Exercises to "Standard Model of Particle Physics II"

Winter 2022/23

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Sheet 12 - January 25, 2023

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in of solutions:
February 1, 2023 - 11:15, Phil. 12, R105

Discussion of solutions:
February 1, 2023 - 11:15, Phil. 12, R105

Problem 1: *Gauge Unification* [**10 Points**]

At the unification scale M_{GUT} the Standard Model gauge couplings are unified to one single coupling g_5 so that

$$g_5 = g_3 = g_2 = g_1 \ ,$$

where $g_3 = g_s$ is the strong coupling, $g_2 = g$ is the coupling of weak interactions and $g_1 = \sqrt{\frac{5}{3}} g'$ with g' the hypercharge coupling.

a) Which value do you expect for the running weak mixing angle $\sin^2\theta_w(M^2_{GUT})$ at the unification scale?

b) To compare this result with experiments we need to run the couplings to low energies i.e. to $Q^2 = M_Z^2$. The one-loop renormalization group equations for the gauge couplings (neglecting small Higgs contributions) are given by

$$\alpha_i^{-1}(Q^2) = \alpha_i^{-1}(M^2_{GUT}) + \frac{b_i}{4\pi}\ln\left(\frac{M^2_{GUT}}{Q^2}\right) \ , \tag{1}$$

where $b_1 = \frac{4F}{3}$, $b_2 = -\left(\frac{22}{3} - \frac{4F}{3}\right)$, $b_3 = -\left(11 - \frac{4F}{3}\right)$ and F is the number of families in the Standard Model.

Using the fine structure constant from electromagnetism $\alpha \equiv e^2/(4\pi)$, express Eq.(1) for α_1 and α_2 in terms of α and the weak mixing angle.

c) What is M_{GUT} in terms of the measured quantities $\alpha(M_Z^2) = 1/128$ and $\alpha_s(M_Z^2) = 0.118$?

d) Show that the weak mixing angle at the scale M_Z is given by

$$\sin^2\theta_w(M_Z^2) = \frac{1}{6} + \frac{5}{9}\frac{\alpha(M_Z^2)}{\alpha_s(M_Z^2)} \ ,$$

and calculate its explicit value. Does your result agree with the measured value?

Problem 2: *$SU(5)$ adjoint Higgs field* **[10 Points]**

We consider the symmetry breaking of $SU(5)$ gauge symmetry by a scalar field H in the adjoint representation **24**. The scalar potential of the $SU(5)$ model, neglecting the Higgs in the **5** representation, is given by

$$V(H) = -m^2 \mathrm{Tr}(H^2) + \lambda_1 \left(\mathrm{Tr}(H^2)\right)^2 + \lambda_2 \mathrm{Tr}(H^4)\,. \tag{2}$$

For simplicity we impose a symmetry $H \to -H$ to remove the cubic term. From representation theory we know that H can be represented as a 5×5 hermitian traceless matrix.

a) Show that H can be transformed into a real diagonal matrix

$$H_d = UHU^\dagger = \mathrm{Diag}\left(h_1,\ h_2,\ h_3,\ h_4,\ h_5\right)\ ,$$

where $h_1 + h_2 + h_3 + h_4 + h_5 = 0$.

b) Write down the potential $V(H)$ with H in diagonal form and find the minimizing condition with respect to the h_is.
Hint: Since the h_is are not all independent you need to use a Lagrange multiplier.

c) How many different values can the h_is take at most? From this result discuss the most general form of symmetry breakings that can be induced by a **24** Higgs field.

d) In terms of matrix multiplication, the covariant derivative of H is given by

$$D_\mu H = \partial_\mu H + i g_5 \left[W_\mu, H\right]\ , \tag{3}$$

where W_μ is a 5×5 traceless hermitian matrix of all $SU(5)$ gauge bosons and g_5 is the unified gauge coupling. The gauge boson masses come from the covariant derivatives

$$\mathcal{L} \supset \mathrm{Tr}\left(D_\mu \langle H \rangle (D_\mu \langle H \rangle)^\dagger\right)\ .$$

Calculate the gauge boson masses with the vacuum expectation value of H given by

$$\langle H \rangle = v \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}\ .$$