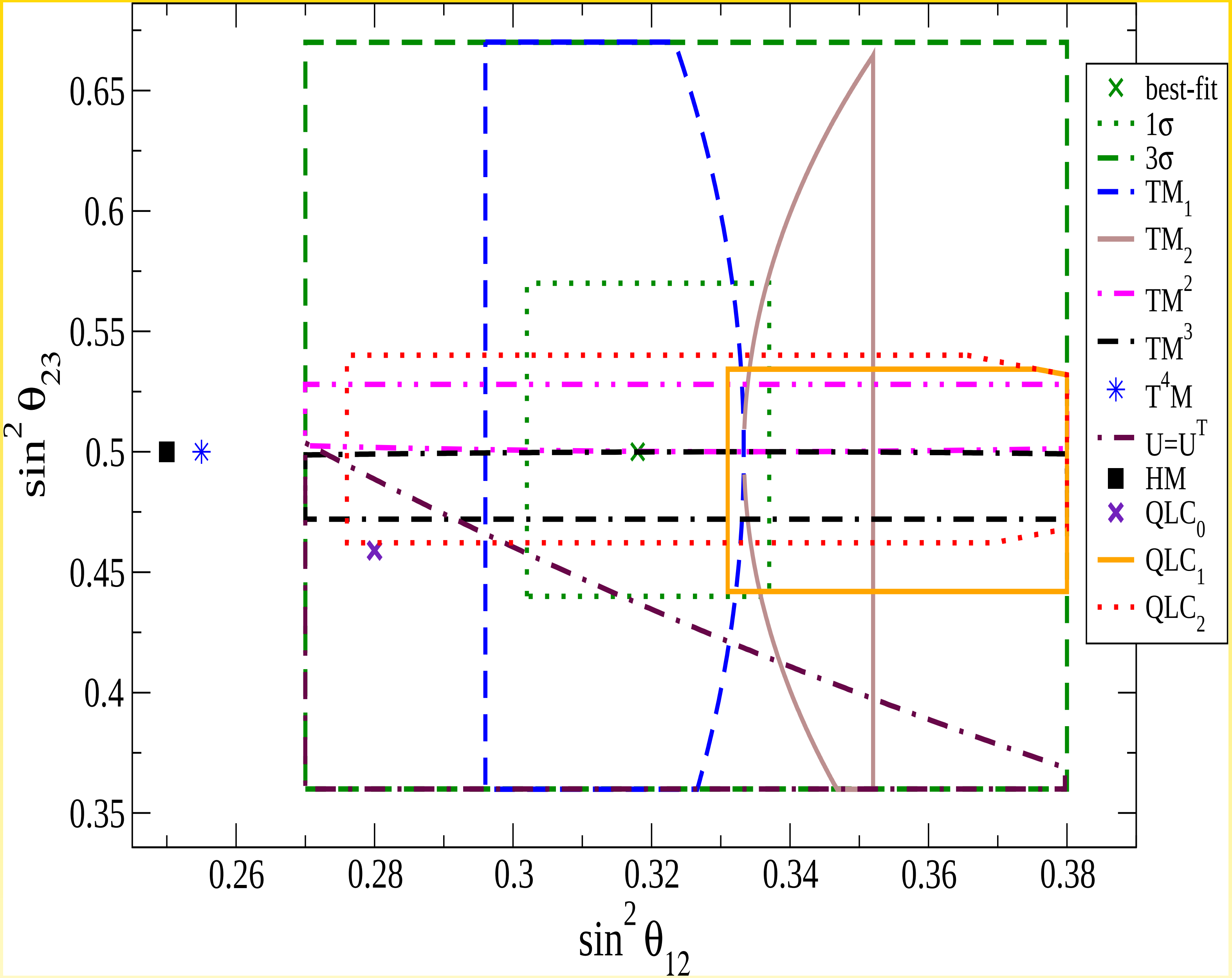
Scenario	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\sin^2\theta_{13}$	T2K/DC
TBM	0.333	0.500	0.000	–
$\mu-\tau$	—	0.500	0.000	–
TM_1	0.296 0.333	**	—	✓
TM_2	0.333 0.352	**	—	✓
TM_3	—	0.500	0.000	–
TM^1	0.333	—	0.000	–
TM^2	**	0.500 0.528	—	✓
TM^3	**	0.472 0.500	—	✓
T^4M	0.255	0.500	0.021	✓
$U=U^T$	0.000 0.389	0.000 0.504	0.0343 0.053	✓
BM	0.500	0.500	0.000	–
HM	0.250	0.500	0.000	–
φ_1	0.276	0.500	0.000	–
φ_2	0.345	0.500	0.000	–
QLC_0	0.280	0.459	—	–
QLC_1	0.331 0.670	0.442 0.534	0.023 0.029	✓
QLC_2	0.276 0.726	0.462 0.540	0.0005 0.0016	–

