Three-Body Entanglement in Particle Decays

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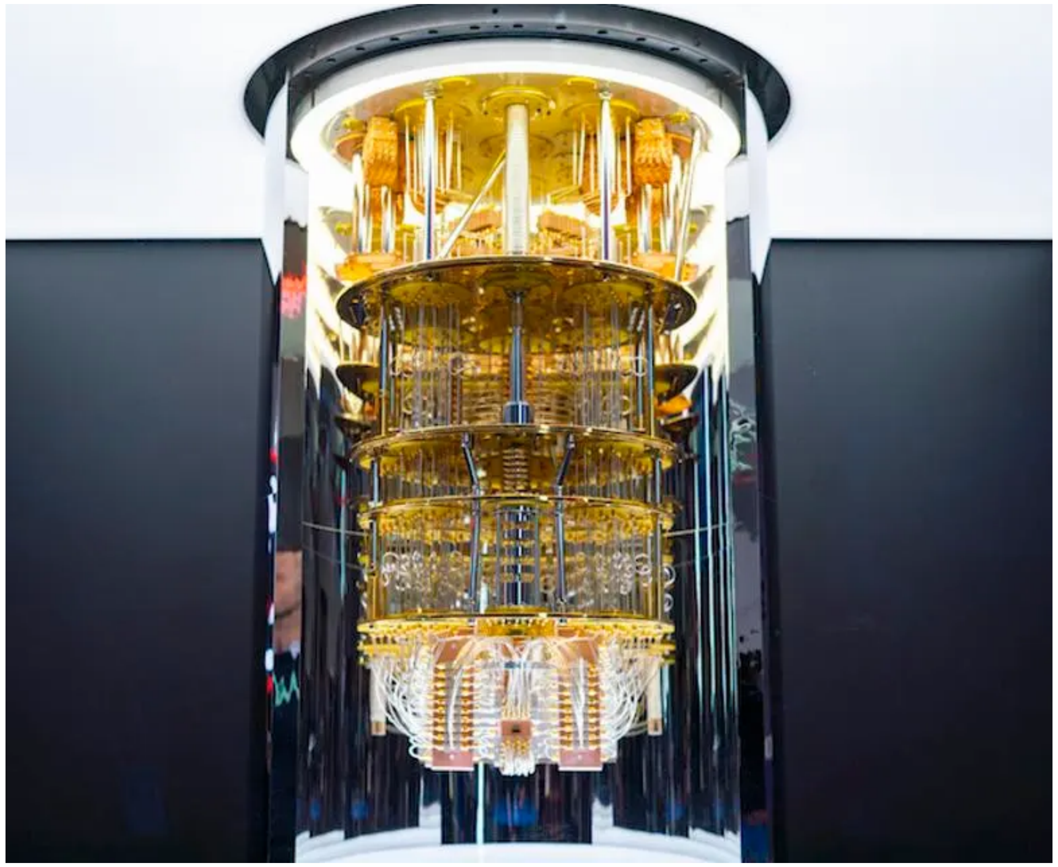
Entanglement

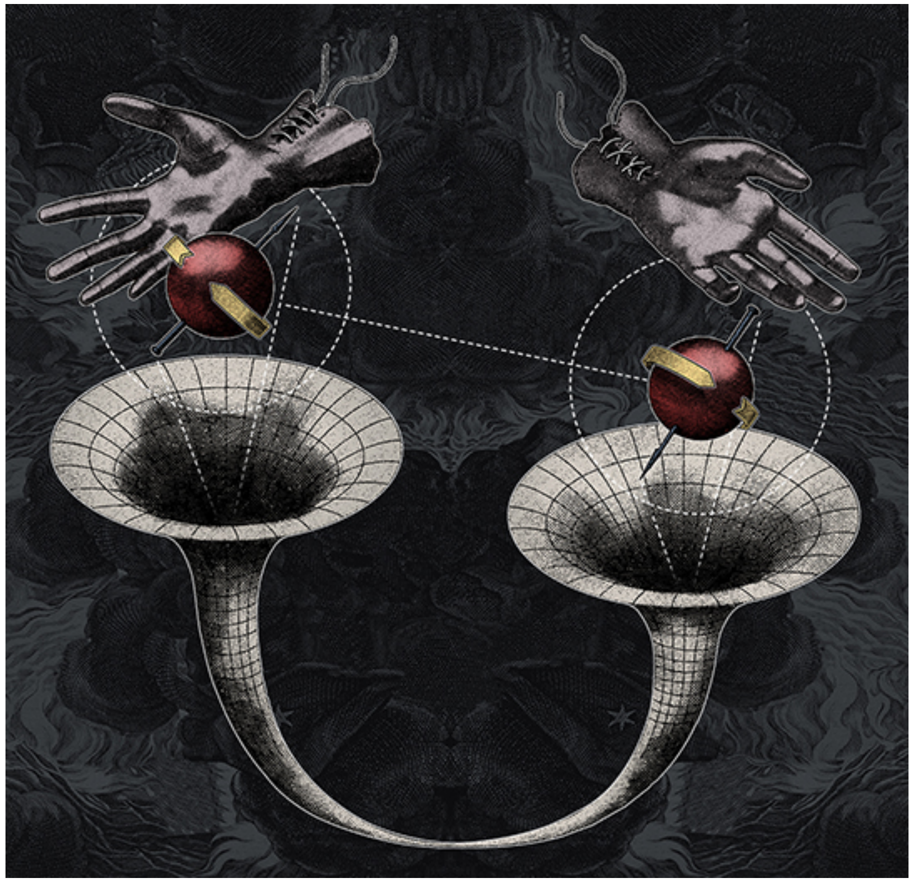
“not one but rather *the* characteristic trait of quantum mechanics”

[Erwin Schrödinger, 1935]

- Entanglement (entropy) plays an important role in developing quantum devices and understanding quantum gravity, AdS/CFT (Holography)



IBM Q system



What is the role of Entanglement in QFT and Particle Physics?

Plan

- Entanglement and Bell’s inequality in the 2-qubit system (Review)
- The status of quantum information measurements at the LHC
- 3-particle entanglement
- Conclusions

Qubit

- The qubit system has two basis kets: $|0\rangle$ and $|1\rangle$. The general state is

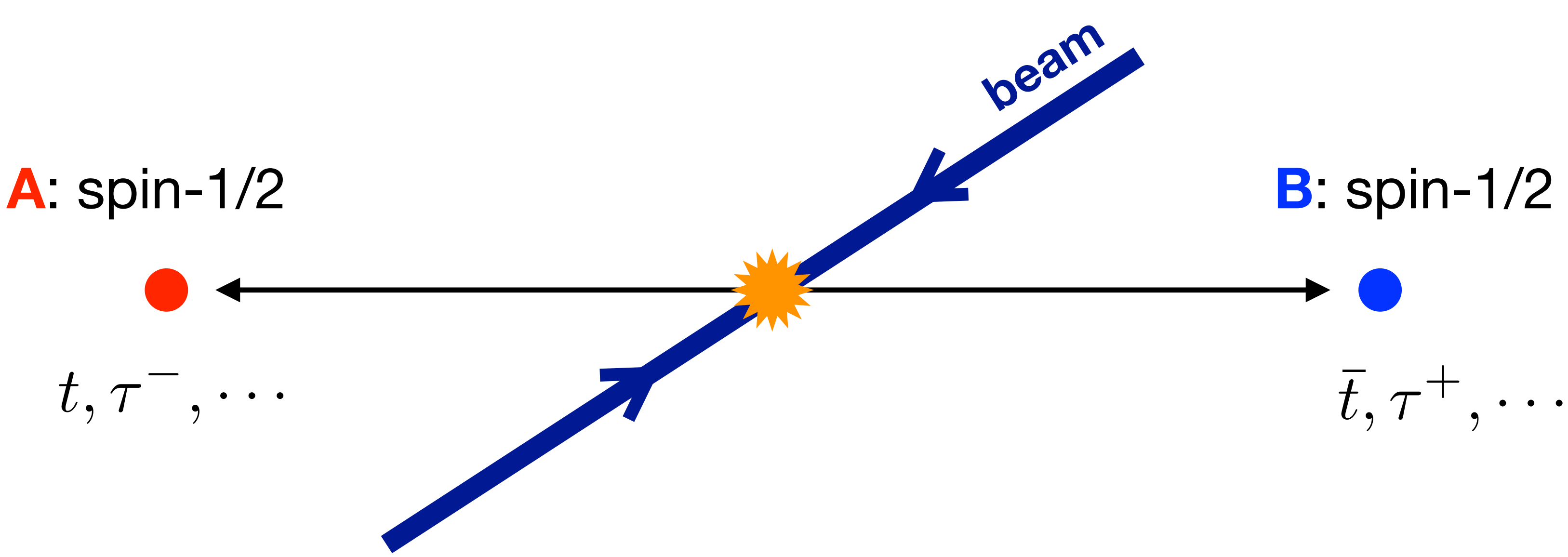
$$|\Psi\rangle = c_0|0\rangle + c_1|1\rangle \qquad (c_i \in \mathbb{C},\ |c_1|^2 + |c_2|^2 = 1)$$

- The n-qubit system has 2^n basis kets:

$$|\psi_1\psi_2\cdots\psi_n\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle \qquad |\psi_i\rangle \in \{|0\rangle, |1\rangle\}$$

Ex.) The general state in the 2-qubit system:

$$|\Psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle \qquad \left(c_{ij} \in \mathbb{C},\ \sum_{i,j}|c_{ij}|^2 = 1\right)$$



Separable or Entangled

Q.) Can we write

$$|\Psi\rangle_{AB} = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

$$\stackrel{?}{=} (\underbrace{\alpha_0|0\rangle + \alpha_1|1\rangle}_{|\Psi\rangle_A}) \otimes (\underbrace{\beta_0|0\rangle + \beta_1|1\rangle}_{|\Psi\rangle_B})$$

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\begin{aligned}
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\end{aligned}
$$

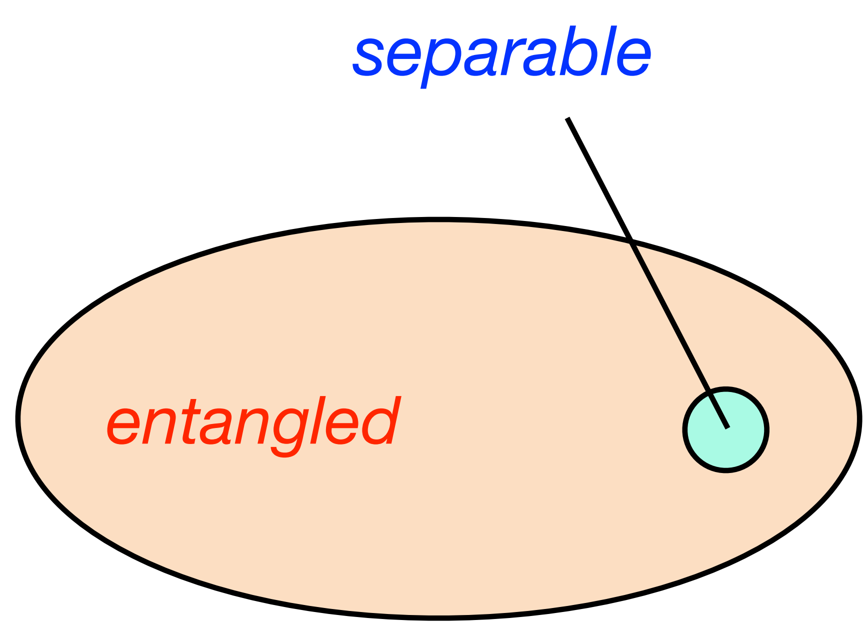
If the answer is

- **Yes** → $|\Psi\rangle_{AB}$ is *separable*
- **No** → $|\Psi\rangle_{AB}$ is *entangled*

$$|\Psi\rangle_{AB} = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

is separable iff: $c_{00}c_{11} - c_{01}c_{10} = 0$

→ Separable states are rare!



Mixed States

- We are uncertain about the quantum state and only know it is $|\Psi_i\rangle$ with probability p_i .

$(i = 1, \cdots, N)$

→ We say the state is ***mixed*** and describe it by the ***density operator*** :

$$\rho = \sum_{i=1}^{N} p_i |\Psi_i\rangle\langle\Psi_i| \qquad \left(p_i \geq 0,\ \sum_i p_i = 1\right)$$

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- The density operators are ***Hermitian***, ***Tr = 1***, ***positive definite*** :

$$\rho = \rho^\dagger,\ \ \mathrm{Tr}\rho = 1,\ \ \rho \geq 0$$

- The expectation value of an observable $\hat{\mathcal{O}}$ is can be computed as:

$$\langle\hat{\mathcal{O}}\rangle = \mathrm{Tr}\left[\hat{\mathcal{O}}\rho\right]$$

- The mixed state ρ_{AB} of the A-B system is saied to be ***separable***, if it's possible to write

$$\rho_{AB} = \sum_i q_i \cdot \rho_A^i \otimes \rho_B^i \qquad \left(q_i \geq 0,\ \sum_i q_i = 1\right)$$

Otherwise, the state is said to be ***entangled***.

