Exercises for Neutrino Physics: Theory and Experiment

Teresa Marrodán Undagoitia
Werner Rodejohann

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Tutors:
Janine Hempfling: janine.hempfling@mpi-hd.mpg.de
Oliver Scholer: scholer@mpi-hd.mpg.de

Sheet 4: Theory

1. **Neutrino oscillations in matter [20 Points]**

The MSW effect describes oscillations of neutrinos in matter. Consider the case of two flavour oscillations in a realistic (non-constant) mass distribution.
The time evolution of the eigenfunctions of the Hamiltonian in matter is described by the following expression:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}|\nu_A\rangle\\|\nu_B\rangle\end{pmatrix}=\begin{pmatrix}E_A(t) & -i\dot{\theta}(t)\\ i\dot{\theta}(t) & E_B(t)\end{pmatrix}\begin{pmatrix}|\nu_A\rangle\\|\nu_B\rangle\end{pmatrix} \tag{1}$$

where

$$-E_A(t)=E_B(t)=\frac{\Delta m^2\sin(2\theta_0)}{4E\sin(2\theta)} \tag{2}$$

with the (constant) vacuum mixing angle θ_0. The eigenstates in matter are $|\nu\rangle=(|\nu_A\rangle,|\nu_B\rangle)^T$ and at any given time they are connected to the flavor states via: $|\tilde{\nu}\rangle=\tilde{U}(t)\,|\nu\rangle$ with

$$|\tilde{\nu}\rangle=\begin{pmatrix}|\tilde{\nu}_e\rangle\\|\tilde{\nu}_\mu\rangle\end{pmatrix},\qquad \tilde{U}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}, \tag{3}$$

and the effective (time-dependent) mixing angle

$$\tan 2\theta=\frac{\frac{\Delta m^2}{2E}\sin(2\theta_0)}{\frac{\Delta m^2}{2E}\cos(2\theta_0)-\sqrt{2}G_F N_e(t)}, \tag{4}$$

where $N_e(t)$ is the electron number density in matter. The effective Hamiltonian is not diagonal, since matter eigenstates and thus the mixing angle θ are not constant. If, however, $\dot{\theta}$ is small, i.e. the change in θ occurs slowly, we obtain the adiabatic approximation, where we can neglect $\dot{\theta}$. Suppose that at the time $t=t_i$ an electron neutrino is produced,

$$|\tilde{\nu}_\alpha(t_i)\rangle\equiv\delta_{\alpha e}\,|\tilde{\nu}_\alpha\rangle \tag{5}$$

(a) What is the expression for $|\tilde{\nu}_\alpha(t_f)\rangle$ at some later time t_f in the adiabatic approximation?

(b) Show that in the adiabatix approximation the result for the transition probability is given by

$$P(\nu_e\to\nu_\mu)=\frac{1}{2}-\frac{1}{2}\cos(2\theta_i)\cos(2\theta_f)-\frac{1}{2}\sin(2\theta_i)\sin(2\theta_f)\cos(\Phi_{AB}) \tag{6}$$

with $\theta_i\equiv\theta(t_i),\theta_f\equiv\theta(t_f)$, and

$$\Phi_{IJ}=\int_{t_i}^{t_f}[E_I(t)-E_J(t)]\,\mathrm{d}t, \tag{7}$$

where $I,J\in\{A,B\}$.

(c) Show that in the case of constant electron density, Eq. (6) reduces to the standard formula of two-flavor oscillations

$$P(\nu_e \to \nu_\mu)(t) = \sin^2(2\theta)\sin^2\left(\frac{\pi t}{L_M^{\mathrm{osc}}}\right), \tag{8}$$

and determine L_M^{osc}. What happens in the limit $N_e \to 0$?

(d) Use Eq.(4) to find an expression for $\sin^2(2\theta)$. The formula for $\sin^2(2\theta)$ describes a Breit-Wigner distribution. We define the resonance width at half height, Δr, as the width at which the curve of the distribution is larger than one half $\left[\sin^2(2\theta) > 1/2\right]$. Find an expression for Δr in terms of θ_0.

(e) The *adiabatic parameter* is defined as

$$\gamma_r \equiv \frac{|E_A|}{\left|\dot{\theta}\right|}, \tag{9}$$

which becomes large if the adiabatic approximation is good. To interpret this, first show that

$$\dot{\theta} = \frac{1}{2}\frac{\sin^2(2\theta)}{\Delta m^2 \sin(2\theta_0)} 2\sqrt{2} E G_F \dot{N}_e. \tag{10}$$

Now assume that the change in matter density is approximately linear, $\dot{N}_e \approx \mathrm{const}$, and that the density passes exactly through the resonance, i.e. $\Delta N_e = N_e(t_f) - N_e(t_i) \propto \Delta r$ to formulate γ_r in terms of L_M^{osc} and $\Delta t = t_f - t_i$. Discuss the adiabaticity condition $\gamma_r > 3$.