Exercises for Neutrino Physics: Theory and Experiment

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Sheet 1: Theory

You have to solve at least 50% of the exercises to pass the lecture. Please send your solutions until the submission date to one of the tutors via email. You can take a picture or a scan of your solutions but make sure that it is readable.

1. **Discrete symmetries [5 Points]**

   A CP transformation takes a left-handed neutrino and transforms it into a right-handed anti-neutrino. Consider the probability $P(\nu_\alpha \to \nu_\beta, t)$ of a neutrino flavor α to oscillate into a flavor β after time t of propagation.

   (a) How do the discrete symmetries CP, T, and CPT change the initial and final states of the process given above? That is, give

   $$\begin{aligned} (\mathrm{CP})^{-1} P(\nu_\alpha \to \nu_\beta, t)\,(\mathrm{CP})\,, \\ (\mathrm{T})^{-1} P(\nu_\alpha \to \nu_\beta, t)\,(\mathrm{T})\,, \\ (\mathrm{CPT})^{-1} P(\nu_\alpha \to \nu_\beta, t)\,(\mathrm{CPT}) \end{aligned} \tag{1}$$

   in terms of other transition probabilities.

   (b) The CPT theorem states that every QFT described by a Lorentz invariant, local Lagrangian is invariant under CPT transformations. How could the above result be used to test this theorem in Nature? (Hint: Consider the case $\alpha = \beta$, i.e. the survival probability of (anti-) neutrinos.)

2. **Neutrino Oscillations and CP violation [15 Points]**

   Neutrinos are produced and detected in weak interactions, but the neutrino mass matrix is diagonal in a different basis. The unitary matrix U rotates the flavor basis into the mass eigenbasis. Consider a neutrino of flavor α produced at timet= 0,

   $$|\nu_\alpha(t=0)\rangle = |\nu_\alpha\rangle = U^*_{\alpha k}\,|\nu_k\rangle \tag{2}$$

   as a superposition of mass eigenstates $|\nu_k\rangle$

   (a) Using approximations i. and ii. below, and the fact that the time evolution of $|\nu_k\rangle$ is governed by the Schrödinger equation to derive the general expression for neutrino oscillations in vacuum,

   $$P_{\alpha\beta} = P(\nu_\alpha \to \nu_\beta, t) = \delta_{\alpha\beta} - 4\sum_{k>j} \mathrm{Re}\left(\mathcal{I}^{kj}_{\alpha\beta}\right)\sin^2\left(\frac{\Delta_{kj}}{2}\right) + 2\sum_{k>j} \mathrm{Im}\left(\mathcal{I}^{kj}_{\alpha\beta}\right)\sin\left(\Delta_{kj}\right), \tag{3}$$

where $\left(\mathcal{I}_{\alpha\beta}^{kj}\right) = U_{\alpha k}^* U_{\beta k} U_{\beta j}^* U_{\alpha j}$ and $\Delta_{kj} = \frac{m_k^2 - m_j^2}{2E} L$. To arrive at this expression, you have to make the following assumptions:

i. Neutrino are highly relativistic, such that the distance travelled is $L \simeq t$ $(c = 1)$ and $E_k = \sqrt{p_k^2 + m_k^2} \simeq p_k \left(1 + \frac{m_k^2}{2p_k^2}\right)$

ii. The central momentum of the neutrino beam is given by $p_k \simeq p \approx E$ for all k

(b) Derive the expression for the CP asymmetry $\mathcal{A}_{\alpha\beta} = P\left(\nu_\alpha \to \nu_\beta, t\right) - P\left(\overline{\nu}_\alpha \to \overline{\nu}_\beta, t\right)$. (Hint: How does the state (2) change under a CP transformation? Use this fact and Eq. (3)

(c) Consider now the special case of two neutrino flavors, say e and μ, hence

$$U = \begin{pmatrix} \cos\theta & \exp(-i\delta)\sin\theta \\ -\exp(-i\delta)\sin\theta & \cos\theta \end{pmatrix}. \tag{4}$$

Calculate the CP asymmetry for $e - \mu$ oscillations. What are the conditions for CP violation to occur?

(d) Consider now the special case of three neutrino flavors whose mixing in general can be parametrised as

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\exp(-i\delta) \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\exp(i\delta) & c_{12}c_{23} - s_{12}s_{23}s_{13}\exp(i\delta) & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\exp(i\delta) & -c_{12}s_{23} - s_{12}c_{23}s_{13}\exp(i\delta) & c_{23}c_{13} \end{pmatrix} \tag{5}$$

where $c_{ab} = \cos(\theta_{ab}), s_{ab} = \sin(\theta_{ab}), 0 \leq \theta_{ab} \leq \pi/2$, and $0 \leq \delta \leq 2\pi$. Calculate the CP asymmetry $\mathcal{A}_{e\mu}$ explicitly and show that it can only be non-vanishing if all of the following conditions are satisfied: 1. None of the mixing angles is 0 or $\pi/2$. 2. None of the mass-square differences (i.e. Δ_{ab}) are zero. 3. The phase δ is neither 0 nor π.