Exercises to "Standard Model of Particle Physics II"

Winter 2015/16

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Exercise 11: *Stückelberg Mechanism* **[10 Points]**

For a gauged abelian symmetry $U(1)'$ (it does not extend to non-abelian symmetries) there exists an interesting mechanism to generate a massive gauge boson, while retaining renormalizability. The method contains a real scalar field σ together with the Z'-boson associated to $U(1)'$.

Consider the Lagrangian

$$\mathscr{L} = -\frac{1}{4}Z'^{\mu\nu}Z'_{\mu\nu} + \frac{1}{2}(M_{Z'}Z'_\mu + \partial_\mu\sigma)(M_{Z'}Z'^\mu + \partial^\mu\sigma) + i\overline{\psi}\gamma^\mu(\partial_\mu - ig'Y'Z'_\mu)\psi - m\overline{\psi}\psi.$$

The gauge transformations for the Dirac fermion (with $U(1)'$ charge Y') and gauge boson are given by

$$\psi \to e^{-ig'Y'\theta(x)}\psi, \qquad Z'_\mu \to Z'_\mu - \partial_\mu\theta(x).$$

Calculate the gauge transformation of the real scalar σ that makes the Lagrangian invariant and show the invariance of the other terms. Can you fix a gauge to eliminate σ from the theory? Count degrees of freedom in both gauges.

Exercise 12: *Optical Theorem* **[10 Points]**

The scattering of particles $a + b \to c + d$ is described by the amplitude

$$f(\theta) = \frac{1}{2ki}\sum_l (2l+1)(\eta_l \mathrm{e}^{2i\delta_l} - 1)P_l(\cos\theta),$$

where P_l are the Legendre-polynomials, θ is the scattering angle, k is the wavenumber in the incident direction and δ_l and η_l are both real functions. δ_l denotes the phase difference and η_l was introduced to describe inelastic scattering. We have $\eta_l = 1$ for elastic and $\eta_l < 1$ for inelastic scattering.

The optical theorem states that the cross section in a forward scattering process is given by

$$\sigma_{\text{tot}} = \frac{4\pi}{k}\text{Im}\left[f(0)\right].$$

a) Show with the help of the optical theorem that

$$\sigma_{\text{tot}} = \frac{2\pi}{k^2}\sum_l (2l+1)(1 - \eta_l\cos(2\delta_l)).$$

b) The differential cross section for elastic scattering is given by

$$\frac{d\sigma_{\text{el}}}{d\Omega} = |f(\theta)|^2.$$

From this, derive the following expression for the elastic scattering cross section

$$\sigma_{\text{el}} = \frac{\pi}{k^2}\sum_l (2l+1)|\eta_l \mathrm{e}^{2i\delta_l} - 1|^2.$$

c) From a) and b) it follows that

$$\sigma_{\text{inel}} = \frac{\pi}{k^2}\sum_l (2l+1)(1-\eta_l^2).$$

Show with this equation that for the reaction $\nu_\mu + e^- \to \mu^- + \nu_e$ we obtain the relation

$$\sigma(\nu_\mu + e^- \to \mu^- + \nu_e) \leq \frac{2\pi}{E_{\text{cm}}^2}, \tag{1}$$

where E_{cm} denotes the center-of-mass energy. Note that this is an $l = 0$ scattering process and that a spin factor $(2s+1)$ should be taken into account.

d) In Fermi theory the cross section is given by

$$\sigma = \frac{G_{\text{F}}^2 s}{\pi}, \tag{2}$$

where G_{F} is Fermi's constant and $\sqrt{s}$ denotes the invariant mass.

Use Eqs. (1) and (2) to find the energy at which Fermi theory breaks down.

Exercise 13: *Mass matrices and mixing angles* [**10 Points**]

A general (Dirac) mass term for fermions is given by

$$\mathscr{L}_M = \overline{\psi}_{i,L} M_{ij} \psi_{j,R} + \text{h.c.}$$

where M is hermitian and given by a $n \times n$ Yukawa coupling matrix Y times the Higgs vev.

a) Show that one can choose a bi-unitary transformation $UMV^\dagger$ to diagonalize M, such that no diagonal element $UMV^\dagger = D := diag(m_1, m_2, ..., m_n)$ is negative. The matrices U and V are unitary.

b) Show that for a real mass matrix M one can choose orthogonal diagonalization matrices.

c) As an example for calculable mixing angles consider a simple 2×2 mass matrix of the form

$$M = \begin{bmatrix} 0 & a \\ a^* & b \end{bmatrix}.$$

The unitary matrix that diagonalizes M can be described by a single parameter: a *mixing angle* θ. Show that the following relation between mixing angle and masses holds:

$$\tan\theta = \sqrt{\frac{m_1}{m_2}}.$$

Compare this with the Cabibbo angle and the down and strange quark masses.

d) A completely different situation holds for the symmetric mass matrix

$$M = \begin{bmatrix} a & b & b \\ b & \frac{1}{2}(a+b+d) & \frac{1}{2}(a+b-d) \\ b & \frac{1}{2}(a+b-d) & \frac{1}{2}(a+b+d) \end{bmatrix}.$$

Give the mixing matrix for this mass matrix (*Hint:* try first a 23-rotation).

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Lecture webpage: `http://www.mpi-hd.mpg.de/manitop/StandardModel2/index_WS15.html`

Hand-in and discussion of sheet:
Tuesday, 17.11.15, 16.15 am, INF 501 / CIP R. 103.