Exercises to "Standard Model of Particle Physics II"

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in of solutions:	**Discussion of solutions:**
October 29, 2024 - 14:00, INF 227, SR 1.403	October 29, 2024 - 14:00, INF 227, SR 1.403

Problem 1: *From QED to QCD* [**8 points**]

In **Problem 2** on the last sheet we showed that in QED $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is *invariant* under the U(1) gauge transformation $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha(x)$. For QCD, this gauge transformation is generalised to $A_\mu \to U(x)\left(A_\mu + \frac{i}{g}\partial_\mu\right)U(x)^\dagger$, while the quarks obey $\psi \to U(x)\psi$, where ψ carries one SU(N) index, $A_\mu \equiv A_\mu^a T^a$ is now matrix-valued, and $U(x) \in \mathrm{SU}(N)$.

a) Using the U(1) covariant derivative $D_\mu = \partial_\mu - ieA_\mu$, show that

$$[D_\mu, D_\nu]\,\psi = -ie\,F_{\mu\nu}\,\psi\,,$$

where ψ is the electron field.

b) For a set of parameters α^a, the transformation matrix of QCD is $U(x)_{ij} = \exp(i\alpha^a T^a)_{ij}$. Using the infinitesimal version of the gauge transformation of the gluon field, $A_\mu \to A_\mu + \frac{1}{g}(\partial_\mu \alpha^a)T^a + i\left[\alpha^a T^a, A_\mu^b T^b\right]$, show that

$$D_\mu \psi \to (1 + i\alpha^a T^a) D_\mu \psi\,,$$

with the QCD covariant derivative $D_\mu = \partial_\mu - igA_\mu^a T^a$.

c) In analogy to QED, we can define the QCD field strength matrix $F_{\mu\nu} = F_{\mu\nu}^a T^a$ via $[D_\mu, D_\nu]\,\psi = -igF_{\mu\nu}^a T^a \psi$, which is no longer invariant. Compute $F_{\mu\nu}^a$.

d) Show that the QCD Lagrangian is gauge-invariant.

$$\mathcal{L}_{\mathrm{QCD}} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}^a F^{a\,\mu\nu}\,.$$

Hint: This is a one-line proof if you use b) and c).

Problem 2: *Goldstone's theorem* [**12 Points**]

Goldstone's theorem states:

> *Every generator of a continuous global symmetry, which does not annihilate the vacuum, corresponds to a massless scalar particle.*

To demonstrate this, consider the Lagrangian $\mathscr{L}(\phi_i, \partial_\mu \phi_i)$ with n real scalar fields ϕ_i, invariant under the global transformation $\phi_i \to \{\exp(i\theta^a T^a)\}_{ij}\, \phi_j$. Defining $\Phi \equiv (\phi_1, \ldots, \phi_n)^T$, we can write this transformation as $\Phi \to \exp(i\theta^a T^a)\, \Phi$.

a) What are the properties of the iT^a, if the ϕ_i are real, and if $\Phi^T \Phi$ is invariant under the global transformation?

b) Show that from the conservation of the Noether current, it follows that:

$$\frac{\partial \mathscr{L}}{\partial \phi_i}\, (iT^a)_{ij}\, \phi_j + \frac{\partial \mathscr{L}}{\partial(\partial^\mu \phi_i)}\, (iT^a)_{ij}\, \partial^\mu \phi_j = 0\,.$$

c) Let $\mathscr{L} = \frac{1}{2}(\partial_\mu \Phi)^T\, (\partial^\mu \Phi) - V(\Phi)$. Show with the help of the first two results that

$$\frac{\partial V}{\partial \phi_i}\, (iT^a)_{ij}\, \phi_j = 0\,.$$

d) Let $\vec{v}$ be the minimum of $V(\Phi)$. Show with the results from above that $M^2\, (iT^a)\, \vec{v} = 0$, where M^2_{ij} is given by

$$(M^2)_{ij} \equiv \left. \frac{\partial^2 V}{\partial \phi_i\, \partial \phi_j} \right|_{\Phi = \vec{v}}\,.$$

e) The matrix M^2 is interpreted as a mass matrix. The broken vacuum $\vec{v}$ is in general not invariant under the transformations $\exp(i\theta^a T^a)$. If $T^a \vec{v} = 0$, one has a *Wigner–Weyl realization* of the symmetry, if $T^a \vec{v} \neq 0$ one has a realization à la *Nambu–Goldstone*. With the result from d) it follows that for all a with $T^a \vec{v} \neq 0$ the mass matrix has an eigenvector with eigenvalue 0.

   As an explicit example, consider the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \Phi)^T (\partial^\mu \Phi) - V(\Phi^T \Phi)$$

   with the potential

$$V(\Phi^T \Phi) = \frac{1}{2}\mu^2 \Phi^T \Phi + \frac{1}{4}\lambda(\Phi^T \Phi)^2\,,$$

   where $\Phi = (\phi_1, \phi_2, \phi_3)^T$ is a triplet of real and scalar particles. The coupling obeys $\lambda > 0$.

   (i) Show that $\mathscr{L}$ is invariant under an SU(2) transformation $\Phi \to \exp(i\theta^a T^a)\, \Phi$. The three generators T^a of SU(2) are written here as $(T^a)_{ij} = -i\epsilon_{aij}$ (this is called the adjoint representation of SU(2)).

   (ii) For $\mu^2 < 0$ the potential has a minimum $\vec{v}^T = (0, 0, v)$ with $v \neq 0$. Derive a relation between v, μ^2, and λ. Show that the vacuum state $\vec{v}$ is invariant under a transformation generated by T^3, but not T^1 and T^2. That is, the vacuum state does *not* possess the symmetry of the Lagrangian (spontaneous symmetry breaking). Why do we speak in this case of spontaneous symmetry breaking from SU(2) to U(1)?

   (iii) The scalar fields will be expanded around the minimum, $\Phi^T \equiv (\phi_1, \phi_2, v + \phi_3)$, i.e. we treat ϕ_3 as a field fluctuating around v instead of 0. Show by inserting into the above Lagrangian that there is one massive scalar field and two massless Goldstone bosons. What is the mass of ϕ_3? Convince yourself that the matrix M^2 defined in d) is really the mass matrix.