Entanglement as a Resource

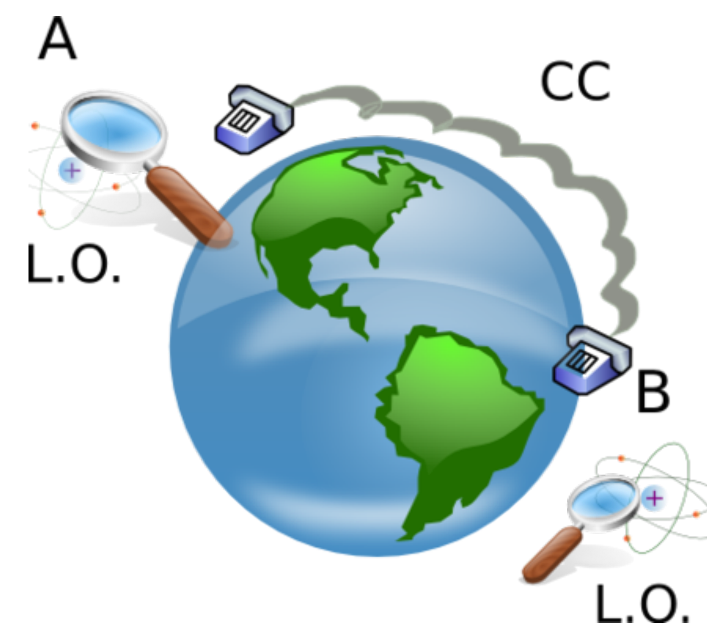
- Separable states will never be entangled by Local Operation and Classical Communication (LOCC)

- LOCC introduces an *order* among states

$\Rightarrow$ Entanglement is *defined* s.t. it decreases under non-invertible LOCC:

maximally entangled $\rho_{\max} \xrightarrow{\text{LOCC}} \rho \xrightarrow{\text{LOCC}} \rho_{\text{sep}}$ separable state

$$\frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$



ex) $|00\rangle + |11\rangle \xrightarrow[\text{local measurement}]{} |00\rangle$

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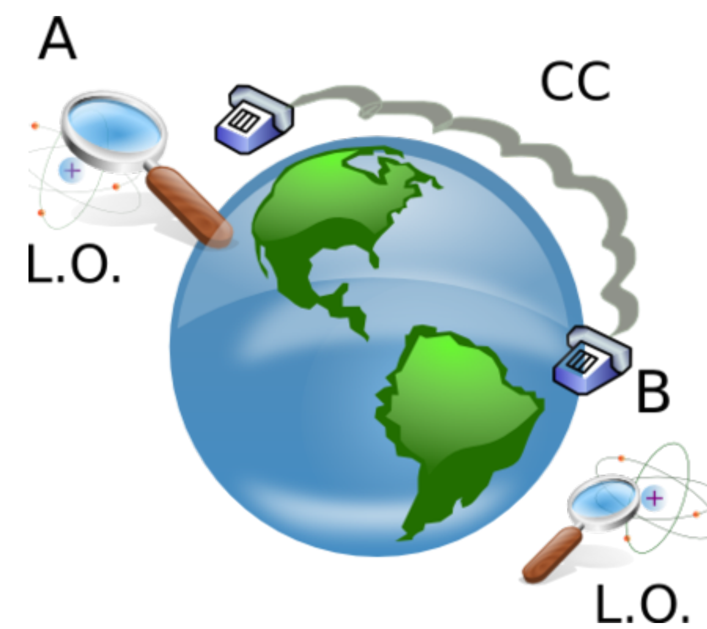

ex) $|00\rangle + |11\rangle \xrightarrow[\text{local measurement}]{} |00\rangle$

- Entanglement measures (monotones), $E(\rho)$

  - monotonically decreases under LOCC
  - $E(\rho) = 0$ if ρ is separable and $E(\rho) > 0$ otherwise
  - $E(\rho) = 1$ if ρ is maximally entangled states

Entanglement Measures

- Entanglement measures are often defined nicely for pure states $|\Psi\rangle_{AB}$

Von Neumann Entropy: $S_V(|\Psi\rangle_{\mathrm{AB}}) = -\mathrm{Tr}[\rho_A \log_2 \rho_A]$

$$\rho_A = \mathrm{Tr}_B\left(|\Psi\rangle\langle\Psi|_{AB}\right)$$

Linear Entropy: $S_L(|\Psi\rangle_{\mathrm{AB}}) = 2[1 - \mathrm{Tr}(\rho_A^2)]$

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- Entanglement measures for mixed states $\rho_{AB} = \sum_i p_k |\Psi_k\rangle\langle\Psi_k|$

Entanglement of Formation: $E_F(\rho_{\mathrm{AB}}) = \inf_{p_k, |\Psi_k\rangle} [p_k S_V(\Psi_k)]$

Concurrence: $\mathcal{C}(\rho_{\mathrm{AB}}) = \inf_{p_k, |\Psi_k\rangle} \left[p_k \sqrt{S_L(\Psi_k)}\right]$

minimising over all possible decompositions

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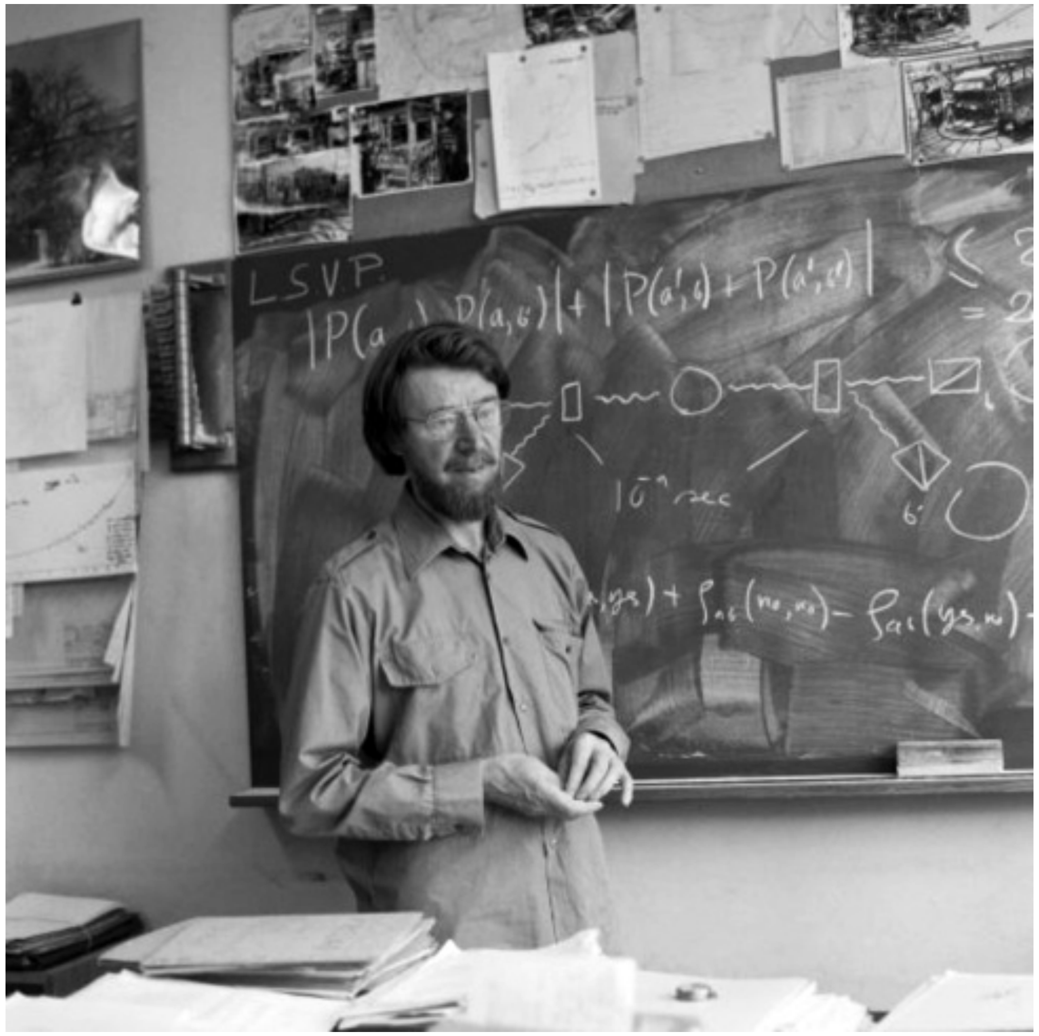
minimising over all possible decompositions

- For 2-qubit systems, the concurrence admits the analytical expression: [Wootters '98]

$$\mathcal{C}[\rho] = \max(0, \eta_1 - \eta_2 - \eta_3 - \eta_4) \in [0, 1]$$

$\eta_1 \geq \eta_2 \geq \eta_3 \geq \eta_4$ are eigenvalues of $\sqrt{\rho\tilde{\rho}}$ with $\tilde{\rho} \equiv (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

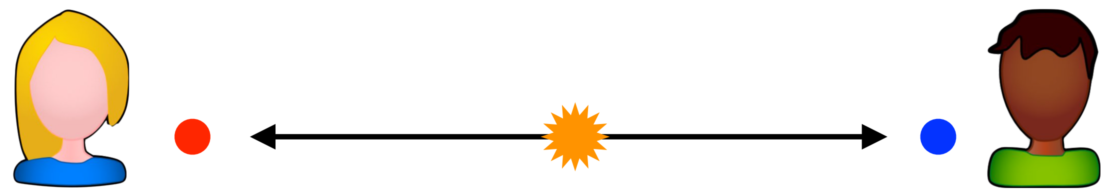
Bell-nonlocality



measures axis: $x \in \{\vec{n}\}$

outcome: $a \in \{-1, +1\}$

probability for a: $p_A(a \mid x)$



$y \in \{\vec{n}\}$

$b \in \{-1, +1\}$

$p_B(b \mid y)$

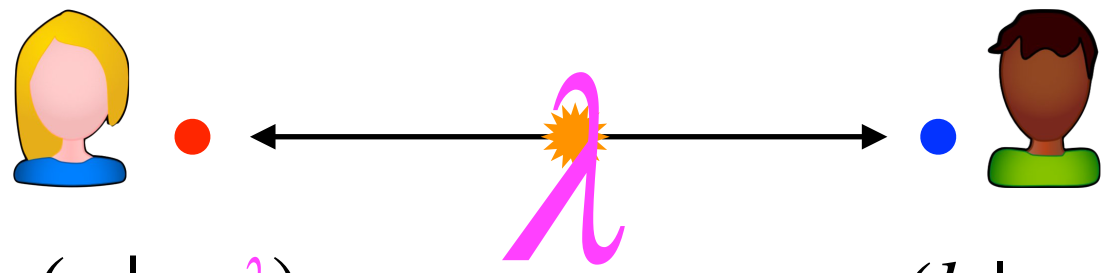
Experimental data is described by the *joint distribution* $p(a, b \mid x, y)$

measures axis: $x \in \{\vec{n}\}$ | $y \in \{\vec{n}\}$

outcome: $a \in \{-1, +1\}$ | $b \in \{-1, +1\}$

probability for a: $p_A(a\,|\,x)$ | $p_B(b\,|\,y)$

q_λ



$p_A(a\,|\,x, \lambda)$ $p_B(b\,|\,y, \lambda)$

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probability for λ

Local theories

$$p_L(a, b\,|\,x, y) = \sum_{\lambda} q_\lambda \cdot p_A(a\,|\,x, \lambda) \cdot p_B(b\,|\,y, \lambda)$$

measures axis: $x \in \{\vec{n}\}$ $\qquad$ $y \in \{\vec{n}\}$

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probability for a: $p_A(a|x)$ $\qquad$ $p_B(b|y)$

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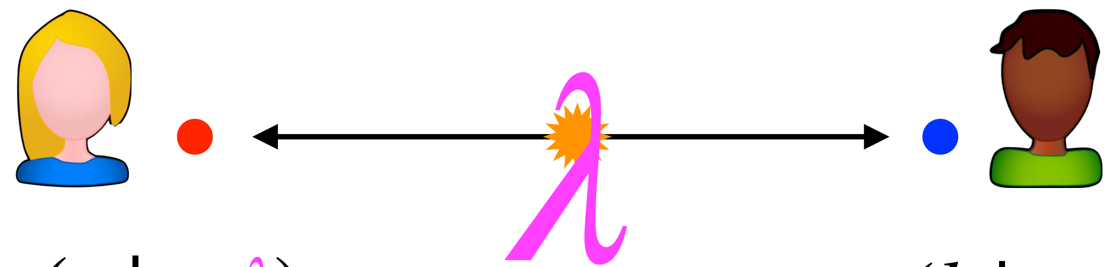

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probability for λ

$$p_L(a, b\,|\,x, y) = \sum_\lambda q_\lambda \cdot p_A(a\,|\,x, \lambda) \cdot p_B(b\,|\,y, \lambda)$$

Quantum Mechanics

density operator

$$p_Q(a, b\,|\,x, y) = \mathrm{Tr}\left[\rho_{AB}\left(M_{a|x} \otimes M_{b|y}\right)\right]$$

For projective measurement:

$$M_{a|x} = |a_x\rangle\langle a_x|$$

$$\hat{s}_x\,|a_x\rangle = a_x\,|a_x\rangle$$

measures axis: $x \in \{\vec{n}\}$

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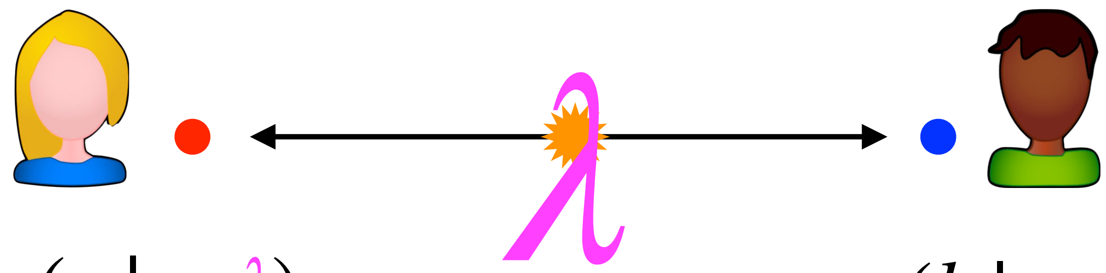

