Exercises to "Standard Model of Particle Physics II"

Winter 2022/23

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in of solutions:
November 16, 2022 - 11:15, Phil. 12, R105

Discussion of solutions:
November 16, 2022 - 11:15, Phil. 12, R105

Problem 1: *Running couplings* [**10 Points**]

a) Derive the running of the QED coupling $\alpha_{\text{QED}}(\mu)$ from its $\beta-$function

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi} ,$$

where $\beta(\alpha) \equiv d\alpha/d\ln\mu$.

b) In QED the coupling $\alpha = e^2/4\pi$ is a running coupling. For low energies one obtains $\alpha(\mu = m_e) = 1/137$. The running of the QED coupling with only one lepton is given by

$$\alpha_{\text{QED}}(\mu) = \frac{\alpha(m_e)}{1 - \frac{2\alpha(m_e)}{3\pi}\ln(\frac{\mu}{m_e})} .$$

At which scale does $\alpha_{\text{QED}}(\mu)$ become infinite? (If further charged leptons and quarks are considered, the coupling increases even faster.)

c) In QCD, with $\alpha_s = g_3^2/4\pi$, the running coupling

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{(33-2f)\alpha_s(\mu_0)}{6\pi}\ln(\frac{\mu}{\mu_0})}$$

is obtained, where f denotes the respective number of quark flavours with mass $2m_q \leq \mu$ in the considered energy range.

The experimental boundary conditions are $\alpha_s(\mu_0 = m_Z = 91\text{ GeV}) = 0.12$, $m_t = 175$ GeV, $m_b = 4.8$ GeV, $m_c = 1.4$ GeV and $m_u \simeq m_d \simeq m_s \simeq 0$.

i) Determine the pole $\mu = \Lambda_{\text{QCD}}$ of $\alpha_s(\mu)$.

ii) For which μ does the coupling $\alpha_s(\mu)$ become very small (asymptotic freedom), and where does perturbation theory break down?

iii) Determine the value of $\alpha_s(\mu)$ in the different energy ranges ($2m_q \leq \mu \leq 2m_t$ etc.) at the thresholds.

iv) $\alpha_s(\mu)$ should be continuous. However, from the threshold values calculated in the last part we saw that this is not yet true. To make $\alpha_s(\mu)$ continuous, promote μ_0 to a function $\mu_0(f)$ with $\mu_0(f = 5) = 91$ GeV. Find a relation between $\mu_0(f)$ and $\mu_0(f-1)$ assuming that $\alpha_s(\mu_0(f)) = 0.12$ for all f. With this relation determine the values of $\mu_0(4)$ and $\mu_0(6)$.

d) Draw $\alpha_s^{-1}(\mu)$ and $\alpha_{\text{QED}}^{-1}(\mu)$ as functions of $\ln(\mu)$. What could the intersection of the curves indicate?

Problem 2: *Optical Theorem* [**10 Points**]

In the context of partial wave expansion, the scattering amplitude at an angle θ for the process $a + b \to c + d$ is given by

$$f(\theta) = \frac{1}{2ki} \sum_l (2l+1)(\eta_l e^{2i\delta_l} - 1) P_l(\cos\theta),$$

where P_l are the Legendre-polynomials, θ is the scattering angle, k is the wavenumber in the incident direction and δ_l and η_l are both real functions. δ_l denotes the phase difference and η_l was introduced to describe inelastic scattering. We have $\eta_l = 1$ for elastic and $\eta_l < 1$ for inelastic scattering.
The optical theorem states that the cross section in a forward scattering process is given by

$$\sigma_{\text{tot}} = \frac{4\pi}{k} \text{Im}\left[f(0)\right].$$

a) Show with the help of the optical theorem that

$$\sigma_{\text{tot}} = \frac{2\pi}{k^2} \sum_l (2l+1)(1 - \eta_l \cos(2\delta_l)).$$

b) The differential cross section for elastic scattering is given by

$$\frac{d\sigma_{\text{el}}}{d\Omega} = |f(\theta)|^2.$$

From this, derive the following expression for the elastic scattering cross section

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_l (2l+1)|\eta_l e^{2i\delta_l} - 1|^2.$$

c) From a) and b) it follows that

$$\sigma_{\text{inel}} = \frac{\pi}{k^2} \sum_l (2l+1)(1 - \eta_l^2).$$

Show with this equation that for the reaction $\nu_\mu + e^- \to \mu^- + \nu_e$ we obtain the relation

$$\sigma(\nu_\mu + e^- \to \mu^- + \nu_e) \leq \frac{2\pi}{E_{\text{cm}}^2}, \tag{1}$$

where E_{cm} denotes the center-of-mass energy (k should be considered in the center-of-mass system). Note that this is an $l = 0$ scattering process and that a spin factor $1/(2s+1)$ should be taken into account.

d) In Fermi theory the cross section is given by

$$\sigma = \frac{G_{\text{F}}^2 s}{\pi}, \tag{2}$$

where G_{F} is Fermi's constant and $\sqrt{s}$ denotes the invariant mass.

Use Eqs. (1) and (2) to find the energy at which Fermi theory breaks down.