# Exercises to "Standard Model of Particle Physics II"

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**Lecture webpage:** `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

**Hand-in of solutions:**
December 14, 2022 - 11:15, Phil. 12, R105

**Discussion of solutions:**
December 14, 2022 - 11:15, Phil. 12, R105

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**Problem 1:** *Neutrino oscillations in matter* **[14 Points]**

The MSW effect describes oscillations of neutrinos in matter. Consider the case of two flavor oscillations in a realistic (non-constant) mass distribution.
The time evolution of the eigenfunctions of the Hamiltonian in matter is described by the following expression:

$$i\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix}|\nu_{\mathrm{A}}\rangle\\|\nu_{\mathrm{B}}\rangle\end{pmatrix}=\begin{pmatrix}E_{\mathrm{A}}(t) & -i\dot{\theta}(t)\\ i\dot{\theta}(t) & E_{\mathrm{B}}(t)\end{pmatrix}\begin{pmatrix}|\nu_{\mathrm{A}}\rangle\\|\nu_{\mathrm{B}}\rangle\end{pmatrix},$$

where

$$-E_{\mathrm{A}}(t)=E_{\mathrm{B}}(t)=\Delta m^2\sin(2\theta_0)/4E\sin(2\theta)$$

with the (constant) vacuum mixing angle $\theta_0$ and electron number density $N_e(t)$ in matter. The eigenstates in matter are $|\nu\rangle=(|\nu_{\mathrm{A}}\rangle,\ |\nu_{\mathrm{B}}\rangle)^{\mathrm{T}}$ and at any given time they are connected to the flavor states via: $|\tilde{\nu}\rangle=\tilde{U}(t)|\nu\rangle$ with

$$|\tilde{\nu}\rangle=\begin{pmatrix}|\tilde{\nu}_e\rangle\\|\tilde{\nu}_\mu\rangle\end{pmatrix},\qquad \tilde{U}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix},$$

and the effective (time-dependent) mixing angle

$$\tan(2\theta)=\frac{\frac{\Delta m^2}{2E}\sin(2\theta_0)}{\frac{\Delta m^2}{2E}\cos(2\theta_0)-\sqrt{2}G_{\mathrm{F}}N_e(t)}\,. \qquad (1)$$

The effective Hamiltonian is not diagonal, since matter eigenstates and thus the mixing angle $\theta$ are not constant. If, however, $\dot{\theta}$ is small, i.e. the change in $\theta$ occurs slowly, we obtain the adiabatic approximation, where we can neglect $\dot{\theta}$. Suppose that at the time $t=t_i$ an electron neutrino is produced: $|\tilde{\nu}_\alpha(t_i)\rangle\equiv\delta_{\alpha e}|\tilde{\nu}_\alpha\rangle$.

a) What is the expression for $|\tilde{\nu}_\alpha(t_f)\rangle$ at some later time $t_f$ in the adiabatic approximation?

b) Show that in the adiabatic approximation the result for the transition probability is given by

$$P(\nu_e\to\nu_\mu)=\tfrac{1}{2}-\tfrac{1}{2}\cos(2\theta_i)\cos(2\theta_f)-\tfrac{1}{2}\sin(2\theta_i)\sin(2\theta_f)\cos\Phi_{\mathrm{AB}}\,, \qquad (2)$$

with $\theta_i\equiv\theta(t=t_i)$, $\theta_f\equiv\theta(t=t_f)$ and $\Phi_{\mathrm{IJ}}\equiv\int_{t_i}^{t_f}[E_{\mathrm{I}}(t)-E_{\mathrm{J}}(t)]\,\mathrm{dt}$, where $\mathrm{I},\ \mathrm{J}\in\{\mathrm{A},\ \mathrm{B}\}$.

c) Show that in the case of constant electron density, Eq. (2) reduces to the standard formula of two-flavor oscillations

$$P(\nu_e\to\nu_\mu)(t)=\sin^2(2\theta)\sin^2\left(\frac{\pi t}{L_M^{\mathrm{osc}}}\right),$$

and determine $L_M^{\mathrm{osc}}$. What happens in the limit $N_e\to 0$?

d) Use Eq. (1) to find an expression for $\sin^2(2\theta)$. The formula for $\sin^2(2\theta)$ describes a Breit-Wigner distribution. We define the resonance width at half height, $\Delta r$, as the width at which the curve of the distribution is larger than one half [$\sin^2(2\theta) > 1/2$]. Find an expression for $\Delta r$ in terms of $\theta_0$.

e) The *adiabatic parameter* is defined as $\gamma_r \equiv |E_{\mathrm{A}}|/|\dot{\theta}|$ which becomes large if the adiabatic approximation is good. To interpret this, first show that

$$\dot{\theta} = \frac{1}{2}\frac{\sin^2(2\theta)}{\Delta m^2 \sin(2\theta_0)} 2\sqrt{2} E G_F \dot{N}_e \,.$$

Now assume that the change in matter density is approximately linear, $\dot{N}_e \approx$ const, and that the density passes exactly through the resonance, i.e. $\Delta N_e = N_e(t_f) - N_e(t_i) \propto \Delta r$ to formulate $\gamma_r$ in terms of $L_M^{\mathrm{osc}}$ and $\Delta t = t_f - t_i$. Discuss the adiabatic condition $\gamma_r > 3$.

