

# Exercises to “Standard Model of Particle Physics II”

Winter 2022/23

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**Lecture webpage:** <https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html>

**Hand-in of solutions:**

November 23, 2022 - 11:15, Phil. 12, R105

**Discussion of solutions:**

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**Problem 1:** *Electroweak symmetry breaking by a Higgs triplet* [10 Points]

a) Verify that

$$t_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

form a triplet representation of the SU(2) algebra  $[t_a, t_b] = i\epsilon_{abc}t_c$ .

b) Consider the complex triplet Higgs field  $\Phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x))^T$  coupled covariantly to an SU(2)<sub>L</sub> gauge group with

$$D_\mu \Phi = \partial_\mu \Phi - ig t_a W_\mu^a(x) \Phi,$$

where  $W_\mu^a$  denote the gauge bosons of the SU(2)<sub>L</sub> group. We introduce a potential  $V(\Phi)$  such that the gauge symmetry breaks spontaneously. The precise form of the potential is not interesting for the moment. As a result, the Higgs field acquires a vacuum expectation value (VEV), which can be written  $\Phi_0 = (0, 0, v)^T$ . Derive the gauge boson mass eigenstates of the broken SU(2)<sub>L</sub> symmetry.

c) In the next step we extend the gauge group of the theory to SU(2)<sub>L</sub> × U(1)<sub>Y</sub> so that the covariant derivative now reads

$$D_\mu \Phi = \partial_\mu \Phi - ig t_a W_\mu^a(x) \Phi - ig' \mathbb{1} B_\mu(x) \Phi,$$

where  $B_\mu$  denotes the gauge boson of the U(1)<sub>Y</sub> group. Using the same VEV as in part b), derive the gauge boson mass eigenstates and the electroweak mixing angle. What are the consequences for the  $\rho = m_W^2/(m_Z^2 \cos^2 \theta_W)$  parameter compared to the SM?

d) How does the particle content of part b) and c) differ from the Standard Model? How can these theories be distinguished in an experiment?

**Problem 2: The Rho-parameter [10 Points]**

We encountered the Rho-parameter defined by

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2(\theta_W)}$$

whose value predicted by the SM is 1. Now we generalize our findings while keeping the above definition.

- a) From representation theory we know that in a  $n$ -dimensional representation (i.e.  $n$ -plet, or spin- $(n-1)/2$  representation), an eigenbasis to the generators  $t^3$  of  $SU(2)$  can be found, such that  $t_3$  acts like  $t^3 = \text{diag}(j, j-1, \dots, -j)$ , where  $j = (n-1)/2$ . Show that if we expand a Higgs  $n$ -plet  $\Phi$  with a hypercharge of  $y$  in this basis its components will be eigenstates of  $Q = t^3 + \hat{Y}$ , where  $\hat{Y}\Phi = y\mathbb{1}\Phi = y\Phi$ . What is the condition on  $y$  to ensure that there is a neutral component?
- b) In this basis the raising and lowering operators defined from  $t^\pm = (t^1 \pm it^2)$  can be described by their action on the orthonormal basis vectors  $e_m$  (labelled by their  $t^3$  eigenvalues  $m$ ):

$$t^+ e_m = \sqrt{j(j+1) - m(m+1)} e_{m+1}, \quad t^- e_m = \sqrt{j(j+1) - m(m-1)} e_{m-1}.$$

Assume that the hypercharge is such that  $\phi_m$  is neutral and that only this neutral component of  $\Phi$  acquires a vev,  $\vec{v} = v e_m$ . Recall that the covariant derivative reads

$$D_\mu \Phi = (\partial_\mu - i g t^i W_\mu^i - i g' \hat{Y} B_\mu) \Phi$$

and, show that

$$m_W^2 = \frac{g^2}{2} \vec{v}^\dagger (t^+ t^- + t^- t^+) \vec{v}.$$

- c) Show that this leads to

$$m_W^2 = g^2 v^2 (j(j+1) - m^2).$$

- d) With the standard mixing angle  $\tan \theta_W = g'/g$ , derive the mass of  $Z$  and use it to show that

$$\rho = \frac{v^2 (j(j+1) - y^2)}{2v^2 y^2}.$$

- e) Check that the Higgs doublet with hypercharge  $1/2$  satisfies the condition that  $\rho = 1$ . What would be the next combination of weak isospin and (rational) hypercharge that can satisfy this bound?