Albright, Dueck, W.R., 1004.2798



0.4
0.38
0.36
0.34
0.32
0.3
0.28
0.26
0.24
$\sin^2\theta_{12}$
0
0.05
0.1
0.15
0.2
0.25
0.3
$|U_{e3}|$
best-fit
1σ
3σ
TBM
TM_1
TM_2
TM^2
TM^3
T^4M
$U=U^T$
HM
QLC_1
QLC_2


best-fit
1σ
3σ
TBM
TM1
TM2
TM2
TM3
T4M
U=UT
HM
QLC1
QLC2
sin2 θ23
|Ue3|
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0
0.05
0.1
0.15
0.2
0.25
0.3


best-fit
1σ
3σ
TM1
TM2
TM2
TM3
T4M
U=UT
HM
QLC0
QLC1
QLC2
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.26
0.28
0.3
0.32
0.34
0.36
0.38
sin2 θ23
sin2 θ12

Symmetry and Flavor Symmetry

Each Majorana mass matrix is invariant under $Z_2 \times Z_2$ (Grimus, Lavoura; Lam)

$$m_\nu = \begin{pmatrix} a & b & b \\ \cdot & d & e \\ \cdot & \cdot & d \end{pmatrix} \text{ invariant under } R_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

look for S such that

$$[S, R_{\mu\tau}] = 0 \; , \quad SS = \mathbb{1}$$

Solvable for general θ_{12} and for special scenarios, e.g.

$d + e = a$	bimaximal
$d + e = a + b$	tri-bimaximal
$d + e = a + \sqrt{2}\, b$	golden ratio$_1$
$d + e = a + 2\sqrt{2}\, b \cot 2\theta_{12}$	general

Symmetry and Flavor Symmetry

$$S = \begin{pmatrix} \cos 2\theta_{12} & -\sqrt{2}\,\cos\theta_{12}\,\sin\theta_{12} & \sqrt{2}\,\cos\theta_{12}\,\sin\theta_{12} \\ \cdot & \sin^2\theta_{12} & \cos^2\theta_{12} \\ \cdot & \cdot & \sin^2\theta_{12} \end{pmatrix}$$

charged leptons are diagonal

$$T^\dagger\, m_\ell^\dagger\, m_\ell\, T = m_\ell^\dagger\, m_\ell$$

With $T = \mathrm{diag}(-1, i, -i)$ it follows for $\theta_{12} = \pi/4$ (bimaximal)

$$S^2 = T^4 = (S\,T)^3 = \mathbb{1} \Longrightarrow S_4$$

Interpretation: flavor symmetry G_f generated by S, T broken such that m_ν invariant under S and charged leptons under T ($R_{\mu\tau}$ is accidental)

What's special about TBM?

TBM mass matrix

$$\begin{pmatrix} A & B & B \\ \cdot & \frac{1}{2}(A+B+D) & \frac{1}{2}(A+B-D) \\ \cdot & \cdot & \frac{1}{2}(A+B+D) \end{pmatrix}$$

is invariant under

$$R_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and } S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

Note: S and $T = \mathrm{diag}(1, \omega^2, \omega)$ generate A_4 via $S^2 = T^3 = (ST)^3 = \mathbb{1}$

in many A_4 models: $R_{\mu\tau}$ accidental, charged leptons preserve Z_3 invariance via T, neutrinos preserve Z_2 invariance via S

