Exercises to “Standard Model of Particle Physics II”

Winter 2024/25

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in of solutions:	**Discussion of solutions:**
December 3, 2024 - 14:00	December 3, 2024 – 14:00, INF 227, SR 1.404

Problem 1: *Neutrino oscillations in matter* **[14 Points]**

The MSW effect describes oscillations of neutrinos in matter. Consider the case of two flavor oscillations in a realistic (non-constant) mass distribution.

The time evolution of the eigenfunctions of the Hamiltonian in matter is described by the following expression:

$$i\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix}|\nu_{\mathrm{A}}\rangle\\|\nu_{\mathrm{B}}\rangle\end{pmatrix} = \begin{pmatrix}E_{\mathrm{A}}(t) & -i\dot{\theta}(t)\\ i\dot{\theta}(t) & E_{\mathrm{B}}(t)\end{pmatrix}\begin{pmatrix}|\nu_{\mathrm{A}}\rangle\\|\nu_{\mathrm{B}}\rangle\end{pmatrix},$$

where

$$-E_{\mathrm{A}}(t) = E_{\mathrm{B}}(t) = \Delta m^2\sin(2\theta_0)/4E\sin(2\theta)$$

with the (constant) vacuum mixing angle θ_0 and electron number density $N_e(t)$ in matter. The eigenstates in matter are $|\nu\rangle = (|\nu_{\mathrm{A}}\rangle,\ |\nu_{\mathrm{B}}\rangle)^{\mathrm{T}}$ and at any given time they are connected to the flavor states via: $|\tilde{\nu}\rangle = \tilde{U}(t)|\nu\rangle$ with

$$|\tilde{\nu}\rangle = \begin{pmatrix}|\tilde{\nu}_e\rangle\\|\tilde{\nu}_\mu\rangle\end{pmatrix}, \qquad \tilde{U} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix},$$

and the effective (time-dependent) mixing angle

$$\tan(2\theta) = \frac{\frac{\Delta m^2}{2E}\sin(2\theta_0)}{\frac{\Delta m^2}{2E}\cos(2\theta_0) - \sqrt{2}G_{\mathrm{F}}N_e(t)}\,. \tag{1}$$

The effective Hamiltonian is not diagonal, since matter eigenstates and thus the mixing angle θ are not constant. If, however, $\dot{\theta}$ is small, i.e. the change in θ occurs slowly, we obtain the adiabatic approximation, where we can neglect $\dot{\theta}$. Suppose that at the time $t = t_i$ an electron neutrino is produced: $|\tilde{\nu}_\alpha(t_i)\rangle \equiv \delta_{\alpha e}|\tilde{\nu}_\alpha\rangle$.

a) What is the expression for $|\tilde{\nu}_\alpha(t_f)\rangle$ at some later time t_f in the adiabatic approximation?

b) Show that in the adiabatic approximation the result for the transition probability is given by

$$P(\nu_e \to \nu_\mu) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta_i)\cos(2\theta_f) - \tfrac{1}{2}\sin(2\theta_i)\sin(2\theta_f)\cos\Phi_{\mathrm{AB}}\,, \tag{2}$$

with $\theta_i \equiv \theta(t=t_i)$, $\theta_f \equiv \theta(t=t_f)$ and $\Phi_{\mathrm{IJ}} \equiv \int_{t_i}^{t_f}\left[E_{\mathrm{I}}(t) - E_{\mathrm{J}}(t)\right]\mathrm{d}t\,,$ where $\mathrm{I},\ \mathrm{J} \in \{\mathrm{A},\ \mathrm{B}\}$.

c) Show that in the case of constant electron density, Eq. (2) reduces to the standard formula of two-flavor oscillations

$$P(\nu_e \to \nu_\mu)(t) = \sin^2(2\theta)\sin^2\left(\frac{\pi t}{L_M^{\mathrm{osc}}}\right),$$

and determine L_M^{osc}. What happens in the limit $N_e \to 0$?

d) Use Eq. (1) to find an expression for $\sin^2(2\theta)$. The formula for $\sin^2(2\theta)$ describes a Breit-Wigner distribution. We define the resonance width at half height, Δr, as the width at which the curve of the distribution is larger than one half $[\sin^2(2\theta) > 1/2]$. Find an expression for Δr in terms of θ_0.

e) The *adiabatic parameter* is defined as $\gamma_r \equiv |E_\mathrm{A}|/|\dot{\theta}|$ which becomes large if the adiabatic approximation is good. To interpret this, first show that

$$\dot{\theta} = \frac{1}{2}\frac{\sin^2(2\theta)}{\Delta m^2 \sin(2\theta_0)} 2\sqrt{2} E G_F \dot{N}_e \,.$$

Now assume that the change in matter density is approximately linear, $\dot{N}_e \approx \text{const}$, and that the density passes exactly through the resonance, i.e. $\Delta N_e = N_e(t_f) - N_e(t_i) \propto \Delta r$ to formulate γ_r in terms of L_M^{osc} and $\Delta t = t_f - t_i$. Discuss the adiabatic condition $\gamma_r > 3$.

Problem 2: *Discrete symmetries* [**6 Points**]

A CP transformation takes a left-handed neutrino and transforms it into a right-handed anti-neutrino. Consider the probability $P(\nu_\alpha \to \nu_\beta,\, t)$ of a neutrino of flavor α to oscillate into a flavor β after time t of propagation.

a) How do the discrete symmetries CP, T, and CPT change the initial and final states of the process given above? That is, give

$$(\mathrm{CP})^{-1} P(\nu_\alpha \to \nu_\beta,\, t)(\mathrm{CP})\,,$$
$$(\mathrm{T})^{-1} P(\nu_\alpha \to \nu_\beta,\, t)(\mathrm{T})\,,$$
$$(\mathrm{CPT})^{-1} P(\nu_\alpha \to \nu_\beta,\, t)(\mathrm{CPT})$$

in terms of other transition probabilities.

b) The CPT theorem states that every QFT described by a Lorentz invariant, local Lagrangian is invariant under CPT transformations. How could the above result be used to test this theorem in Nature? (Hint: Consider the case $\alpha = \beta$, i.e. the survival probability of (anti-) neutrinos.)