$y \in \{\vec{n}\}$

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$p_B(b \mid y)$

$p_A(a \mid x, \lambda)$ $\qquad$ $p_B(b \mid y, \lambda)$

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For projective measurement:

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For separable quantum states:

$$\rho_{AB} = \sum_\lambda q_\lambda \cdot \rho_A^\lambda \otimes \rho_B^\lambda \implies p_{Q_{\text{sep}}}(a, b \mid x, y) = \sum_\lambda q_\lambda \cdot \mathrm{Tr}\left[\rho_A^\lambda M_{a|x}\right] \cdot \mathrm{Tr}\left[\rho_B^\lambda M_{a|x}\right]$$

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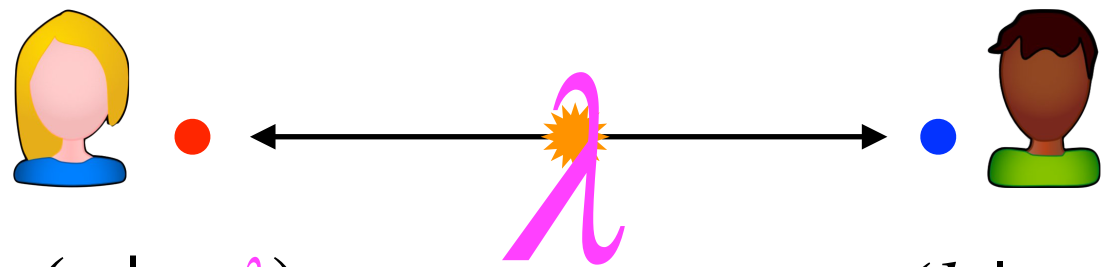

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$p_B(b \mid y)$

$p_A(a \mid x, \lambda)$ $\qquad$ $p_B(b \mid y, \lambda)$

Experimental data is described by the *joint distribution* $p(a, b \mid x, y)$

Local theories $\quad p_L(a, b \mid x, y) = \boxed{\sum_\lambda q_\lambda \cdot p_A(a \mid x, \lambda) \cdot p_B(b \mid y, \lambda)}$

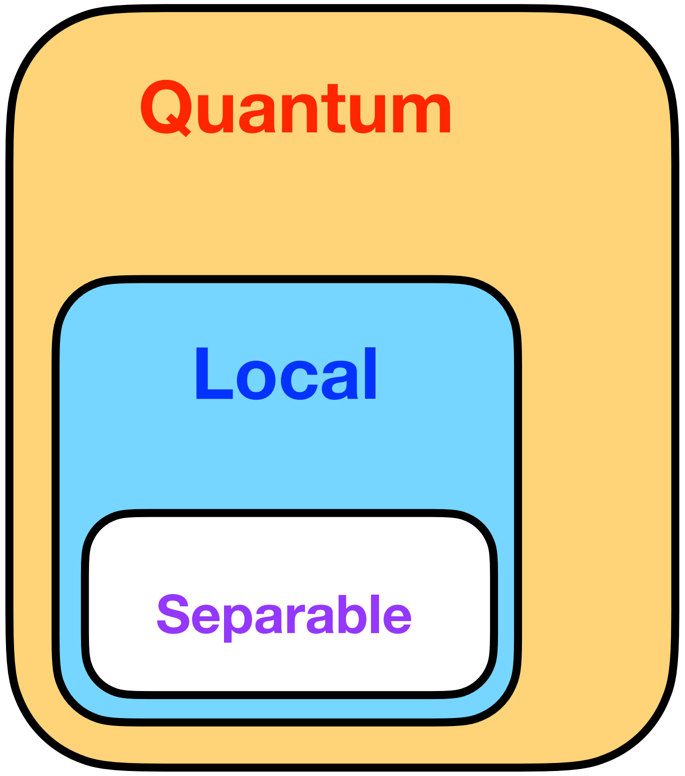
density operator

Quantum Mechanics $\quad p_Q(a, b \mid x, y) = \mathrm{Tr}\left[\rho_{AB}\left(M_{a|x} \otimes M_{b|y}\right)\right]$

For separable quantum states:

Local

$$\rho_{AB} = \sum_\lambda q_\lambda \cdot \rho_A^\lambda \otimes \rho_B^\lambda \implies p_{Q_{\mathrm{sep}}}(a, b \mid x, y) = \boxed{\sum_\lambda q_\lambda \cdot \mathrm{Tr}\left[\rho_A^\lambda M_{a|x}\right] \cdot \mathrm{Tr}\left[\rho_B^\lambda M_{a|x}\right]}$$



Quantum ⊃ Local ⊃ Separable

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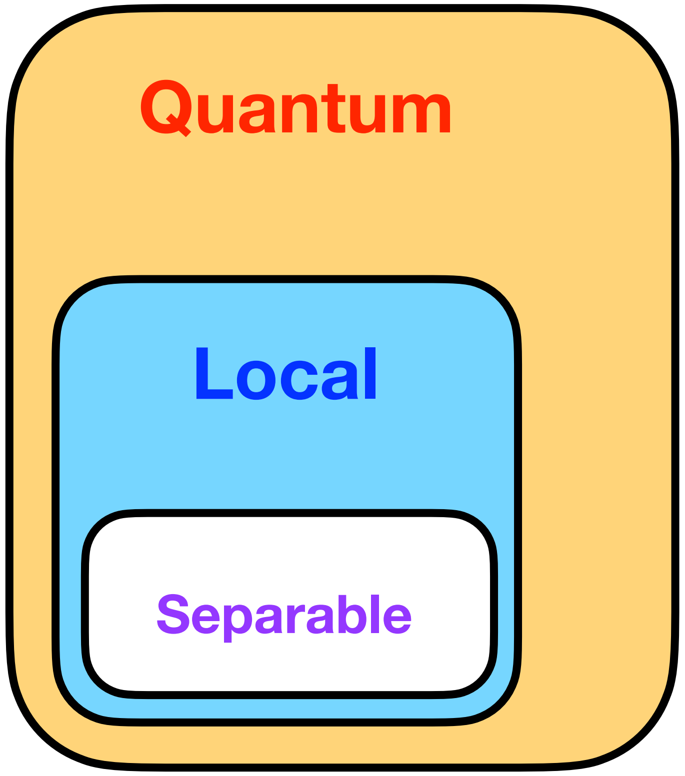

Quantum ⊃ Local ⊃ Separable

Bell-nonlocal ⊂ Entanglement

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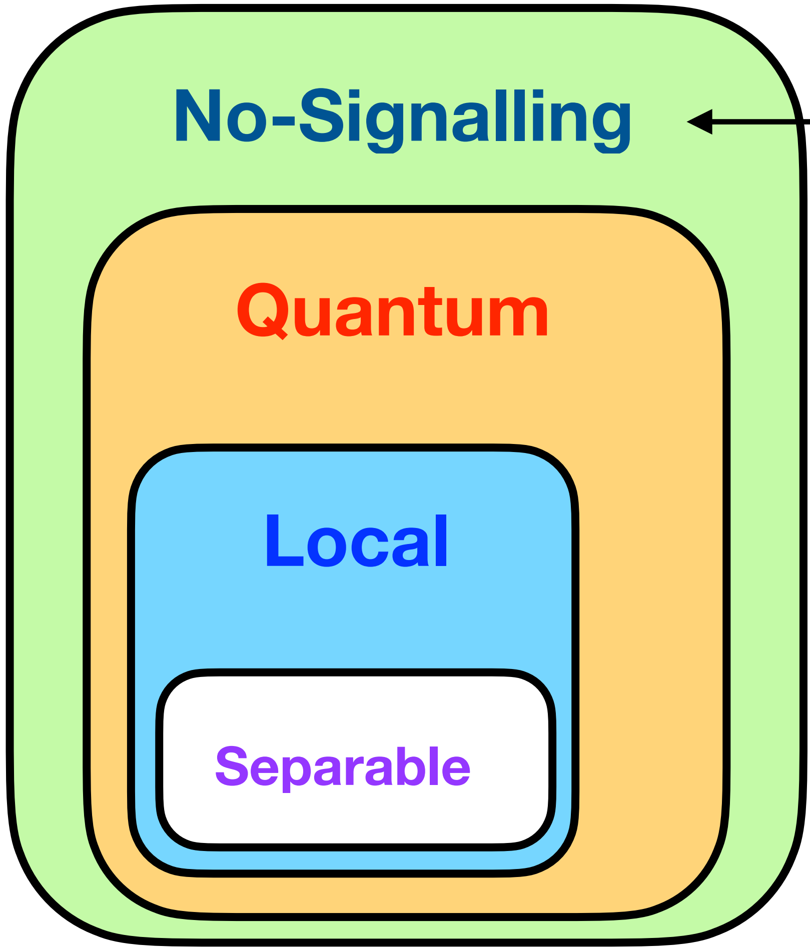
- Nonlocal states in QM does not violate causality

$$p(a\,|\,x,y) \equiv \sum_b p(a,b\,|\,x,y)$$

Condition for no causality violation: No-Signalling [Cirel'son(1980), Popescu, Rohrlich(1994)]

$$^{\forall}a,b,x,x',y,y' \quad \begin{cases} p(a\,|\,x,y) = p(a\,|\,x,y') & \text{Alice's dist. is indep. of Bob's choice for meas. axis} \\ p(b\,|\,x,y) = p(b\,|\,x',y) & \text{Bob's dist. is indep. of Alice's choice for meas. axis} \end{cases}$$

No-Signalling ⊃ Quantum ⊃ Local ⊃ Separable



Beyond QM

(no causality violation)

Bell Inequalities

- Bell-type inequalities (in general) are the inequalities that separate different types of distributions (No-signalling, Quantum, Local).

- Define the correlator $C_{xy} = \langle A_x B_y \rangle \equiv \sum_{a,b} abp(a,b\,|\,x,y)$

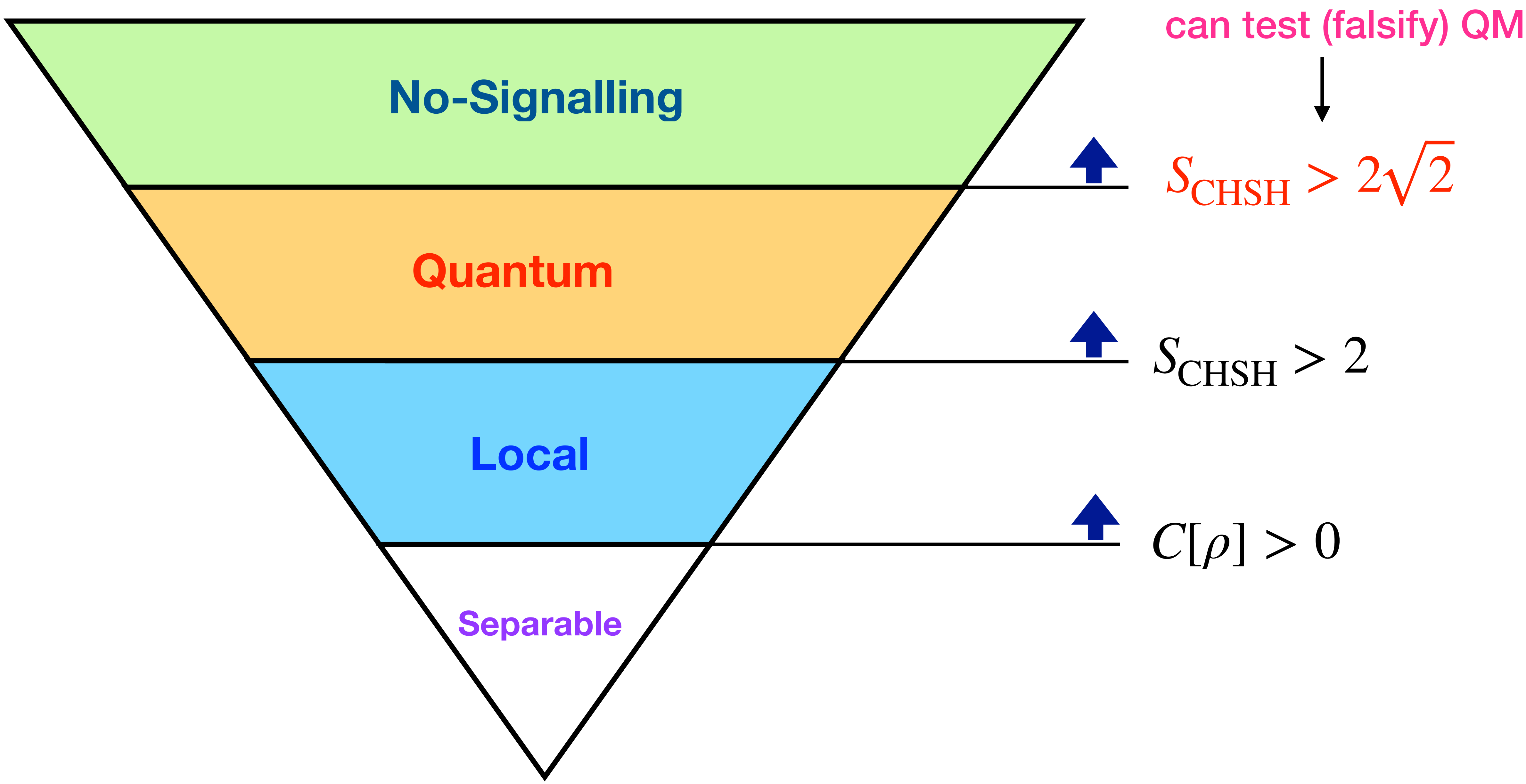
- CHSH quantity [Clauser-Horne-Shimony-Holt(1969)]