Scenario	S	T	relations	group
bimaximal	$\sqrt{\frac{1}{2}}\begin{pmatrix} 0 & -1 & 1 \\ \cdot & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \cdot & \cdot & \sqrt{\frac{1}{2}} \end{pmatrix}$	$\mathrm{diag}(-1, i, -i)$	$T^4 = (S\,T)^3 = \mathbb{1}$	S_4
tri-bimaximal	$\frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ \cdot & -1 & -2 \\ \cdot & \cdot & -1 \end{pmatrix}$	$\mathrm{diag}(e^{-2i\pi/3}, e^{2i\pi/3}, 1)$	$T^3 = (S\,T)^3 = \mathbb{1}$	A_4
golden ratio (A)	$\frac{-1}{\sqrt{5}}\begin{pmatrix} 1 & -\sqrt{2} & \sqrt{2} \\ \cdot & 1/\varphi & \varphi \\ \cdot & \cdot & 1/\varphi \end{pmatrix}$	$\mathrm{diag}(1, e^{-4i\pi/5}, e^{4i\pi/5})$	$T^5 = (S\,T)^3 = \mathbb{1}$	A_5

example: $G_f = \Delta(96)$

generated by $S^2 = (ST)^3 = T^8 = (ST^{-1}ST)^3 = \mathbb{1}$ with

$$S = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \cdot & -1 & 1 \\ \cdot & \cdot & -1 \end{pmatrix} \text{ and } T = \begin{pmatrix} e^{6i\pi/4} & 0 & 0 \\ \cdot & e^{7i\pi/4} & 0 \\ \cdot & \cdot & e^{3i\pi/4} \end{pmatrix}$$

assumption (1): charged leptons invariant under $G_e = Z_3$; neutrinos under $G_\nu = Z_2 \times Z_2$

assumption (2): $G_e = ST$ and $G_\nu = \left\{S, ST^4ST^4\right\}$

$$|U| = \sqrt{\frac{1}{3}}\begin{pmatrix} \frac{1}{2}(\sqrt{3}+1) & 1 & \frac{1}{2}(\sqrt{3}-1) \\ \frac{1}{2}(\sqrt{3}-1) & 1 & \frac{1}{2}(\sqrt{3}+1) \\ 1 & 1 & 1 \end{pmatrix}$$

Toorop, Feruglio, Hagedorn, PLB **703**

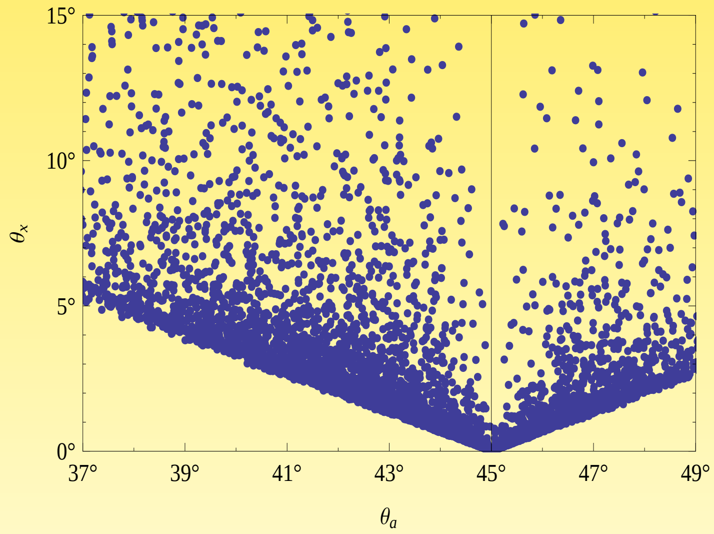
Hidden Z_2

recall: μ–τ symmetric mass matrix is invariant under

$$R_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} -\cos 2\theta_{12} & \sin 2\theta_{12}/\sqrt{2} & \sin 2\theta_{12}/\sqrt{2} \\ \cdot & \cos^2\theta_{12} & -\sin^2\theta_{12} \\ \cdot & \cdot & \cos^2\theta_{12} \end{pmatrix}$$

S is called "hidden Z_2" (Ge, He, Yin; He, Yin; Dicus, Ge, Repko)

invariance under S means $[m_\nu, S(\theta_{12})] = 0$:



Generalization

$$S = \begin{pmatrix} -\cos 2\theta_s & \sin 2\theta_s/\sqrt{2} & \sin 2\theta_s/\sqrt{2} \\ \cdot & \cos^2\theta_s & -\sin^2\theta_s \\ \cdot & \cdot & \cos^2\theta_s \end{pmatrix}$$

is also a reflection, i.e. $S \in O(2) \backslash SO(2)$

in basis in which $(\nu_e, \nu_\mu, \nu_\tau)$ becomes $(\nu_e,\, (\nu_\mu + \nu_\tau)/\sqrt{2}, (\nu_\mu - \nu_\tau)/\sqrt{2})$:

$$G_O \equiv U_{23}^T(-\pi/4)\, S\, U_{23}(-\pi/4) = \begin{pmatrix} -\cos 2\theta_s & \sin 2\theta_s & 0 \\ \sin 2\theta_s & \cos 2\theta_s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

most general reflection!

