

Exercises to “Standard Model of Particle Physics II”

Winter 2023/24

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Lecture webpage: <https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html>

Hand-in of solutions:

November 8, 2023 - 09:15, Phil. 12, kHS

Discussion of solutions:

November 8, 2023 - 11:15, Phil. 12, R105

Problem 1: $SU(N)$ [10 Points]

Let $U \in SU(N)$, i.e. $\det U = 1$ and $U^\dagger U = 1$. Any element of $SU(N)$ can be written as $U = \exp(-i\theta^a T^a)$, where the T^a are generators of the group with normalization $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

- Show that the T^a are traceless hermitian matrices.
- How many linear independent generators are there?
- The structure constants, d_{abc} and f_{abc} , are defined through

$$[T^a, T^b] = if_{abc} T^c, \quad \{T^a, T^b\} = \frac{1}{N}\delta_{ab} + d_{abc} T^c.$$

Show that

$$\begin{aligned} \text{Tr}(T^a T^b T^c) &= \frac{1}{4}(d_{abc} + if_{abc}), \\ \left[\sum_a T^a T^a, T^b \right] &= 0, \\ [T^a, [T^b, T^c]] + [T^c, [T^a, T^b]] + [T^b, [T^c, T^a]] &= 0. \end{aligned}$$

- Show that the structure constants form a representation of $SU(N)$, i.e. take $(T^a)_{bc} = -if_{abc}$ as a generator. This is the so-called adjoint representation.
- Calculate the f_{abc} for

$$T^a = \frac{\sigma^a}{2},$$

where a runs from 1 to 3, and σ^a are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 2: From QED to QCD [10 points]

In **Problem 2** on the last sheet we showed that in QED $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is *invariant* under the U(1) gauge transformation $A_\mu \rightarrow A_\mu + \frac{1}{e}\partial_\mu\alpha(x)$. For QCD, this gauge transformation is generalised to $A_\mu \rightarrow U(x) \left(A_\mu + \frac{i}{g}\partial_\mu \right) U(x)^\dagger$, while the quarks obey $\psi \rightarrow U(x)\psi$, where ψ carries one SU(N) index, $A_\mu \equiv A_\mu^a T^a$ is now matrix-valued, and $U(x) \in \text{SU}(N)$.

- a) Using the U(1) covariant derivative $D_\mu = \partial_\mu - ieA_\mu$, show that

$$[D_\mu, D_\nu] \psi = -ie F_{\mu\nu} \psi,$$

where ψ is the electron field.

- b) For a set of parameters α^a , the transformation matrix of QCD is $U(x)_{ij} = \exp(i\alpha^a T^a)_{ij}$. Using the infinitesimal version of the gauge transformation of the gluon field, $A_\mu \rightarrow A_\mu + \frac{1}{g}(\partial_\mu\alpha^a)T^a + i[\alpha^a T^a, A_\mu^b T^b]$, show that

$$D_\mu \psi \rightarrow (1 + i\alpha^a T^a) D_\mu \psi,$$

with the QCD covariant derivative $D_\mu = \partial_\mu - igA_\mu^a T^a$.

- c) In analogy to QED, we can define the QCD field strength matrix $F_{\mu\nu} = F_{\mu\nu}^a T^a$ via $[D_\mu, D_\nu] \psi = -igF_{\mu\nu}^a T^a \psi$, which is no longer invariant. Compute $F_{\mu\nu}^a$.
- d) Show that the QCD Lagrangian is gauge-invariant.

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}.$$

Hint: This is a one-line proof if you use b) and c).