For $a, b \in \{\pm 1\},\ x \in \{\mathbf{n}_1, \mathbf{n}_2\},\ y \in \{\mathbf{e}_1, \mathbf{e}_2\}$

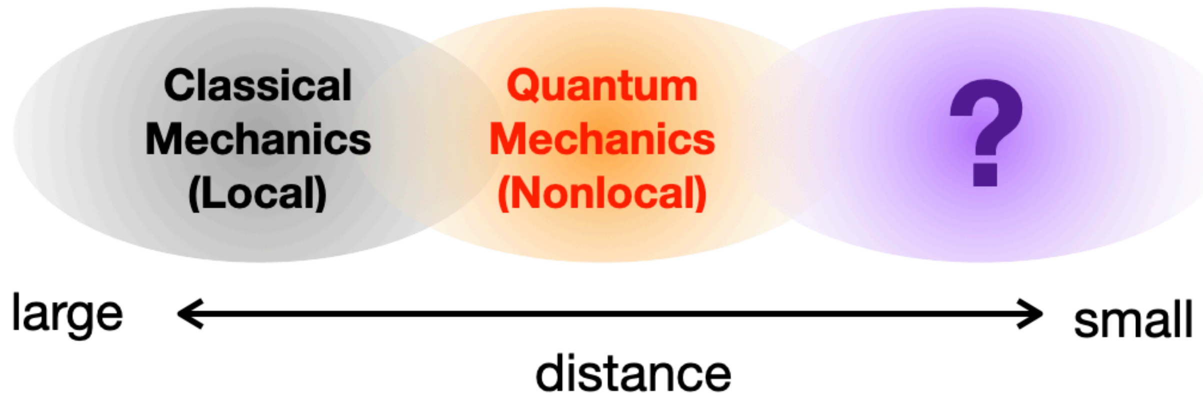
$$S_{\text{CHSH}} \equiv C_{\mathbf{n}_1,\mathbf{e}_1} + C_{\mathbf{n}_1,\mathbf{e}_2} + C_{\mathbf{n}_2,\mathbf{e}_1} - C_{\mathbf{n}_2,\mathbf{e}_2}$$

$$S_{\text{CHSH}} \leq \begin{cases} 2 & \text{Local theories [CHSH(1969)]} \\ 2\sqrt{2} & \text{Quantum Mechanics [Tsirelson(1987)]} \\ 4 & \text{No-signalling [Popescu, Rohrlich(1994)]} \end{cases}$$

No-signalling $\supset$ Quantum $\supset$ Local $\supset$ Separable



High Energy Test of QM



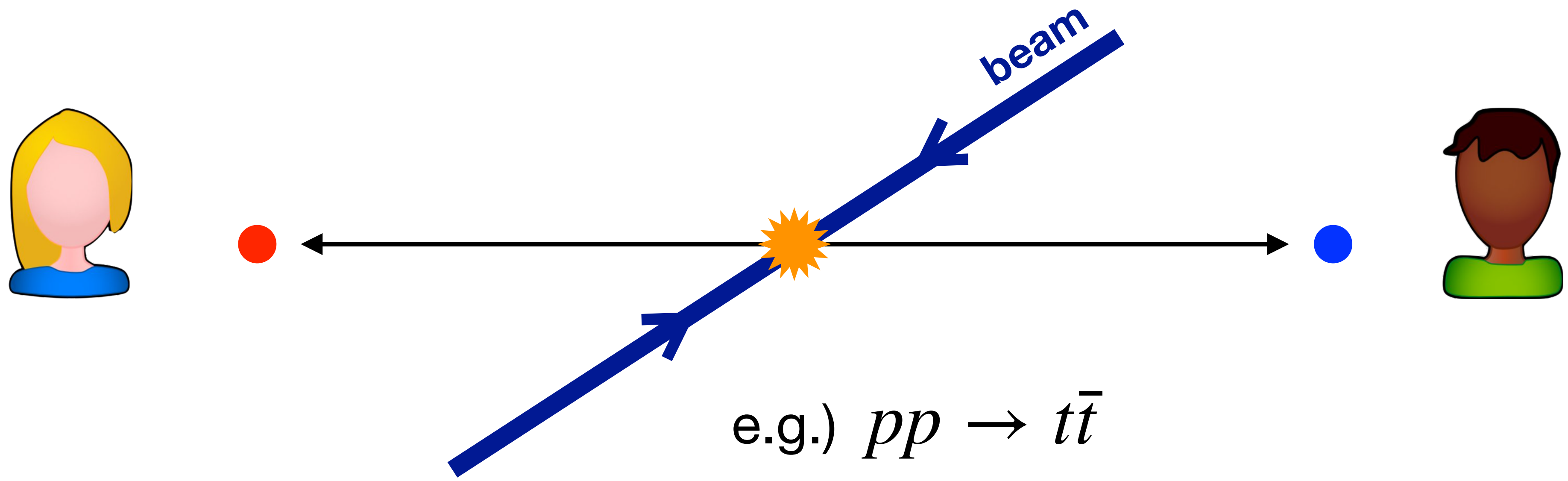
- At very short-distances (high-energies), QM might be modified.
  - sense of locality may change (again)
  - QM might be modified to be married with gravity

- Possible modification of QM?
  - No-signalling theories: $\langle \mathcal{B} \rangle_{\rm NS} \leq 4$ [Cirel'son (1980), Popescu, Rohrlich (1994)]
  - Non-linear extensions of QM: [Weinberg (1989), Polchinski (1991), D.E.Kaplan, S.Rajendran, (2021)]

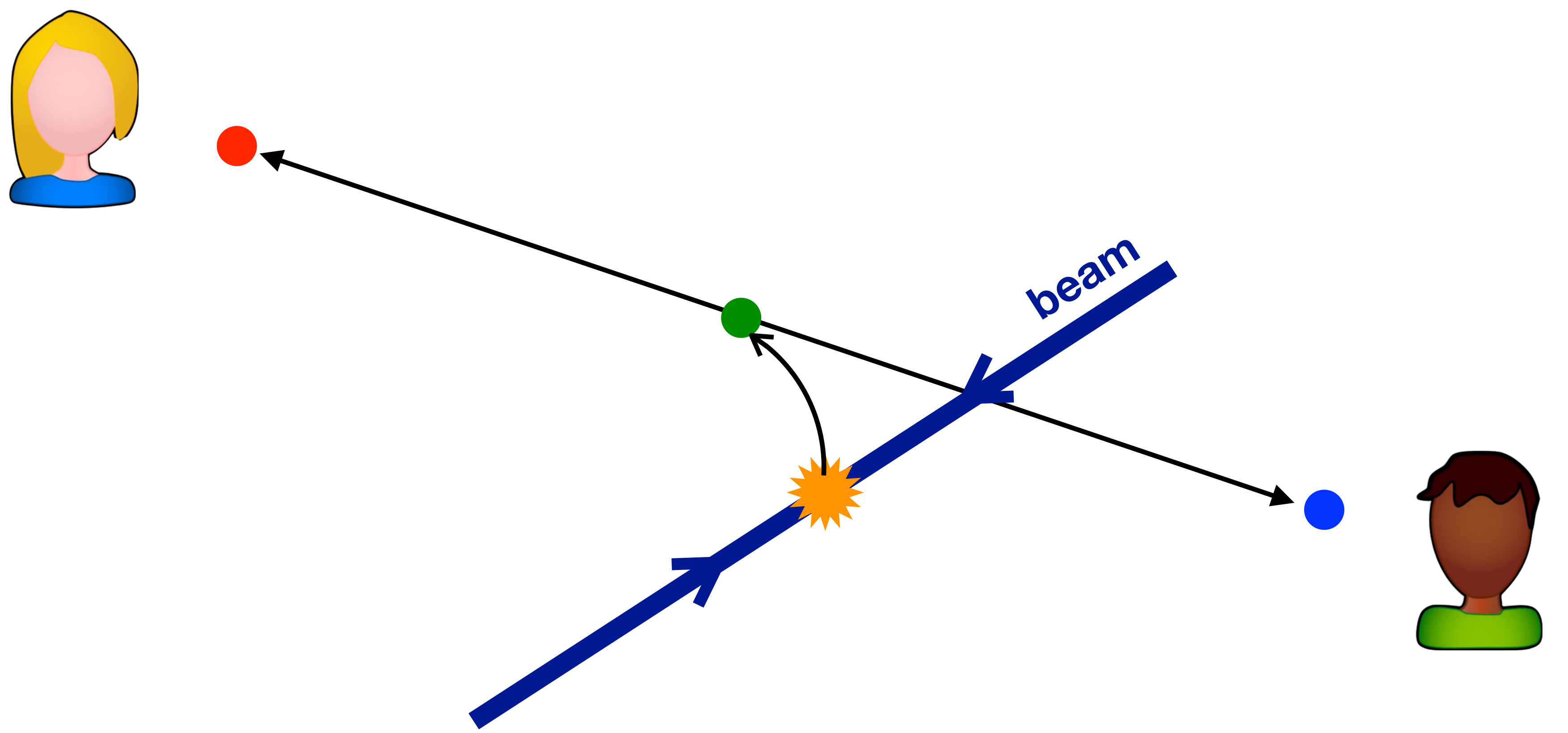
$$i\partial_t|\chi\rangle = \int d^3x \left[\hat{\mathcal{H}}(x) + \langle\chi|\hat{\mathcal{O}}_1(x)|\chi\rangle\hat{\mathcal{O}}_2(x)\right]|\chi\rangle$$

- Quantum Information quantities (entanglement, CHSH quantity) are often sensitive to BSM and non-perturbative effects. They might open a door to "new" physics.

Entangled pairs at Colliders



e.g.) $pp \to t\bar{t}$



e.g.) $H \to \tau^+\tau^-, WW^*, ZZ^*$

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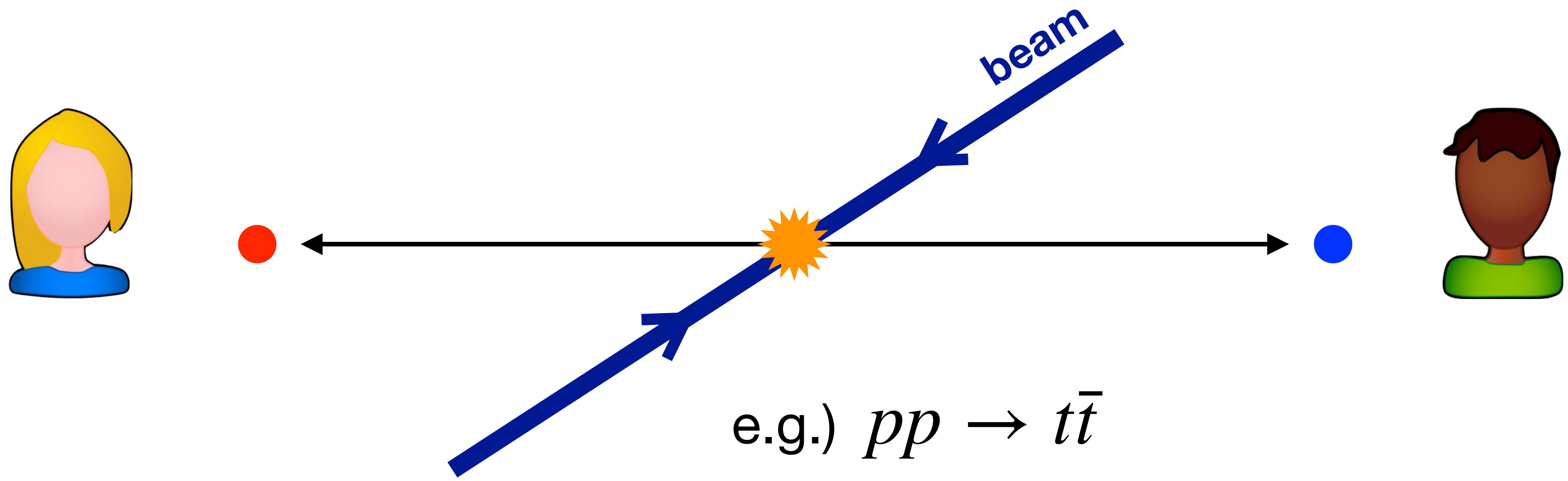