To get $O(2)$, we need to find rotation

To get $O(2)$, we need to find rotation

$$R(\theta_s) = G(\theta_s/2)G(0) = \begin{pmatrix} c_s & s_s/\sqrt{2} & s_s/\sqrt{2} \\ -s_s/\sqrt{2} & \cos^2(\theta_s/2) & -\sin^2(\theta_s/2) \\ -s_s/\sqrt{2} & -\sin^2(\theta_s/2) & \cos^2(\theta_s/2) \end{pmatrix}$$

S and R span the group $O(2)$

The most general $O(2)$ invariant mass matrix comes from $[m_\nu, S(\theta_s)] = 0\ \forall\theta_s$:

$$m_\nu = \begin{pmatrix} m_1 & 0 & 0 \\ \cdot & (m_1+m_3)/2 & (m_1-m_3)/2 \\ \cdot & \cdot & (m_1+m_3)/2 \end{pmatrix}$$

automatically μ–τ symmetric and $\Delta m^2_{12} = 0$

"Hidden $O(2)$" (Heeck, W.R., 1112.3628)

Hidden $O(2)$

$$m_\nu = \begin{pmatrix} m_1 & 0 & 0 \\ \cdot & (m_1+m_3)/2 & (m_1-m_3)/2 \\ \cdot & \cdot & (m_1+m_3)/2 \end{pmatrix}$$

flavor democratic perturbation

$$\mu \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

gives TBM!


And now for
something
completely
different...

Sterile Neutrinos??

- LSND/MiniBooNE
- cosmology
- BBN
- r-process nucleosynthesis in Supernovae
- reactor anomaly (Mention *et al.*, PRD **83**)

	$\Delta m^2_{41}\,[\mathrm{eV}^2]$	$\lvert U_{e4}\rvert$	$\lvert U_{\mu 4}\rvert$	$\Delta m^2_{51}\,[\mathrm{eV}^2]$	$\lvert U_{e5}\rvert$	$\lvert U_{\mu 5}\rvert$
3+2/2+3	0.47	0.128	0.165	0.87	0.138	0.148
1+3+1	0.47	0.129	0.154	0.87	0.142	0.163

or $\Delta m^2_{41} = 1.78$ eV^2 and $|U_{e4}|^2 = 0.151$

Kopp, Maltoni, Schwetz, 1103.4570

How to incorporate $\nu_{\rm st}$ in Flavor Symmetry Models

- add sterile neutrino to an effective theory (Barry, W.R., Zhang)
- seesaw mechanism: how to make $\nu_{\rm st}$ light?
  - extra dimensions (Kusenko, Takahashi, Yanagida)
  - massless at leading order, massive after breaking (Lindner, Merle, Niro)
  - seesaw variants (Zhang)
  - Frogatt-Nielsen (Barry, W.R., Zhang)

Note: sterile neutrino can have eV mass or keV mass (Warm Dark Matter)

Seesaw Model based on A_4

Field	L	e^c	μ^c	τ^c	$h_{u,d}$	φ	φ'	φ''	ξ	ξ'	ξ''	Θ	ν_1^c	ν_2^c	ν_3^c
$SU(2)_L$	2	1	1	1	2	1	1	1	1	1	1	1	1	1	1
A_4	$\underline{3}$	$\underline{1}$	$\underline{1}''$	$\underline{1}'$	$\underline{1}$	$\underline{3}$	$\underline{3}$	$\underline{3}$	$\underline{1}$	$\underline{1}'$	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{1}'$	$\underline{1}$
Z_3	ω	ω^2	ω^2	ω^2	1	1	ω	ω^2	ω^2	ω	1	1	ω^2	ω	1
$U(1)_{\rm FN}$	-	3	1	0	-	-	-	-	-	-	-	-1	F_1	F_2	F_3

various possibilities for the FN-charges:

	$F_1,\ F_2,\ F_3$	Mass spectrum	$\lvert U_{\alpha 4}\rvert$	$\lvert U_{\alpha 5}\rvert$	m_{ee} NO	m_{ee} IO	Phenomenology
I	9, 10, 10	$M_{2,3} = \mathcal{O}(\text{eV})$	$\mathcal{O}(0.1)$	$\mathcal{O}(0.1)$	**0**	**0**	$3+2$ **mixing**
IIA	9, 10, 0	$M_2 = \mathcal{O}(\text{eV})$ $M_3 = \mathcal{O}(10^{11}\ \text{GeV})$	$\mathcal{O}(0.1)$	$\mathcal{O}(10^{-11})$	**0**	$\frac{2\sqrt{\Delta m_{\rm A}^2}}{3}$	$3+1$ **mixing**
IIB	9, 0, 10	$M_2 = \mathcal{O}(10^{11}\ \text{GeV})$ $M_3 = \mathcal{O}(\text{eV})$	$\mathcal{O}(10^{-11})$	$\mathcal{O}(0.1)$	$\frac{\sqrt{\Delta m_\odot^2}}{3}$	$\frac{\sqrt{\Delta m_{\rm A}^2}}{3}$	
III	9, 5, 5	$M_{2,3} = \mathcal{O}(10\ \text{GeV})$	$\mathcal{O}(10^{-6})$	$\mathcal{O}(10^{-6})$	$\frac{\sqrt{\Delta m_\odot^2}}{3}$	$\sqrt{\Delta m_{\rm A}^2}$	**Leptogenesis**

