# Exercises to "Standard Model of Particle Physics II"

Winter 2022/23

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Sheet 09 - December 14, 2022

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**Lecture webpage:** `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

**Hand-in of solutions:** December 21, 2022 - 11:15, Phil. 12, R105

**Discussion of solutions:** December 21, 2022 - 11:15, Phil. 12, R105

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**Problem 1:** *Majorana neutrinos* [**10 Points**]

The Lagrangian for the coupling of a fermion pair $f$ with the $Z$-boson is given by

$$\mathcal{L} = \frac{g}{2\cos\theta_W} \overline{f}\gamma^\mu (v_f - a_f \gamma_5) f Z_\mu \tag{1}$$

For neutrinos we have $a_\nu = v_\nu = 1/2$.

a) Calculate the decay width for $Z \to \overline{\nu}\nu$ in the Standard Model but keeping a possible neutrino mass in the expression.

   **Hint**: The decay width for Dirac neutrinos is given by

$$\Gamma_{\text{Dirac}} = \frac{|\vec{p}|}{32\pi^2 m_z^2} \int \mathrm{d}\Omega |\overline{\mathcal{A}}|^2 \,, \tag{2}$$

   where $\overline{\mathcal{A}}$ is the spin-averaged transition amplitude $\mathcal{A}$ for $Z \to \overline{\nu}\nu$, which may be calculated using the appropriate Feynman rules (i.e. the Lagrangian).

b) Neutrinos could also be Majorana particles, which obey the relation $\nu^C = \nu$. The superscript $C$ denotes charge conjugation

$$\nu^C = C\overline{\nu}^T, \tag{3}$$

   with $C = i\gamma_2\gamma_0$ in the Dirac basis. Show the following properties:

$$-C = C^T = C^{-1} = -C^* = C^\dagger \tag{4}$$

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T \tag{5}$$

$$C^{-1}\gamma_5 C = \gamma_5^T \tag{6}$$

$$\overline{\Psi^C} = -\Psi^T C^{-1} \tag{7}$$

$$(\Psi_L)^C = \left(\Psi^C\right)_R \,, \tag{8}$$

   where $\left(\Psi^C\right)_L = P_L \left(\Psi^C\right)$.

c) Show that for Majorana neutrinos the vector current $\overline{\nu}\gamma_\mu\nu$ vanishes. What happens with $\overline{\nu}\gamma_5\nu, \overline{\nu}\gamma_\mu\gamma_5\nu$ and $\overline{\nu}\left[\gamma_\mu, \gamma_\nu\right]\nu$?

d) Using the previous result calculate the decay width $Z \to \overline{\nu}\nu$ for Majorana neutrinos and compare with part a).

**Hint 1**: Due to the Majorana properties, the final state particles are indistinguishable from each other which results in a factor of 1/2.

**Problem 2:** *Seesaw II* **[10 Points]**

We consider the lepton sector of the Standard Model and expand it by adding a Higgs triplet $\Delta$. The particles considered have the following $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ transformation properties:

$$L_a \sim (2,\ -1); \qquad l_{a\mathrm{R}} \sim (1,\ -2); \qquad \phi \sim (2,\ 1); \qquad \Delta \sim (3,\ 2),$$

where the fields denote respectively the left-handed lepton doublet, the right-handed lepton singlet, the SM Higgs doublet and the (non-SM) Higgs triplet in the convention that $Q = I_3 + Y/2$. The flavour index is denoted as $a$. For the triplet use the representation as a $2\times 2$ matrix with the (electric) charge eigenstates

$$\Delta = \begin{pmatrix} \Delta^+ & \sqrt{2}\Delta^{++} \\ \sqrt{2}\Delta^0 & -\Delta^+ \end{pmatrix}.$$

The mass terms for the leptons arise from the Yukawa Lagrangian

$$\mathscr{L}_\mathrm{Y} = \sum_{a,b} \big[ - y_{ab}\overline{l_{a\mathrm{R}}}\phi^\dagger L_b + \frac{1}{2}\tilde{y}_{ab}\overline{L_a^\mathrm{c}} i\tau_2 \Delta L_b \big] + \mathrm{h.c.}$$

a) Convince yourself that $\mathscr{L}_\mathrm{Y}$ is a singlet under $\mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{U}(1)_\mathrm{Y}$.

b) The introduction of the triplet changes the Higgs potential to

$$\begin{aligned} V(\phi,\ \Delta) =& a\phi^\dagger\phi + \frac{b}{2}\mathrm{Tr}[\Delta^\dagger\Delta] + c(\phi^\dagger\phi)^2 + \frac{d}{4}(\mathrm{Tr}[\Delta^\dagger\Delta])^2 + \frac{e-h}{2}\phi^\dagger\phi\mathrm{Tr}[\Delta^\dagger\Delta] \\ &+ \frac{f}{4}\mathrm{Tr}[\Delta^\dagger\Delta^\dagger]\mathrm{Tr}[\Delta\Delta] + h\phi^\dagger\Delta^\dagger\Delta\phi + (t\phi^\dagger\Delta(\mathrm{i}\tau_2\phi^*) + \mathrm{h.c.})\,. \end{aligned}$$

Use the condition that only neutral components of the Higgs fields can develop non-zero vacuum expectation values (vev's) $\langle\phi^0\rangle = v$ and $\langle\Delta^0\rangle = v_\Delta/\sqrt{2}$ and find $V(\langle\phi\rangle,\ \langle\Delta\rangle)$.

c) Define $t = |t|e^{i\omega}$, $v_\Delta = |v_\Delta|e^{i\gamma}$ and write the conditions of minimization of the potential with respect to $v$, $|v_\Delta|$, and $\gamma$. *Hint*: Start with $\gamma$.

d) Show that, under the assumptions $a,\ b \sim v^2$, and $c,\ d,\ f \sim 1$, together with $|t|, |v_\Delta| \ll v$, the conditions for the minimum are approximately equivalent to

$$v^2 \approx -\frac{a}{2c} \qquad \text{and} \qquad |v_\Delta| = \frac{2|t|v^2}{b + (e-h)v^2}\,.$$

e) What do the approximations and findings of 2d) imply for the masses in the lepton sector?

f) Now assume that $\sqrt{b} = m_\Delta$ is very large compared to the electroweak scale $v$ and $|v_\Delta|$, and that $t^2 \sim b$, while keeping $c,\ d,\ e,\ f,\ h \sim 1$. Find the relations between the vev's under these conditions. What does it mean for the lepton masses?

g) Discuss the difference between this scenario (the so-called *type-II* seesaw) and the type-I seesaw from **problem 2** on **sheet 8**.