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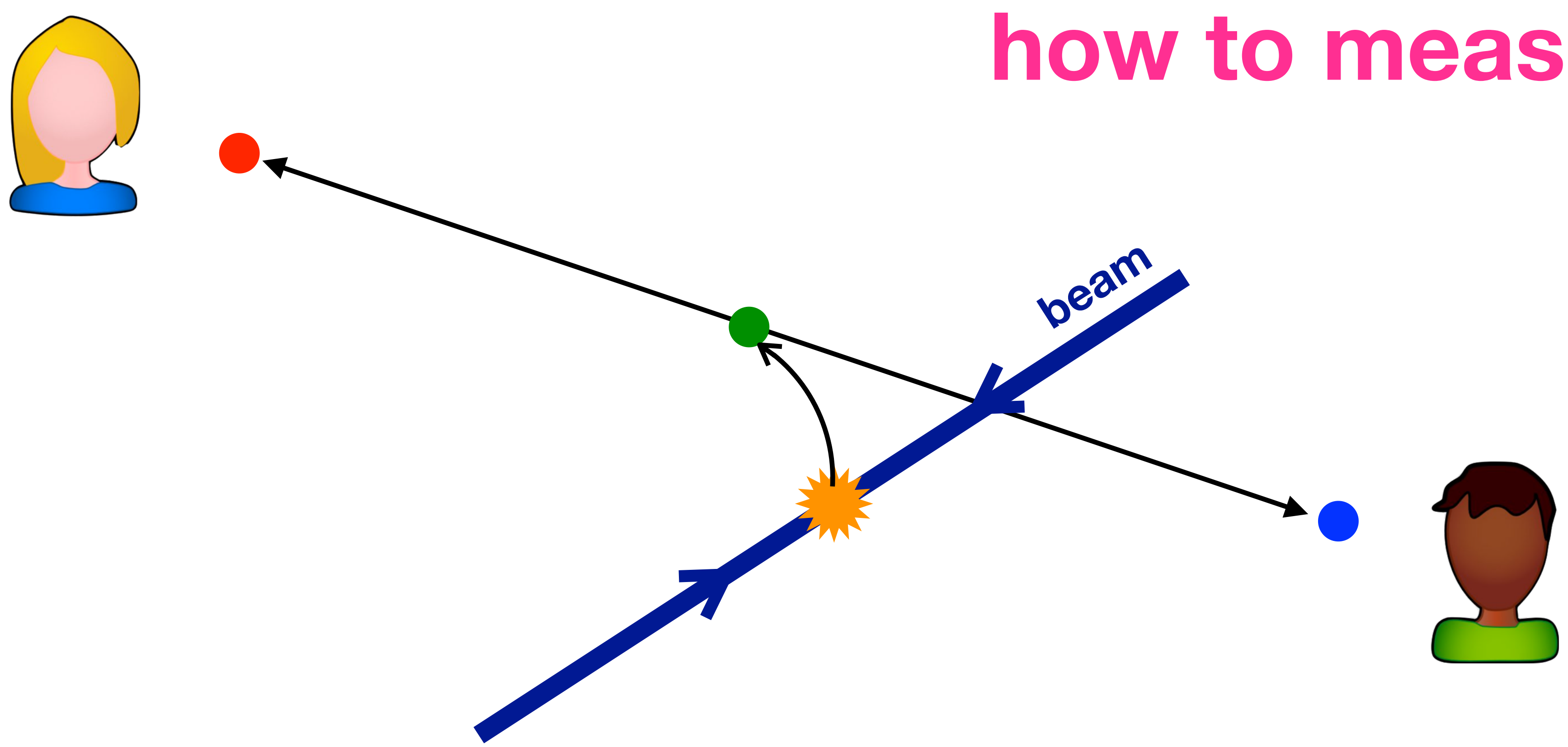
how to measure spin?

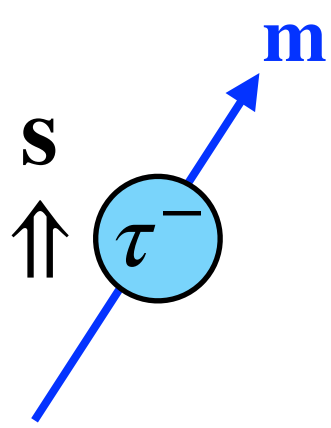


e.g.) $H \to \tau^+\tau^-, WW^*, ZZ^*$

Particles with weak decays are their own polarimeters

e.g.) For $\tau^- \to \pi^- + \nu_\tau$ (τ^- rest frame), the spin of τ^- is measured in the direction of π^- ($\vec{\pi}$) and the outcome is +1.

measurement axis



$$p(+\,|\,\mathbf{m}) = |\langle +_{\mathbf{m}} | +_{\mathbf{s}} \rangle|^2$$

$$= \frac{1 + (\mathbf{m} \cdot \mathbf{s})}{2}$$

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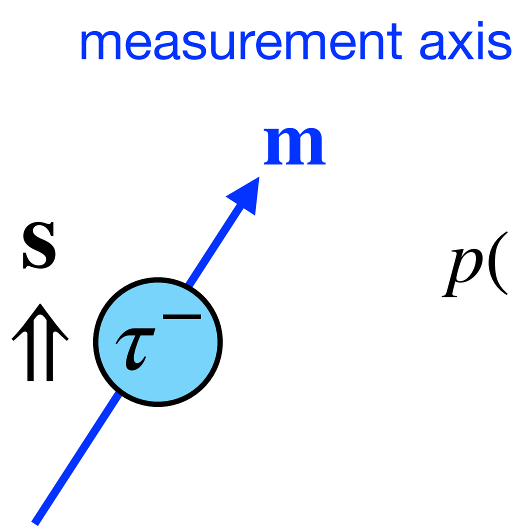

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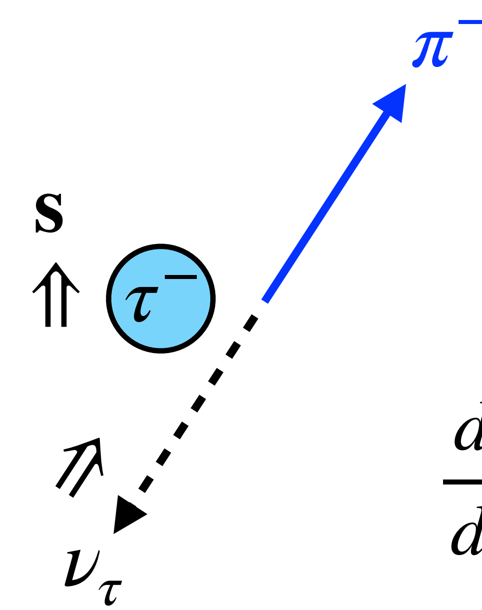
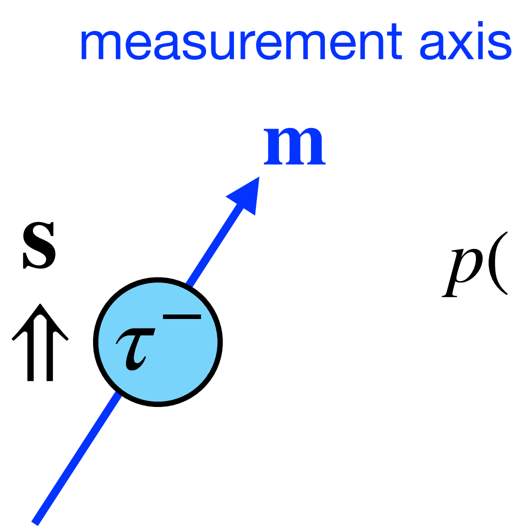


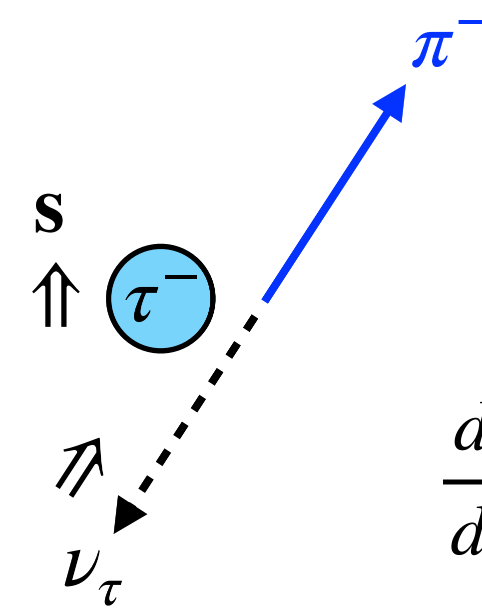
$$\frac{d\Gamma}{d\Omega} = \frac{1 + (\vec{\pi} \cdot \mathbf{s})}{2}$$

Particles with weak decays are their own polarimeters

e.g.) For $\tau^- \to \pi^- + \nu_\tau$ (τ^- rest frame), the spin of τ^- is measured in the direction of π^- ($\vec{\pi}$) and the outcome is +1.



$$p(+\,|\,\mathbf{m}) = |\langle +_{\mathbf{m}}\,|\,+_{\mathbf{s}}\rangle|^2 = \frac{1 + (\mathbf{m}\cdot\mathbf{s})}{2}$$



$$\frac{d\Gamma}{d\Omega} = \frac{1 + (\vec{\pi}\cdot\mathbf{s})}{2}$$

Weakly decaying particles in the SM:

$$\tau,\ t,\ W^{\pm},\ Z^0$$

Particles with weak decays are their own polarimeters

More generally,

$$\frac{d\Gamma}{d\Omega} = \frac{1 + \alpha_x \cdot (\vec{x} \cdot \mathbf{s})}{2}$$

$\alpha_x \in [-1, +1]$: spin analyzing power

- tau decay

$\alpha_x = 1$ for ($x = \pi^-$ in $\tau^- \to \pi^- \nu$)

- top decay

decay product x	α_x
b	$-0.3925(6)$
W^+	$0.3925(6)$
ℓ^+ (from a W^+)	$0.999(1)$
$\bar{d}$, $\bar{s}$ (from a W^+)	$0.9664(7)$
u, c (from a W^+)	$-0.3167(6)$

Spin correlation:

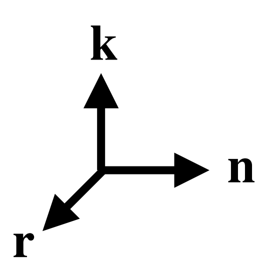
$\mathbf{n}, \mathbf{n}'$: spin measurement axes

$\vec{x}, \vec{y}$: direction of decay products

$$C_{\mathbf{n},\mathbf{n}'} \equiv \langle (\mathbf{s}_A \cdot \mathbf{n})(\mathbf{s}_B \cdot \mathbf{n}') \rangle = \frac{9}{\alpha_x \alpha_y} \langle (\vec{x} \cdot \mathbf{n})(\vec{y} \cdot \mathbf{n}') \rangle$$

$$S_{\mathrm{CHSH}} \equiv C_{\mathbf{n}_1,\mathbf{n}_1'} + C_{\mathbf{n}_1,\mathbf{n}_2'} + C_{\mathbf{n}_2,\mathbf{n}_1'} - C_{\mathbf{n}_2,\mathbf{n}_2'} \begin{cases} > 2 & \to \text{Bell-nonlocal} \\ > 2\sqrt{2} & \to \text{QM violation} \end{cases}$$

$$D \equiv \mathrm{Tr}[C]/3 < -\frac{1}{3}$$ → sufficient cond. for entanglement



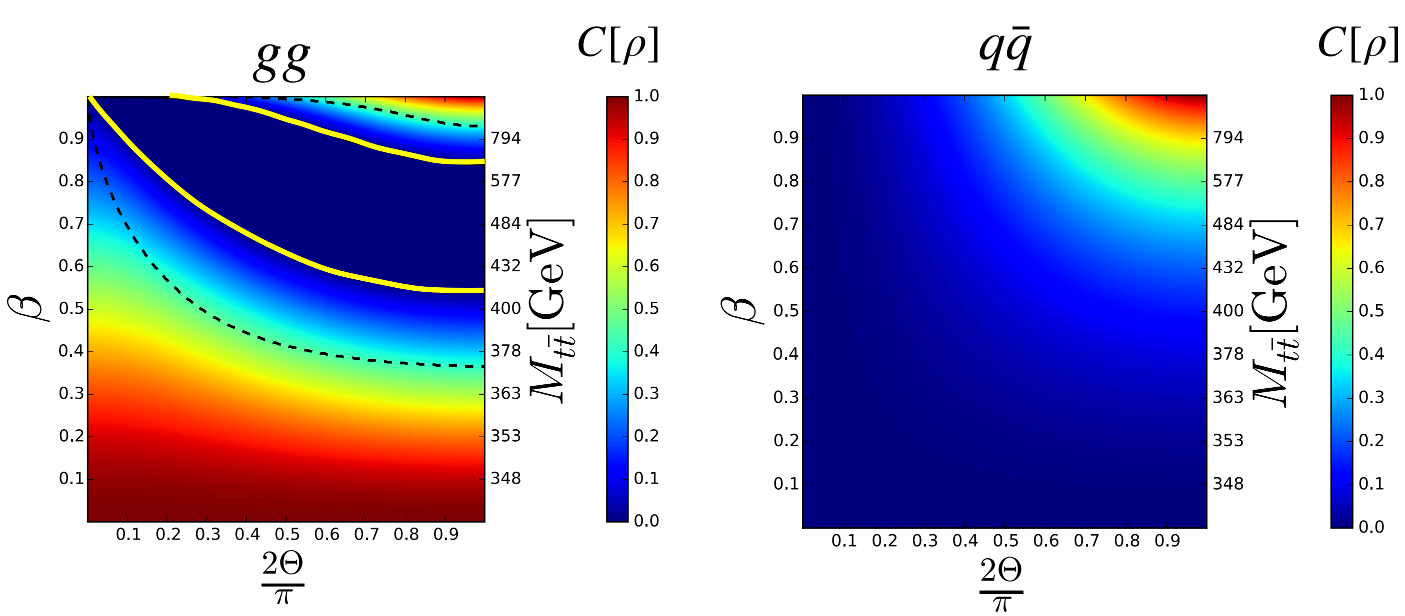
Recent activities to look into entanglements, etc. in HEP

