Exercises to “Standard Model of Particle Physics II”

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Hand-in of solutions:
December 10, 2024 - 14:00

Discussion of solutions:
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Problem 1: *Seesaw I* [**6 Points**]

The Higgs mechanism generates Dirac masses for the active neutrinos when right-handed sterile neutrinos are introduced. The following term appears in the Lagrange density:

$$\mathscr{L}_{\text{Dirac}} = -\overline{\nu}_{\text{L}} M_{\text{D}} N_{\text{R}} + \text{h.c.}\,,$$

where $\nu_{\text{L}} = (\nu_{\text{L}}^1,\ \nu_{\text{L}}^2,\ \nu_{\text{L}}^3)^{\text{T}}$ is the column vector of the active neutrinos and N_{R} the corresponding vector for the sterile neutrinos. The matrix M_{D} is $-$ in general $-$ a complex 3×3 matrix.
The right-handed sterile neutrinos can, furthermore, have a Majorana mass with the Lagrange density

$$\mathscr{L}_{\text{Majorana}} = -\frac{1}{2}\overline{(N_{\text{R}})^{\text{c}}} M_{\text{R}} N_{\text{R}} + \text{h.c.}\,,$$

where M_{R} is a symmetric 3×3 matrix, and $\psi^{\text{c}} = \mathcal{C}\overline{\psi}^T$ for a general Dirac spinor ψ and the charge conjugation matrix $\mathcal{C} = i\gamma^2\gamma^0$. Let m_{D} and m_{R} denote the mass scale of M_{D} and M_{R}, respectively. Suppose the entries of M_{R} are much larger than the ones of M_{D} ($m_{\text{R}} \gg m_{\text{D}}$).

a) Show that it is possible to rewrite the whole mass matrix in the flavor basis in the following way

$$\mathscr{L}_{\text{mass}} \equiv \mathscr{L}_{\text{Dirac}} + \mathscr{L}_{\text{Majorana}} = -\frac{1}{2}\overline{\Psi^{\text{c}}} M \Psi + \text{h.c.}\,,$$

with

$$\Psi \equiv \begin{pmatrix} (\nu_{\text{L}})^{\text{c}} \\ N_{\text{R}} \end{pmatrix} \qquad \text{and} \qquad M \equiv \begin{pmatrix} 0 & M_{\text{D}} \\ M_{\text{D}}^{\text{T}} & M_{\text{R}} \end{pmatrix}.$$

For the solution prove and use the identity $\overline{\nu_{\text{L}}} M_{\text{D}} N_{\text{R}} = \overline{(N_{\text{R}})^{\text{c}}} M_{\text{D}}^{\text{T}} (\nu_{\text{L}})^{\text{c}}$.

b) Using the (unitary) transformation $\Psi = U\chi$ with $U = \begin{pmatrix} 1 & \rho \\ -\rho^\dagger & 1 \end{pmatrix}$ (change of basis) it is possible to convert the 6×6 matrix M into a block diagonal form, i.e. that it takes on the following form

$$U^{\text{T}} M U \simeq \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \tag{1}$$

with symmetric 3×3 matrices $M_1,\ M_2$. The matrix ρ in the transformation matrix U is assumed to be proportional to the scale m_{R}^{-1} and terms of order m_{R}^{-2} (and smaller) can be neglected in the calculation. Determine ρ, and M_1 and M_2 from Eq. (1). (You may assume that M_{R} is invertible.)

c) Where does the name “seesaw” come from?

Problem 2: *Seesaw II* **[14 Points]**

We consider the lepton sector of the Standard Model and expand it by adding a Higgs triplet Δ. The particles considered have the following $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ transformation properties:

$$L_a \sim (2,\ -1); \qquad l_{a\mathrm{R}} \sim (1,\ -2); \qquad \phi \sim (2,\ 1); \qquad \Delta \sim (3,\ 2),$$

where the fields denote respectively the left-handed lepton doublet, the right-handed lepton singlet, the SM Higgs doublet and the (non-SM) Higgs triplet in the convention that $Q = I_3 + Y/2$. The flavour index is denoted as a. For the triplet use the representation as a 2×2 matrix with the (electric) charge eigenstates

$$\Delta = \begin{pmatrix} \Delta^+ & \sqrt{2}\Delta^{++} \\ \sqrt{2}\Delta^0 & -\Delta^+ \end{pmatrix}.$$

The mass terms for the leptons arise from the Yukawa Lagrangian

$$\mathscr{L}_\mathrm{Y} = \sum_{a,b} \left[- y_{ab} \overline{l_{a\mathrm{R}}} \phi^\dagger L_b + \frac{1}{2} \tilde{y}_{ab} \overline{L_a^\mathrm{c}} i\tau_2 \Delta L_b \right] + \mathrm{h.c.}$$

a) Convince yourself that $\mathscr{L}_\mathrm{Y}$ is a singlet under $\mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{U}(1)_\mathrm{Y}$.

b) The introduction of the triplet changes the Higgs potential to

$$\begin{aligned} V(\phi,\ \Delta) =& a\phi^\dagger\phi + \frac{b}{2}\mathrm{Tr}[\Delta^\dagger\Delta] + c(\phi^\dagger\phi)^2 + \frac{d}{4}(\mathrm{Tr}[\Delta^\dagger\Delta])^2 + \frac{e-h}{2}\phi^\dagger\phi\mathrm{Tr}[\Delta^\dagger\Delta] \\ &+ \frac{f}{4}\mathrm{Tr}[\Delta^\dagger\Delta^\dagger]\mathrm{Tr}[\Delta\Delta] + h\phi^\dagger\Delta^\dagger\Delta\phi + (t\phi^\dagger\Delta(\mathrm{i}\tau_2\phi^*) + \mathrm{h.c.})\,. \end{aligned}$$

Use the condition that only neutral components of the Higgs fields can develop non-zero vacuum expectation values (vev’s) $\langle\phi^0\rangle = v$ and $\langle\Delta^0\rangle = v_\Delta/\sqrt{2}$ and find $V(\langle\phi\rangle,\ \langle\Delta\rangle)$.

c) Define $t = |t|e^{i\omega}$, $v_\Delta = |v_\Delta|e^{i\gamma}$ and write the conditions of minimization of the potential with respect to v, $|v_\Delta|$, and γ. *Hint*: Start with γ.

d) Show that, under the assumptions $a,\ b \sim v^2$, and $c,\ d,\ f \sim 1$, together with $|t|, |v_\Delta| \ll v$, the conditions for the minimum are approximately equivalent to

$$v^2 \approx -\frac{a}{2c} \qquad \text{and} \qquad |v_\Delta| = \frac{2|t|v^2}{b + (e-h)v^2}\,.$$

e) What do the approximations and findings of 2d) imply for the masses in the lepton sector?

f) Now assume that $\sqrt{b} = m_\Delta$ is very large compared to the electroweak scale v and $|v_\Delta|$, and that $t^2 \sim b$, while keeping $c,\ d,\ e,\ f,\ h \sim 1$. Find the relations between the vev’s under these conditions. What does it mean for the lepton masses?

g) Discuss the difference between this scenario (the so-called *type-II* seesaw) and the type-I seesaw from **problem 1**.