Barry, W.R., Zhang, 1110.6382

Technicalities of low scale seesaw

$$M_\nu^{6\times 6} = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix}$$

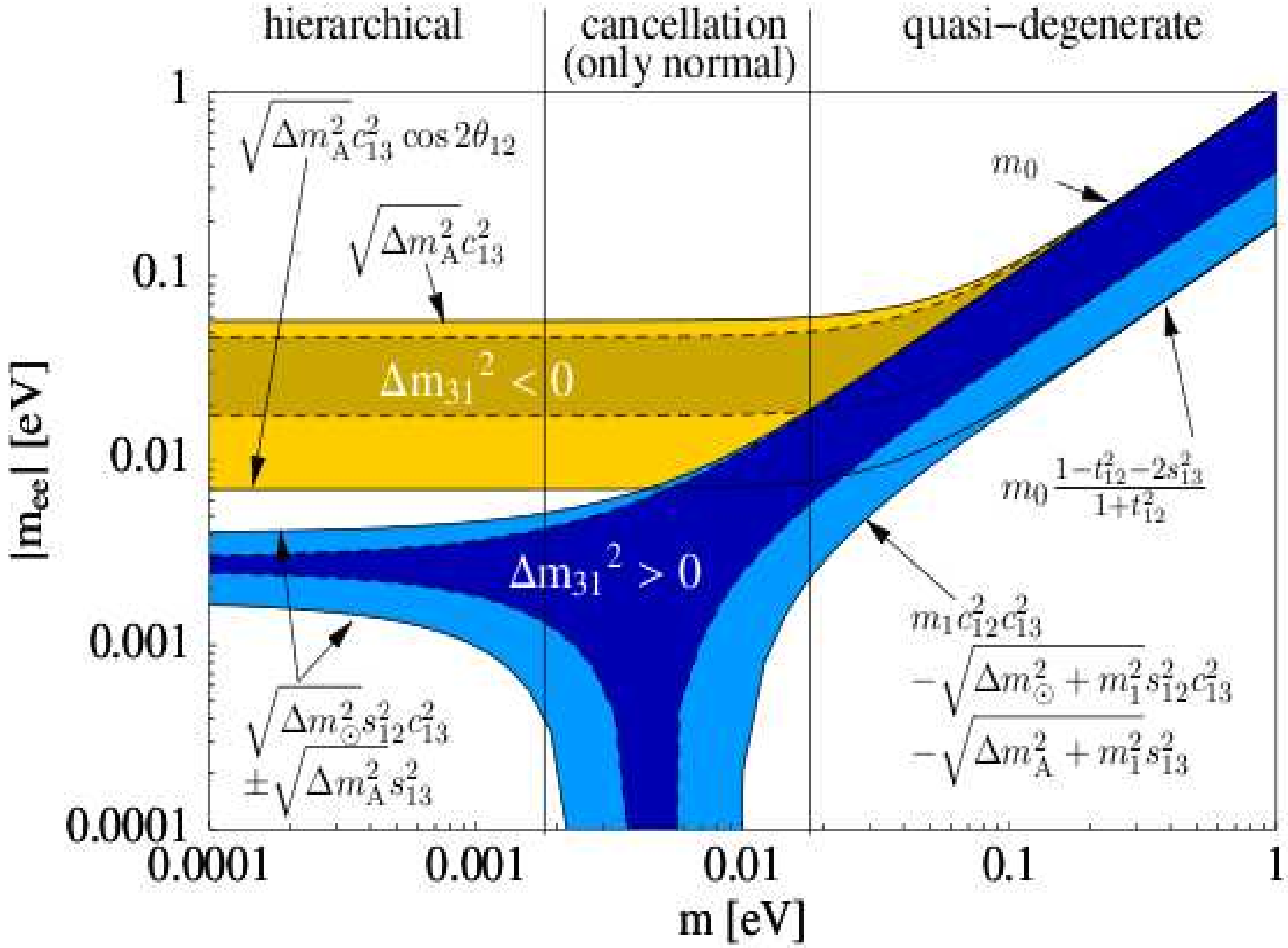
diagonalized by 6×6 matrix

$$U_\nu \simeq \begin{pmatrix} 1 - \frac{1}{2}BB^\dagger & B \\ -B^\dagger & 1 - \frac{1}{2}B^\dagger B \end{pmatrix} \begin{pmatrix} V_\nu & 0 \\ 0 & V_R \end{pmatrix}$$

where $B = M_D M_R^{-1}$ governs "NLO seesaw corrections"

- if $M_R \simeq$ eV: $M_D \simeq 0.1\ \mathrm{eV}$ and $B \simeq 0.1$
- if all $M_i \lesssim 100$ MeV: no neutrino-less double beta decay
- keV sterile neutrino with mixing 10^{-4} generates negligible active neutrino mass

The usual plot



Sterile Neutrinos and $0\nu\beta\beta$

- recall $|m_{ee}|_{\rm NH}^{\rm act}$ can vanish and $|m_{ee}|_{\rm IH}^{\rm act} \sim 0.02$ eV cannot vanish
- $|m_{ee}| = | \underbrace{|U_{e1}|^2 m_1 + |U_{e2}|^2 m_2 \, e^{2i\alpha} + |U_{e3}^2| m_3 \, e^{2i\beta}}_{m_{ee}^{\rm act}} + \underbrace{|U_{e4}|^2 m_4 \, e^{2i\Phi_1}}_{m_{ee}^{\rm st}} |$
- $\Delta m_{\rm st}^2 \simeq 1$ eV^2 and $|U_{e4}| \simeq 0.15$
- sterile contribution to $0\nu\beta\beta$:

$$|m_{ee}|^{\rm st} \simeq \sqrt{\Delta m_{\rm st}^2} \, |U_{e4}|^2 \simeq 0.02 \text{ eV} \begin{cases} \gg |m_{ee}|_{\rm NH}^{\rm act} \\ \simeq |m_{ee}|_{\rm IH}^{\rm act} \end{cases}$$

- $\Rightarrow$ $|m_{ee}|_{\rm NH}$ cannot vanish and $|m_{ee}|_{\rm IH}$ can vanish!

`Barry, W.R., Zhang, JHEP` **1107**

Sterile Neutrinos and $0\nu\beta\beta$

contribution to double beta:

$$\langle m_{ee} \rangle = \left| \sum_{i=1}^{3} U_{ei}^2 m_i + \sum_{\text{light}} U_{e,3+i}^2 M_i \right|$$

in our model:

$$M_D = V_\nu \, \text{diag} \left(\sqrt{-m_1 M_1}, \sqrt{-m_2 M_2}, \sqrt{-m_3 M_3} \right)$$

active-sterile mixing is $U_{\alpha,3+i} = (V_\nu)_{\alpha i} \sqrt{-m_i/M_i}$ and therefore

$$U_{e,3+i}^2 M_i = \left[-(V_\nu^2)_{ei} \frac{m_i}{M_i} \right] M_i = -U_{ei}^2 m_i \, , \quad (i = \text{light})$$

sterile RH neutrino contributions cancels exactly active neutrino contribution!