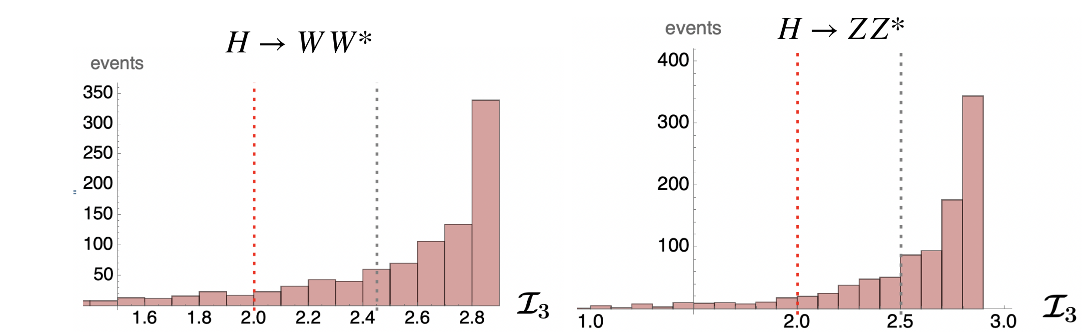
❖ Prospects to measure entanglement and Bell-nonlocality @ LHC, ILC, FCC

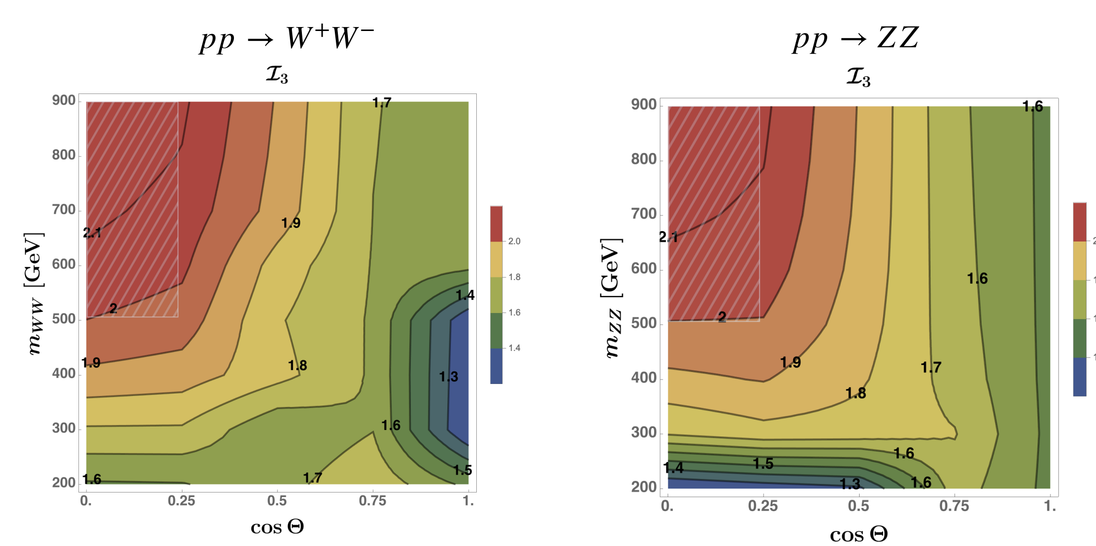
- $pp \to t\bar{t}$ — Y. Afik and J. R. M. de Nova ’21, ’22, M. Fabbrichesi, R. Floreanini, G. Panizzo ‘21, Z. Dong, D. Gonçalves, K. Kong, A. Navarro ‘23

- $H \to WW, ZZ$, $pp \to WW, ZZ$ — A. J. Barr ’21, J.A. Aguilar-Saavedra, A. Bernal, J.A. Casas, J.M. Moreno ’22, A. Bernal, P. Caban, J. Rembieliński ’23, M. Fabbrichesi, R. Floreanini, E. Gabrielli, Luca Marzola ‘23

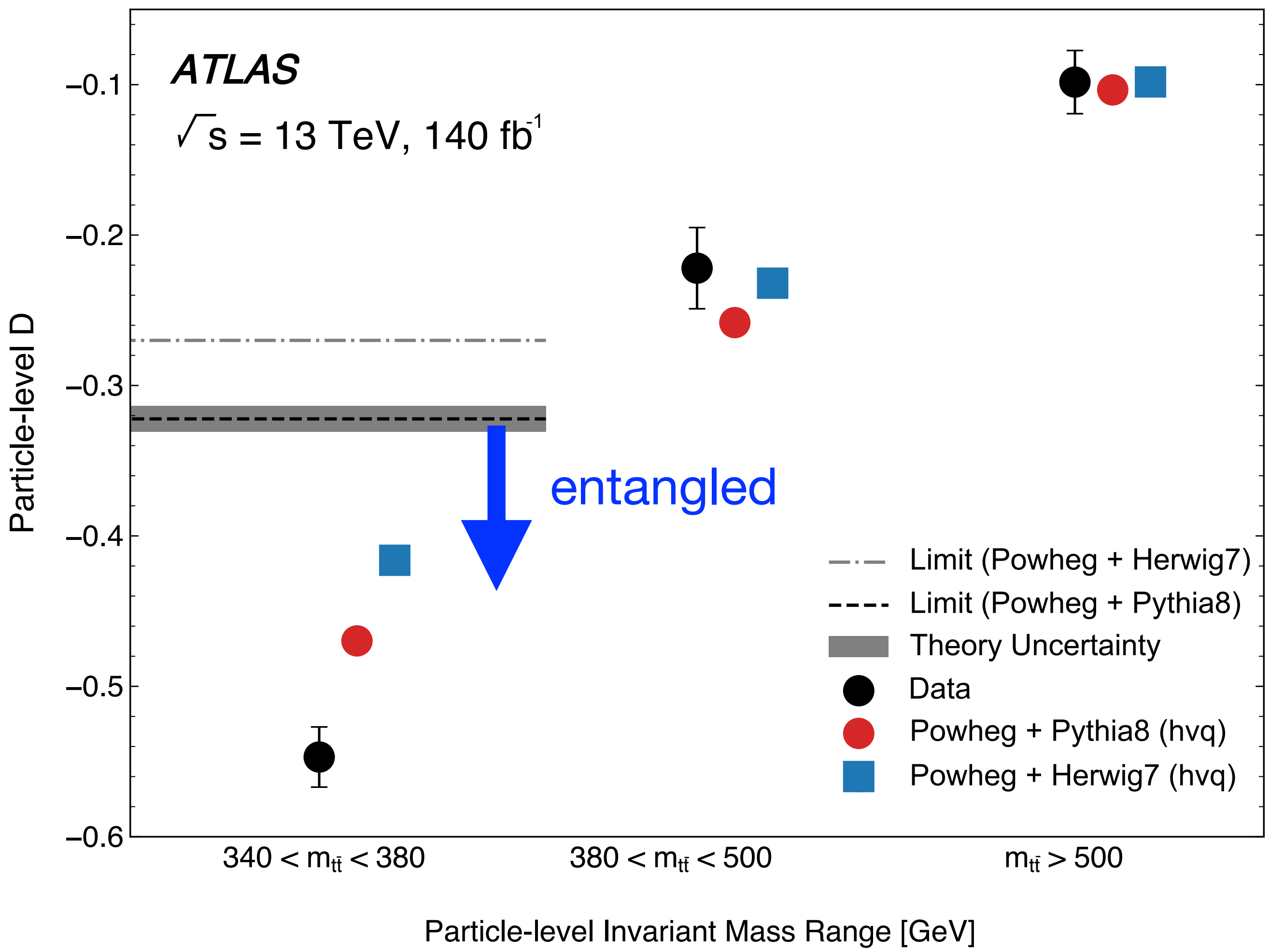
- $H \to \tau^+\tau^-$ (@ e^+e^- colliders) — M. Fabbrichesi, R. Floreanini, E. Gabrielli 22, M. Altakach, P. Lamba, F. Maltoni, K. Mawatari, KS ’22, K. Ma, T. Li ‘23

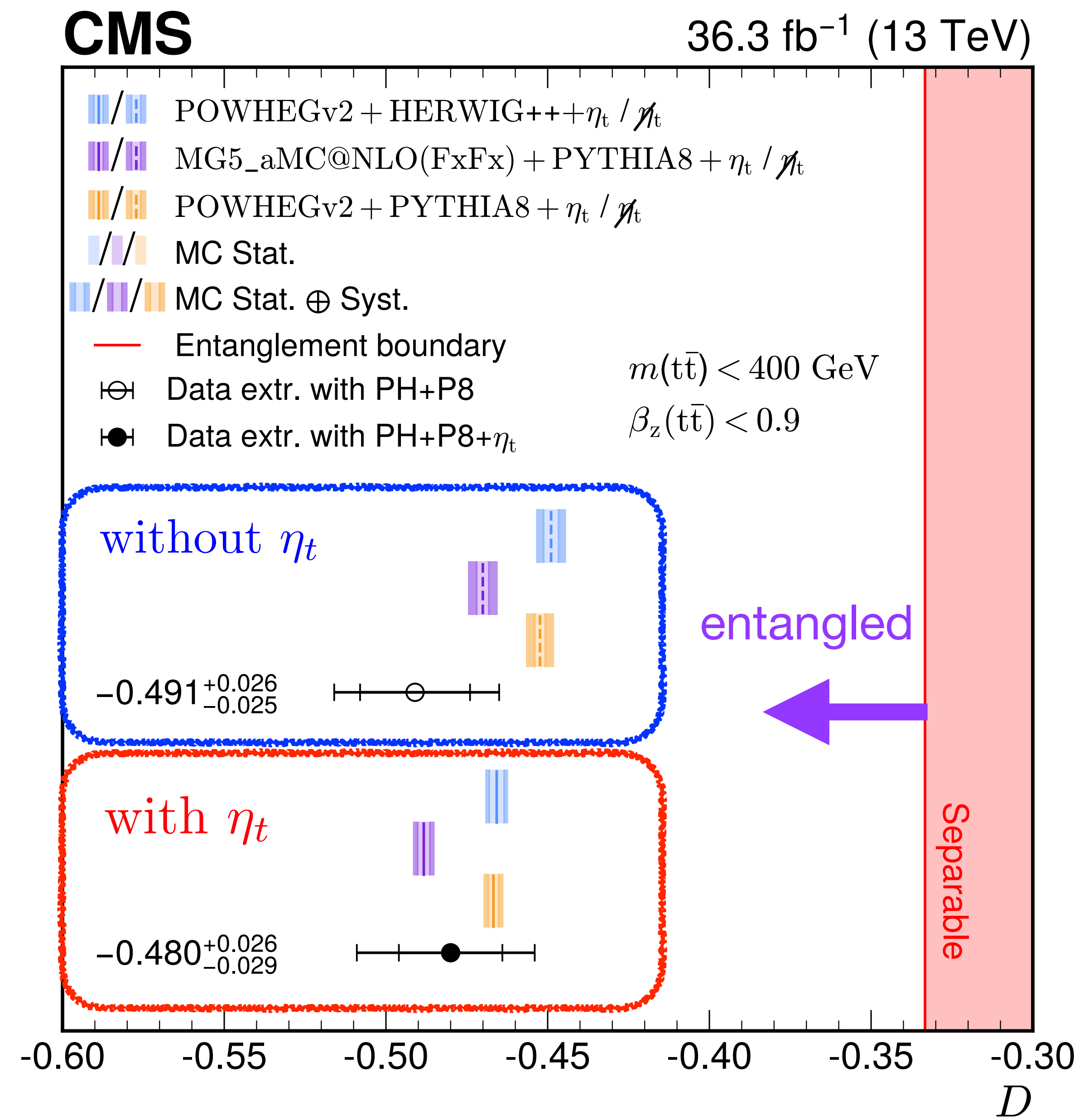






Entanglement Observation at LHC $pp \to t\bar{t}$

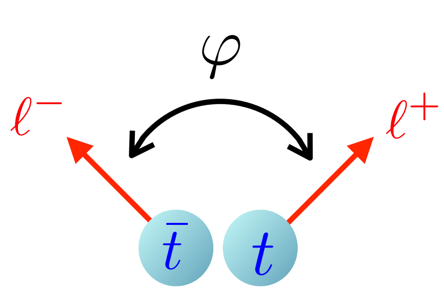




$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\varphi} = \frac{1}{2}(1 - D\cos\varphi)$$

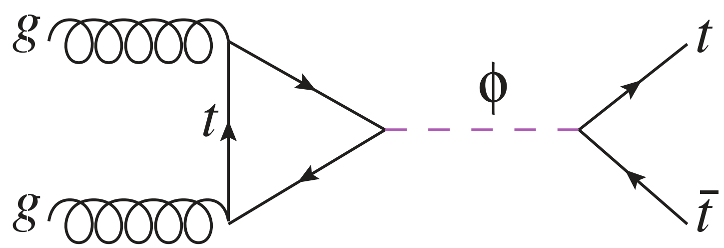
$$\rightarrow \quad D = -3\cdot\langle\cos\varphi\rangle = \frac{\mathrm{tr}[\mathbf{C}]}{3} < -\frac{1}{3}$$

