Exercises for Neutrino Physics: Theory and Experiment

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Sheet 8: Theory

1. **Newtonian "derivation" of the Friedmann equations [10 Points]**

   Consider a homogeneous spherical space of density ρ. Within this space distances scale as

   $$r(t) = a(t)r_0, \qquad (a(0) = 1). \tag{1}$$

   Assuming Newtonian conservation of energy for a mass element of mass m within this space, show that

   $$k := \frac{8\pi}{3}G\rho a^2 - \dot{a}^2 \tag{2}$$

   is constant. With this definition of k derive the Friedmann equation

   $$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho. \tag{3}$$

   How does the system behave for $k > 0$, $k < 0$ and $k = 0$.
   Derive also the second Friedmann equation

   $$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho}{3} \tag{4}$$

   Extra [3 Points]: What are the problems with using Newtonian derivations?

2. **Solving some integrals [10 Points]**

   The Bose-Einstein $(-)$ and Fermi-Dirac $(+)$ distributions given by

   $$f(\vec{p}, T) = \frac{1}{\exp\{\beta(E(\vec{p}) - \mu\} \pm 1} \tag{5}$$

   describe the average number of particles per phase-space volume for bosons and fermions, respectively. Here, μ is the so called chemical potential and $\beta = 1/T$ describes the temperature. Calculate the number density n and energy density ρ for both distributions each in the relativistic limit $m \ll T$ and the non-relativistic limit $m \gg T$. You can assume $\mu = 0$.
   What are the numerical values for the number densities of photons n_γ and neutrinos n_ν assuming $T_\gamma = 2.725\,\mathrm{K}$ and $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$?