**Problem 2:** *Seesaw I* **[6 Points]**

The Higgs mechanism generates Dirac masses for the active neutrinos when right-handed sterile neutrinos are introduced. The following term appears in the Lagrange density:

$$\mathscr{L}_{\mathrm{Dirac}} = -\overline{\nu}_{\mathrm{L}} M_{\mathrm{D}} N_{\mathrm{R}} + \text{h.c.}\,,$$

where $\nu_{\mathrm{L}} = (\nu_{\mathrm{L}}^1,\ \nu_{\mathrm{L}}^2,\ \nu_{\mathrm{L}}^3)^{\mathrm{T}}$ is the column vector of the active neutrinos and $N_{\mathrm{R}}$ the corresponding vector for the sterile neutrinos. The matrix $M_{\mathrm{D}}$ is – in general – a complex $3 \times 3$ matrix.
The right-handed sterile neutrinos can, furthermore, have a Majorana mass with the Lagrange density

$$\mathscr{L}_{\mathrm{Majorana}} = -\frac{1}{2}\overline{(N_{\mathrm{R}})^{\mathrm{c}}} M_{\mathrm{R}} N_{\mathrm{R}} + \text{h.c.}\,,$$

where $M_{\mathrm{R}}$ is a symmetric $3 \times 3$ matrix, and $\psi^{\mathrm{c}} = \mathcal{C}\overline{\psi}^T$ for a general Dirac spinor $\psi$ and the charge conjugation matrix $\mathcal{C} = i\gamma^2\gamma^0$. Let $m_{\mathrm{D}}$ and $m_{\mathrm{R}}$ denote the mass scale of $M_{\mathrm{D}}$ and $M_{\mathrm{R}}$, respectively. Suppose the entries of $M_{\mathrm{R}}$ are much larger than the ones of $M_{\mathrm{D}}$ ($m_{\mathrm{R}} \gg m_{\mathrm{D}}$).

a) Show that it is possible to rewrite the whole mass matrix in the flavor basis in the following way

$$\mathscr{L}_{\mathrm{mass}} \equiv \mathscr{L}_{\mathrm{Dirac}} + \mathscr{L}_{\mathrm{Majorana}} = -\frac{1}{2}\overline{\Psi^{\mathrm{c}}} M \Psi + \text{h.c.}\,,$$

with

$$\Psi \equiv \begin{pmatrix} (\nu_{\mathrm{L}})^{\mathrm{c}} \\ N_{\mathrm{R}} \end{pmatrix} \qquad \text{and} \qquad M \equiv \begin{pmatrix} 0 & M_{\mathrm{D}} \\ M_{\mathrm{D}}^{\mathrm{T}} & M_{\mathrm{R}} \end{pmatrix}.$$

For the solution prove and use the identity $\overline{\nu_{\mathrm{L}}} M_{\mathrm{D}} N_{\mathrm{R}} = \overline{(N_{\mathrm{R}})^{\mathrm{c}}} M_{\mathrm{D}}^{\mathrm{T}} (\nu_{\mathrm{L}})^{\mathrm{c}}$.

b) Using the (unitary) transformation $\Psi = U\chi$ with $U = \begin{pmatrix} 1 & \rho \\ -\rho^\dagger & 1 \end{pmatrix}$ (change of basis) it is possible to convert the $6 \times 6$ matrix $M$ into a block diagonal form, i.e. that it takes on the following form

$$U^{\mathrm{T}} M U \simeq \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \tag{3}$$

with symmetric $3 \times 3$ matrices $M_1,\ M_2$. The matrix $\rho$ in the transformation matrix $U$ is assumed to be proportional to the scale $m_{\mathrm{R}}^{-1}$ and terms of order $m_{\mathrm{R}}^{-2}$ (and smaller) can be neglected in the calculation. Determine $\rho$, and $M_1$ and $M_2$ from Eq. (3). (You may assume that $M_{\mathrm{R}}$ is invertible.)

c) Where does the name "seesaw" come from?