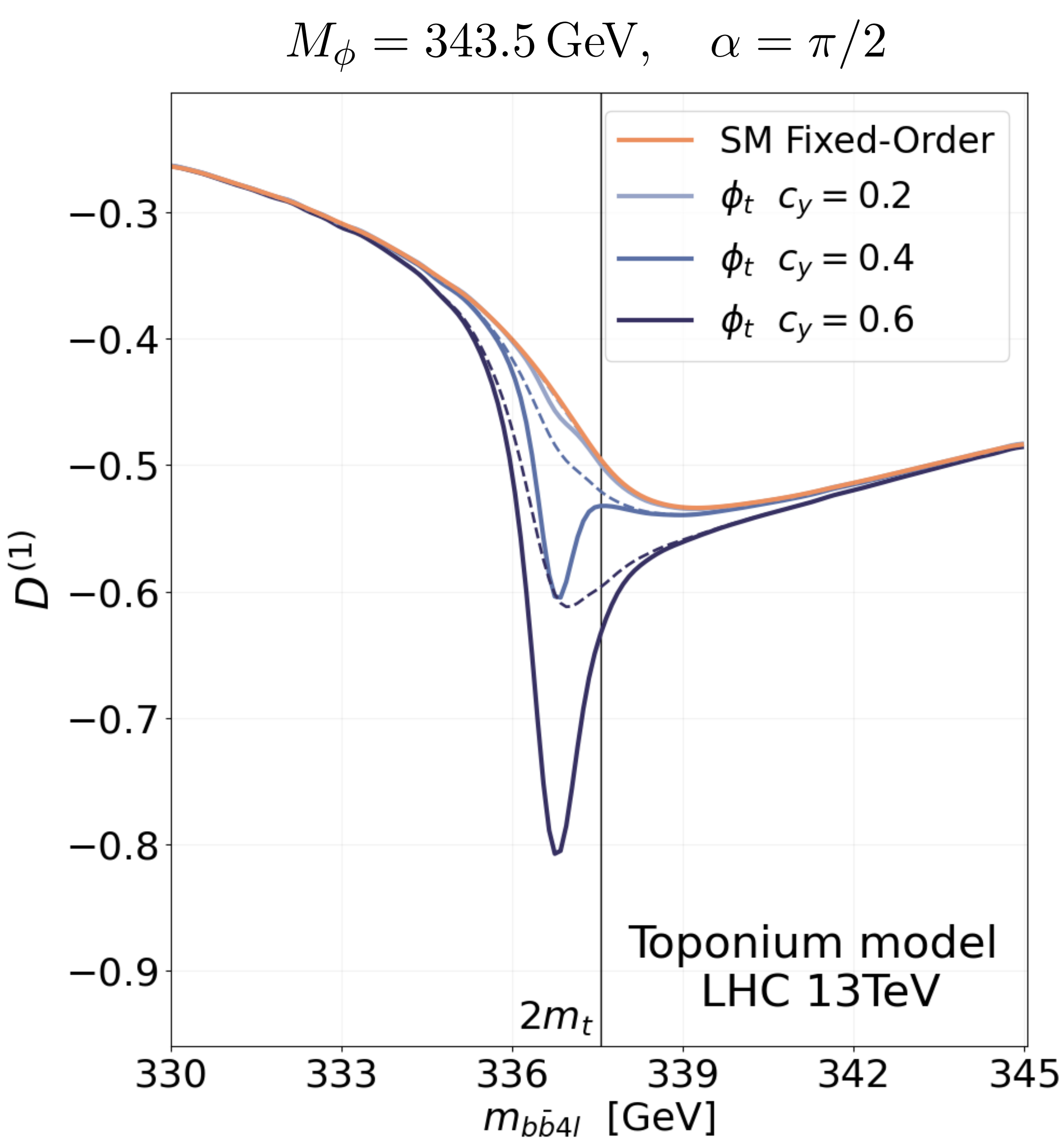
(sufficient condition for entanglement)

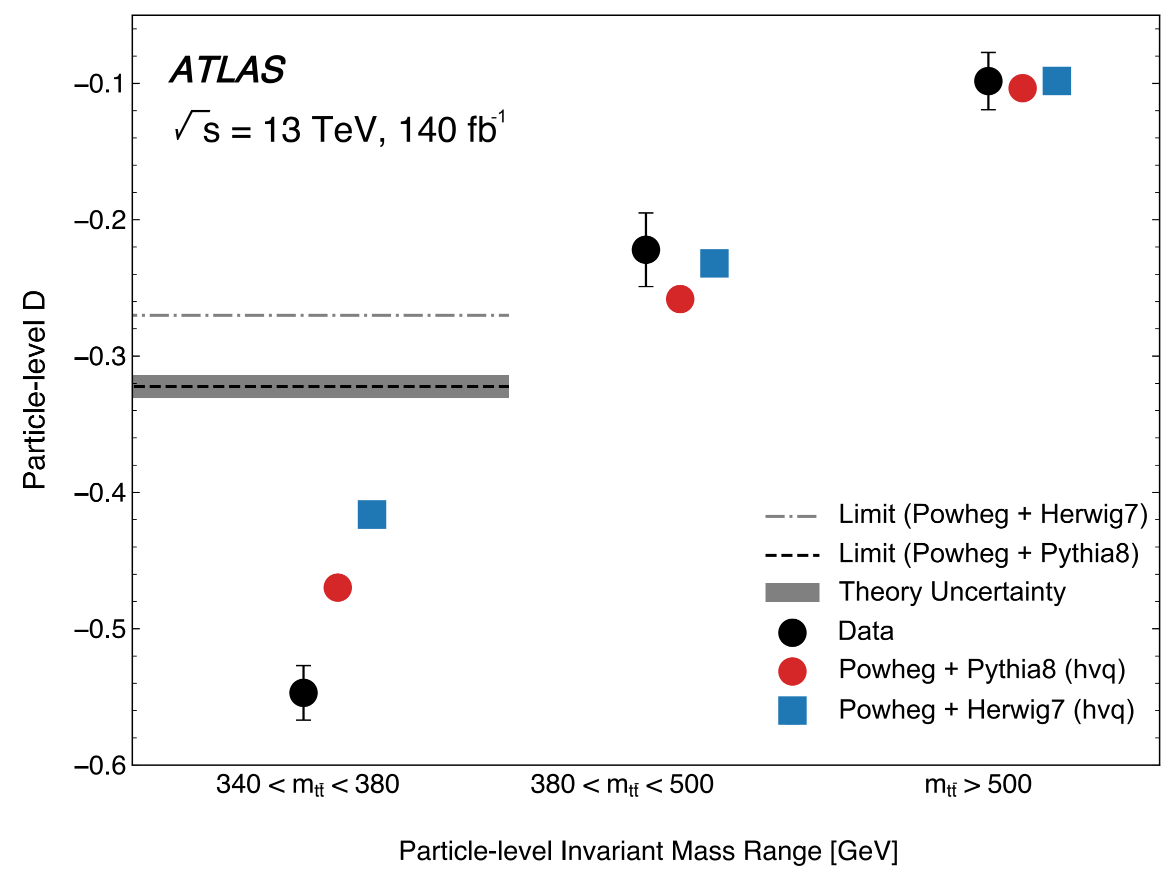


Maltoni, Severi, Tentori, Vryonidou [2401.08751]

$$\mathcal{L} = \mathcal{L}_{\rm SM} - \frac{1}{2}\phi(\partial^2 + M_\phi^2)\phi + c_y\,\frac{y_t}{\sqrt{2}}\,\phi\,\bar{t}\,(\cos\alpha + i\gamma^5\sin\alpha)\,t.$$



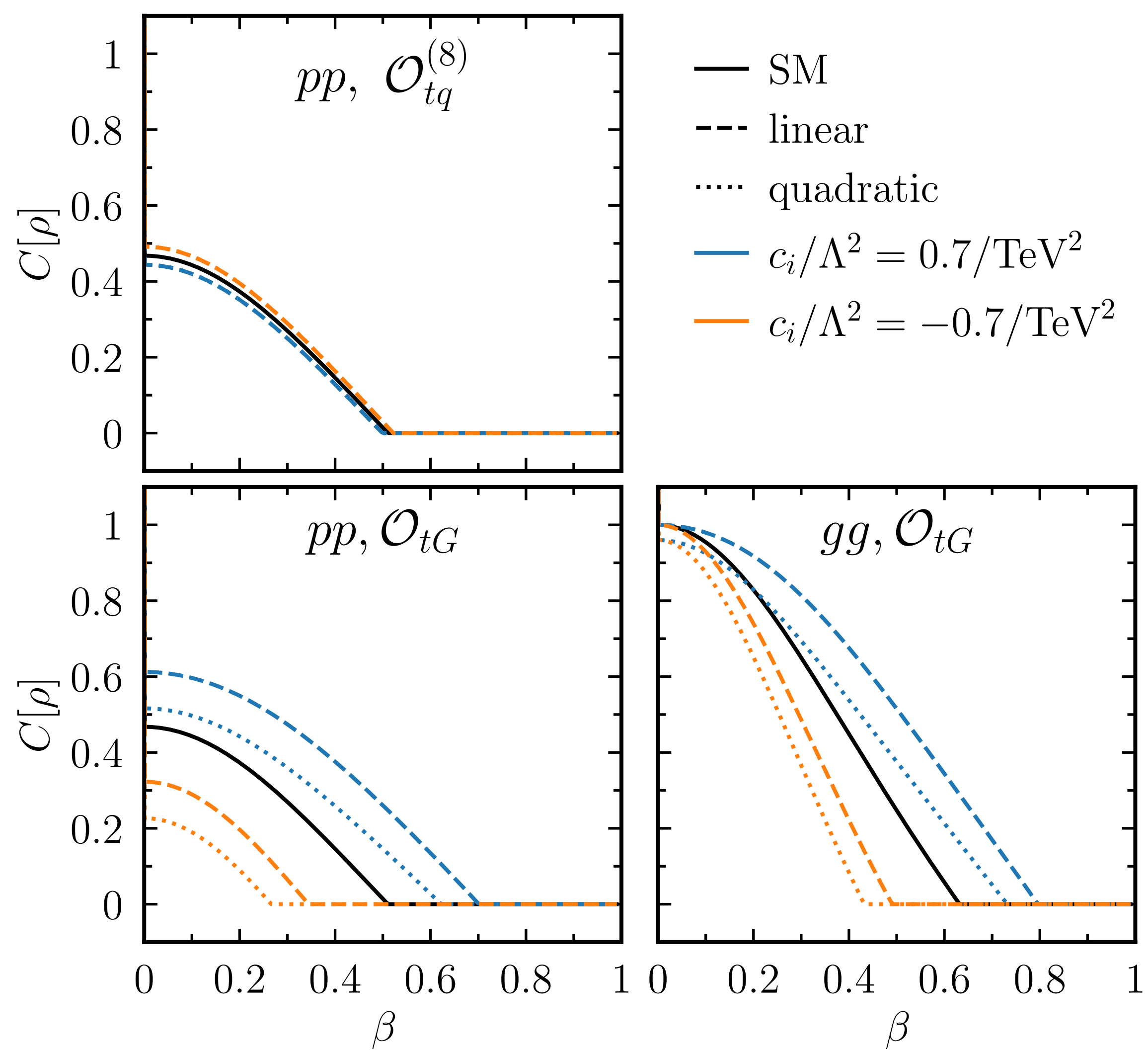




Effect of BSM $pp \to t\bar{t}$

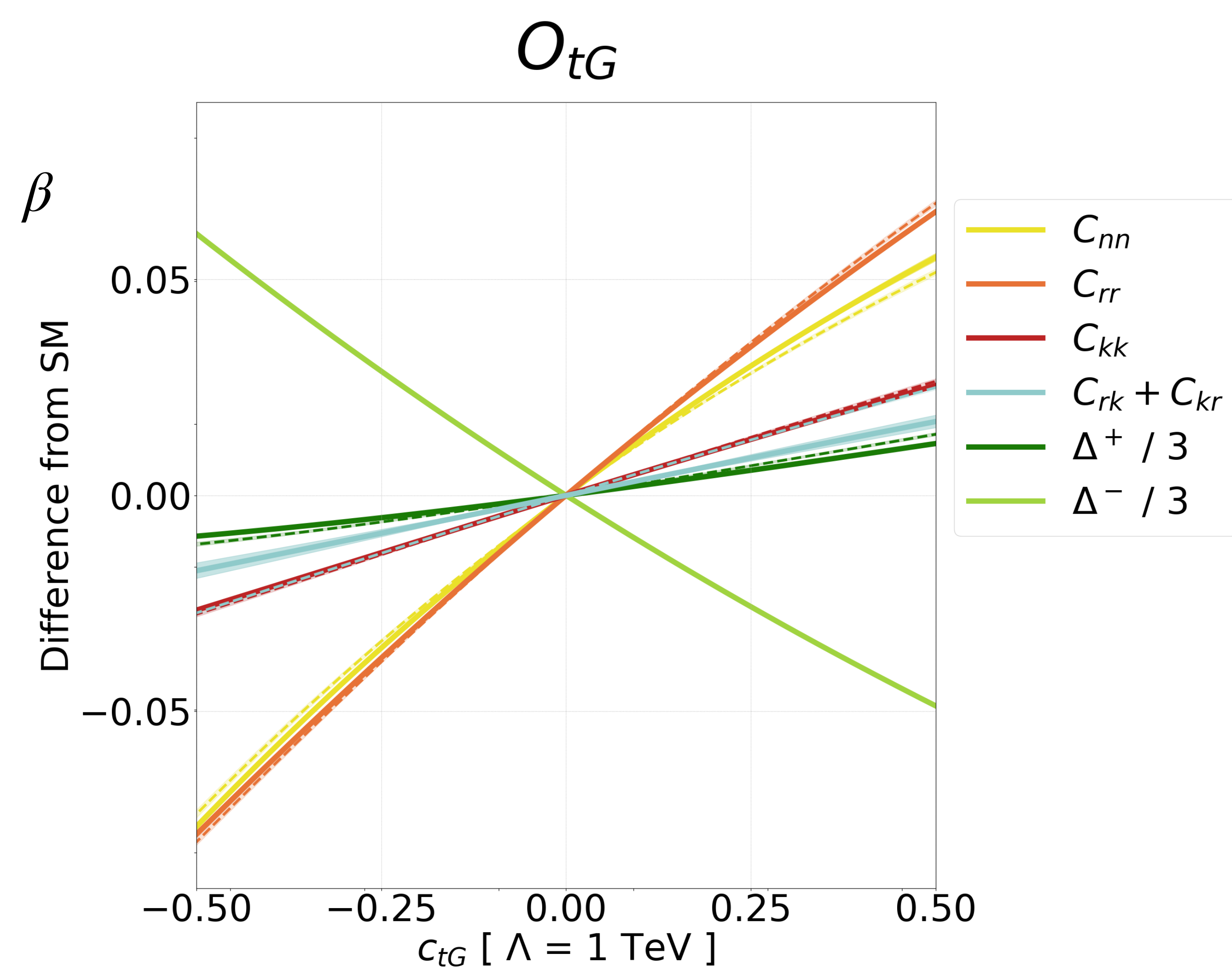
$$\mathcal{O}_{tG} = g_S\,\overline{Q} T_A \tilde{\varphi} \sigma^{\mu\nu} t\, G^A_{\mu\nu} \qquad \mathcal{O}^8_{tq} = \sum_{\mathrm{f}=1}^{2} (\bar{q}_\mathrm{f} \gamma_\mu T_A q_\mathrm{f})(\bar{t}\gamma^\mu T^A t)$$

