Exercises to “Standard Model of Particle Physics II”

Winter 2024/25

Prof. Dr. Manfred Lindner and PD Dr. Werner Rodejohann
Sheet 04 - November 6, 2024

Tutor: Juan Pablo Garcés **e-mail:** juan.garces@mpi-hd.mpg.de
Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in of solutions:
November 12, 2024 – 14:00

Discussion of solutions:
November 12, 2024 – 14:00, INF 227, SR 1.404

Problem 1: *The Rho-parameter* [**10 Points**]

We encountered the Rho-parameter defined by

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2(\theta_W)}$$

whose value predicted by the SM is 1. Now we generalize our findings while keeping the above definition.

a) From representation theory we know that in a n-dimensional representation (i.e. n-plet, or spin-$(n-1)/2$ representation), an eigenbasis to the generators t^3 of SU(2) can be found, such that t_3 acts like $t^3 = \mathrm{diag}(j, j-1, \ldots, -j)$, where $j = (n-1)/2$. Show that if we expand a Higgs n-plet Φ with a hypercharge of y in this basis its components will be eigenstates of $Q = t^3 + \hat{Y}$, where $\hat{Y}\Phi = y\mathbf{1}_{n\times n}\Phi = y\Phi$. What is the condition on y to ensure that there is a neutral component?

b) In this basis the raising and lowering operators defined from $t^{\pm} = (t^1 \pm it^2)$ can be descibed by their action on the orthonormal basis vectors e_m (labelled by their t^3 eigenvalues m):

$$t^+ e_m = \sqrt{j(j+1) - m(m+1)}\, e_{m+1}\,, \qquad t^- e_m = \sqrt{j(j+1) - m(m-1)}\, e_{m-1}\,.$$

Assume that the hypercharge is such that ϕ_m is neutral and that only this neutral component of Φ acquires a vev, $\vec{v} = v e_m$. Recall that the covariant derivative reads

$$D_\mu \Phi = (\partial_\mu - igt^i W_\mu^i - ig'\hat{Y} B_\mu)\Phi$$

and, show that

$$m_W^2 = \frac{g^2}{2} \vec{v}^\dagger (t^+ t^- + t^- t^+) \vec{v}\,.$$

c) Show that this leads to

$$m_W^2 = g^2 v^2 (j(j+1) - m^2)\,.$$

d) With the standard mixing angle $\tan\theta_W = g'/g$, derive the mass of Z and use it to show that

$$\rho = \frac{v^2(j(j+1) - y^2)}{2v^2 y^2}\,.$$

e) Check that the Higgs doublet with hypercharge 1/2 satisfies the condition that $\rho = 1$. What would be the next combination of weak isospin and (rational) hypercharge that can satisify this bound?

Problem 2: *Integrating out the W-boson* **[10 Points]**

Heavy particles decouple from a given theory when the relevant energy scale of a process is well below their mass. An important example are the $W^\pm$ bosons of the SM, which we will consider in this exercise. The relevant part of the SM Lagrangian reads

$$\mathcal{L}_W = -\frac{1}{2} W^-{}_{\mu\nu} W^{+\mu\nu} + m_W^2\, W^+{}_\mu W^{-\mu} + \frac{g}{\sqrt{2}}\, W^+{}_\mu J^{-\mu} + \frac{g}{\sqrt{2}}\, W^-{}_\mu J^{+\mu}\,,$$

with the electroweak charged currents acting as source terms,

$$J^+{}_\mu = \overline{e}_L \gamma_\mu \nu_L\,, \quad J^-{}_\mu = \overline{\nu}_L \gamma_\mu e_L\,.$$

a) Derive the classical equation of motion for the $W^\pm$ bosons and expand to lowest order in $\frac{p}{m_W}$, where p is the momentum of the W (off-shell), i.e. $\partial_\mu W^\pm{}_\nu = -ip_\mu W^\pm{}_\nu$.

b) Solve the equation to lowest order in $\frac{p}{m_W}$ ignoring non-linear interactions among the W's.

c) Show that this solution can be used to obtain the Lagrangian of Fermi's effective theory of weak interactions,

$$\mathcal{L}_{\text{Fermi}} = -2\sqrt{2} G_F J^+{}_\mu J^{-\mu}$$

and determine G_F in terms of the W mass, m_W, and the weak gauge coupling, g.