Exercises to “Standard Model of Particle Physics II”

Winter 2024/25

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in of solutions:
October 23, 2024 - 11:15, Phil. 12, kHS

Discussion of solutions:
~~October 23, 2024 - 11:15, Phil. 12, kHS~~
October 29, 02:15pm, SR 1.403, INF 227

Problem 1: *General Global symmetry transformations* [**5 Points**]

a) Derive the Noether theorem for a global symmetry transformation of the following Lagrange density:

$$\mathscr{L} = \mathscr{L}(u_i, \partial_\mu u_i)$$

In other words, find an expression for the conserved current associated to the symmetry of the theory under the field transformation:
$u_i(x) \to u_i'(x) = e^{i\,\epsilon_k\,T^k_{ij}} u_j(x) \cong u_i(x) + \delta u_i(x)$ with $\delta u_i(x) = i\,\epsilon_k\,T^k_{ij}\,u_j(x)$. Here the T^k_{ij} denote the generators of the algebra of the corresponding symmetry group and ϵ_k the infinitesimal, space-time independent coefficients.

b) Apply your formula to the Lagrange density of a massive complex scalar field (where we consider ϕ and ϕ^* as independent fields):

$$\mathscr{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi^* \phi$$

What is the conserved current in this case?
Hint: Use $\phi(x) \to \phi'(x) = e^{i\epsilon}\phi(x)$ as transformation of the scalar field.

Problem 2: *Local symmetry transformations in QED* [**5 Points**]

Show that the Lagrange density of QED

$$\mathscr{L}_{\text{QED}} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ ,$$

with $D_\mu = \partial_\mu - ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is invariant under the local symmetry transformations

$$\psi \to \psi' = e^{i\epsilon(x)}\psi, \qquad \overline{\psi} \to \overline{\psi'} = \overline{\psi} e^{-i\epsilon(x)},$$

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \epsilon(x).$$

Local symmetry means that now the infinitesimal parameter $\epsilon(x)$ depend on the space-time coordinates.

Problem 3: *General local symmetry transformations* [**10 Points**]

Now, we want the general Lagrange density $\mathscr{L} = \mathscr{L}(u_i, \partial_\mu u_i)$ to be invariant under a local symmetry transformation. In analogy with **Problem 1**, the fields u_i transform as

$$u_i(x) \to u'_i(x) = e^{i\,\epsilon_k(x)\,T^k_{ij}} u_j(x) \cong u_i(x) + \delta u_i(x)$$

with

$$\delta u_i(x) = i\,\epsilon_k(x)\,T^k_{ij}\,u_j(x).$$

a) Considering the transformation properties of the u_i's given above, show that we need to introduce a new field to keep $\mathscr{L}$ invariant under the local symmetry transformation.

b) Extend the system by a set of additional vector fields $A_{\mu k}$ (one for each generator T^k_{ij}), so that the Lagrange density does not depend on their derivatives: $\mathscr{L} = \mathscr{L}(u_i, \partial_\mu u_i, A_{\mu k})$. Assume that the transformation properties of the new fields are of the following form (where P and R denote, at first, unspecified matrices):

$$A_{\mu l}(x) \to A'_{\mu l}(x) = A_{\mu l}(x) + P^k_{lm} A_{\mu m}(x)\epsilon_k(x) + R^k_l \partial_\mu \epsilon_k(x).$$

From the variational principle, derive general conditions for the matrices P and R such that $\mathscr{L}(u_i, \partial_\mu u_i, A_{\mu k})$ is invariant under local symmetry transformations.

c) Consider the following Lagrange density

$$\mathscr{L} = \frac{1}{2}(D_\mu u_i)(D^\mu u_i),$$

where the covariant derivative of the fields u_i is given by $D_\mu u_i = \partial_\mu u_i + g\epsilon_{ijk} A_{\mu j} u_k$ and ϵ_{ijk} denotes the totally anti-symmetric tensor. Taking into account the results from b) find a relation between the matrix R^k_l and the generators T^k_{ij}.