Light sterile neutrinos?

Consider the "role model" Altarelli, Feruglio, NPB 720, 64 (2005)

(effective model)

Field	L	e^c	μ^c	τ^c	$h_{u,d}$	φ	φ'	ξ	ν_s
$SU(2)_L$	2	1	1	1	2	1	1	1	1
A_4	$\underline{3}$	$\underline{1}$	$\underline{1}''$	$\underline{1}'$	$\underline{1}$	$\underline{3}$	$\underline{3}$	$\underline{1}$	$\underline{1}$
Z_3	ω	ω^2	ω^2	ω^2	1	1	ω	ω	1
$U(1)_{\rm FN}$	-	4	2	0	-	-	-	-	6

- Z_3 to separate charged leptons and neutrinos
- $U(1)_{\rm FN}$ for charged lepton mass hierarchy
- ν_s added by us, no extra symmetries or fields

Barry, W.R., Zhang, 1105.3911

allowed terms

$$\mathcal{L}_{\mathrm{Y}_s} = \frac{x_e}{\Lambda^2}\xi(\varphi' L h_u)\nu_s + m_s \nu_s^c \nu_s + \mathrm{h.c.}$$

lies at eV scale due to FN

mass matrix

$$M_\nu^{4\times 4} = \begin{pmatrix} a+\frac{2d}{3} & -\frac{d}{3} & -\frac{d}{3} & e \\ \cdot & \frac{2d}{3} & a-\frac{d}{3} & e \\ \cdot & \cdot & \frac{2d}{3} & e \\ \cdot & \cdot & \cdot & m_s \end{pmatrix}$$

with the usual VEV alignment $\langle\xi\rangle = u$, $\langle\varphi\rangle = (v,0,0)$ and $\langle\varphi'\rangle = (v',v',v')$

diagonalized by

$$U \simeq \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \frac{e}{m_s} \\ 0 & 0 & 0 & \frac{e}{m_s} \\ 0 & 0 & 0 & \frac{e}{m_s} \\ 0 & -\frac{\sqrt{3}e}{m_s} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{\sqrt{3}e^2}{2m_s^2} & 0 & 0 \\ 0 & -\frac{\sqrt{3}e^2}{2m_s^2} & 0 & 0 \\ 0 & -\frac{\sqrt{3}e^2}{2m_s^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{3e^2}{2m_s^2} \end{pmatrix}$$

giving the eigenvalues

$$m_1 = a + d\,, \quad m_2 = a - \frac{3e^2}{m_s}\,, \quad m_3 = -a + d\,, \quad m_4 = m_s + \frac{3e^2}{m_s}$$

and sum-rules

$$\sin^2\theta_{14} \simeq \sin^2\theta_{24} \simeq \sin^2\theta_{34} \simeq \left(\frac{e}{m_s}\right)^2 \simeq \frac{1}{2}(1 - 3\sin^2\theta_{12}) \simeq 2\sin^2\theta_{23} - 1$$

Still $U_{e3} = 0\ldots$

Can also add second ν_s, giving

$$M_\nu^{5\times 5} = \begin{pmatrix} a + \frac{2d}{3} & -\frac{d}{3} & -\frac{d}{3} & e & f \\ \cdot & \frac{2d}{3} & a - \frac{d}{3} & e & f \\ \cdot & \cdot & \frac{2d}{3} & e & f \\ \cdot & \cdot & \cdot & m_{s_1} & 0 \\ \cdot & \cdot & \cdot & \cdot & m_{s_2} \end{pmatrix}$$

Trivial change of FN charge and scales gives keV sterile neutrinos

Summary

- still huge activity in model building, speculations after T2K/DC
- death of TBM flavor symmetry models?
- if $\theta_{13} = 10^\circ$: why initially zero and then corrections?
- return of Gatto-Sartori-Tonin relations?

  e.g. $|U_{e3}| = \mathcal{O}(\Delta m^2_\odot / \Delta m^2_{\rm A})$
- incorporate sterile neutrinos?
- **items not covered:**
  - **quarks**
  - **GUTs**
  - **cosmology**
  - **origin of discrete symmetries**
  - . . .