[Aoude Madge Maltoni Mantani (2022)]



β: top velocity in the $t\bar{t}$ rest frame

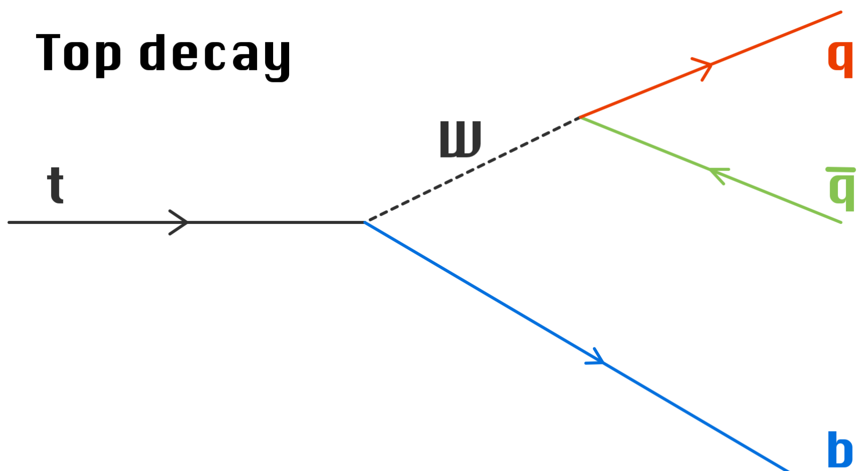
[Severi Vryonidou (2023)]



$$\Delta^\pm = \pm(C_{kk} + C_{rr}) - C_{nn} - 1$$

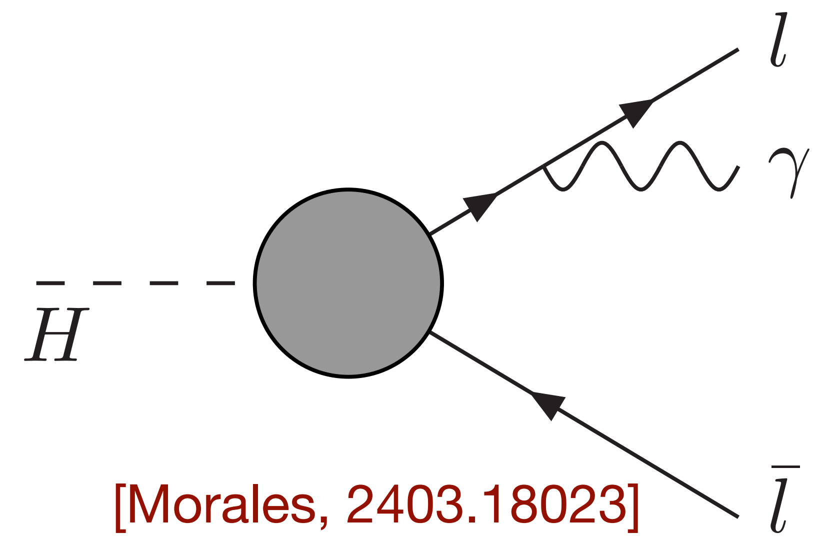
3-particle entanglement

$2 \to 222$



[KS, Spannowsky, 2310.01477]

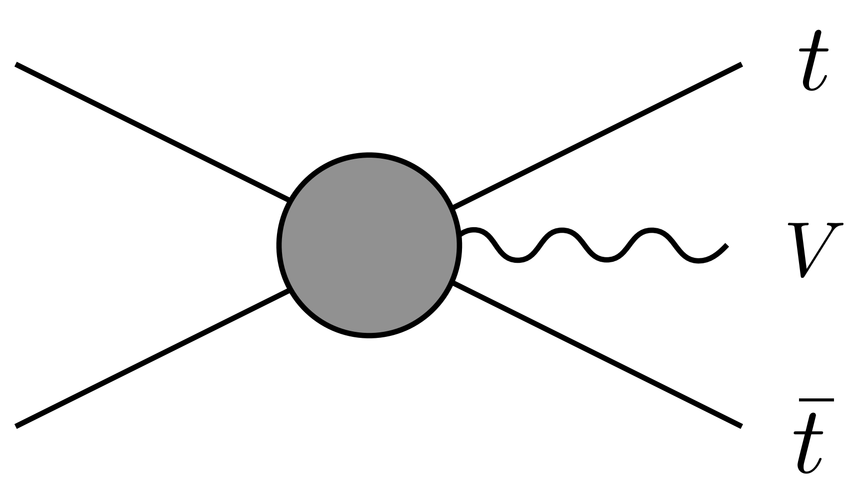
$0 \to 222$



[Morales, 2403.18023]

3-particle entanglement

$22 \to 223$



[Subba, Rahaman, 2404.03292]

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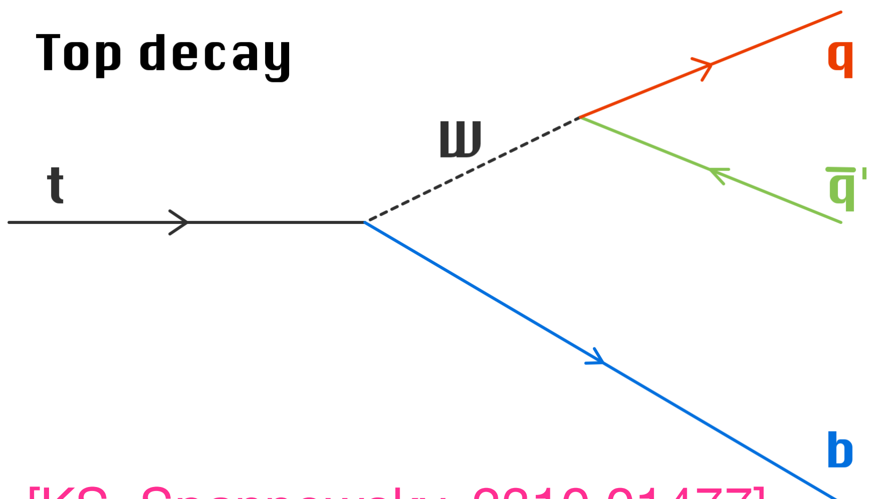

[KS, Spannowsky, 2310.01477]

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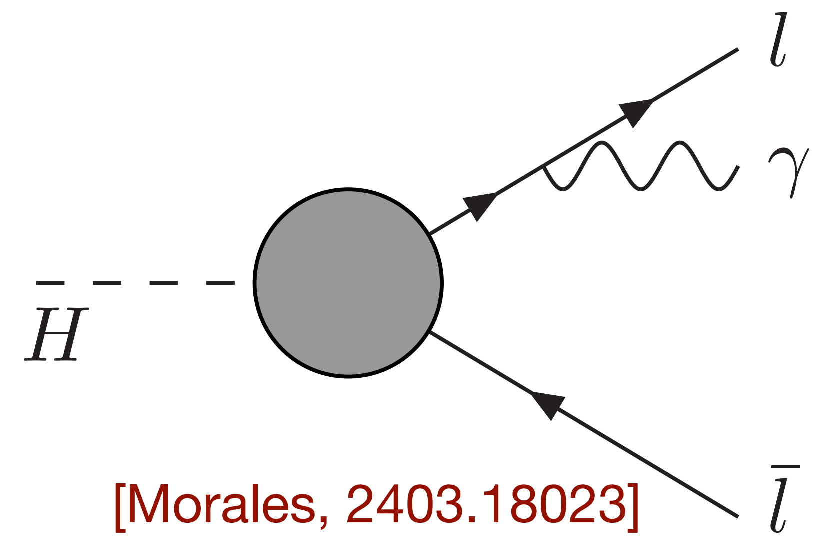

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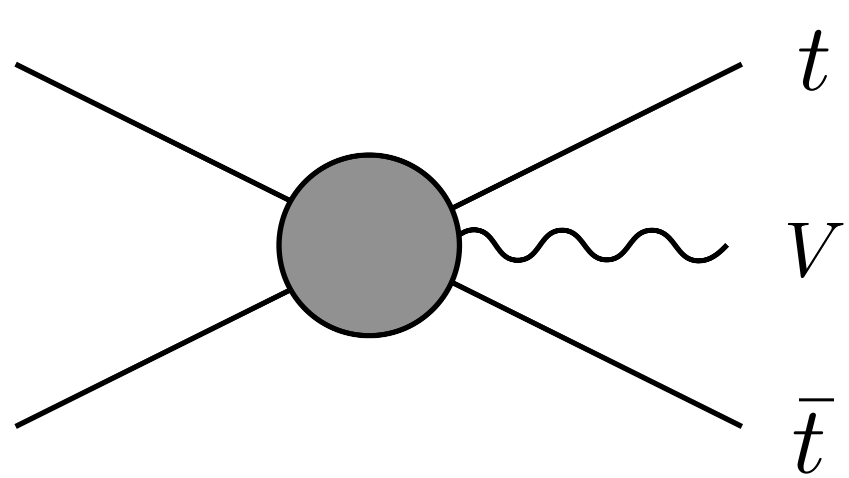


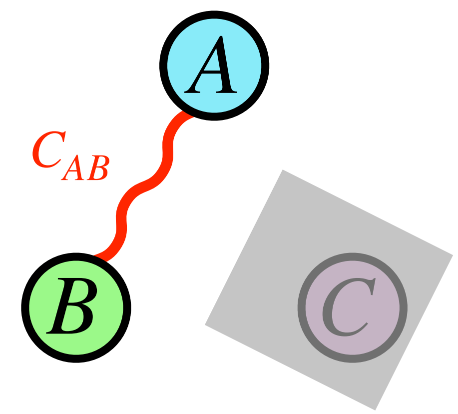
[Subba, Rahaman, 2404.03292]

Three qubit system: 222

- $2^3 = 8$ basis kets:

$$|\Psi_{ABC}\rangle = c_{000}|000\rangle + c_{001}|001\rangle + c_{010}|010\rangle + \cdots$$

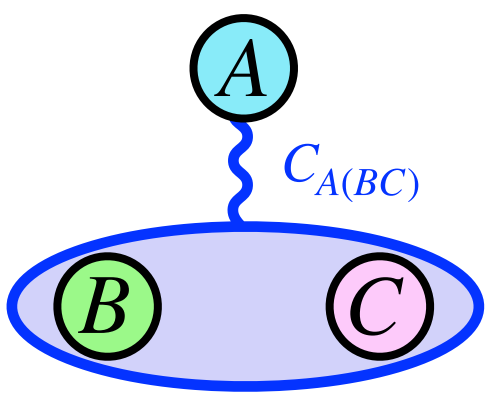
- Entanglement among 2-individual particles:



$$\mathcal{C}_{AB}[|\Psi_{ABC}\rangle] = \mathcal{C}[\rho_{AB}] \qquad \rho_{AB} = \mathrm{Tr}_C|\Psi\rangle\langle\Psi|_{ABC}$$

- Entanglement among one-to-other:

$$|\Psi_{ABC}\rangle = |0\rangle_A \otimes (c_{000}|00\rangle_{BC} + c_{001}|01\rangle_{BC} + \cdots)$$
$$+|1\rangle_A \otimes (c_{100}|00\rangle_{BC} + c_{101}|01\rangle_{BC} + \cdots)$$



$$\mathcal{C}_{A(BC)}[|\Psi_{ABC}\rangle] = \sqrt{2[1 - \mathrm{Tr}\rho_{BC}^2]} \qquad \rho_{BC} = \mathrm{Tr}_A|\Psi\rangle\langle\Psi|_{ABC}$$

Classification

Fully-separable: $|\phi^{\text{fs}}\rangle_{A|B|C} = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C,$

$\mathcal{C}_{ij} = \mathcal{C}_{i(jk)} = 0$

Bi-separable: $|\phi^{\text{bs}}\rangle_{A|BC} = |\alpha\rangle_A \otimes |\delta\rangle_{BC}$

$|\phi^{\text{bs}}\rangle_{B|AC} = |\beta\rangle_B \otimes |\delta\rangle_{AC}$

$|\phi^{\text{bs}}\rangle_{C|AB} = |\gamma\rangle_C \otimes |\delta\rangle_{AB}$

$\mathcal{C}_{A(BC)} = 0$

$\mathcal{C}_{BC}, \mathcal{C}_{B(AC)}, \mathcal{C}_{C(AB)} \neq 0$

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complement

Genuinely **M**ultipartite **E**ntangled (**GME**):

$$|GHZ_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle$$

maximally GME

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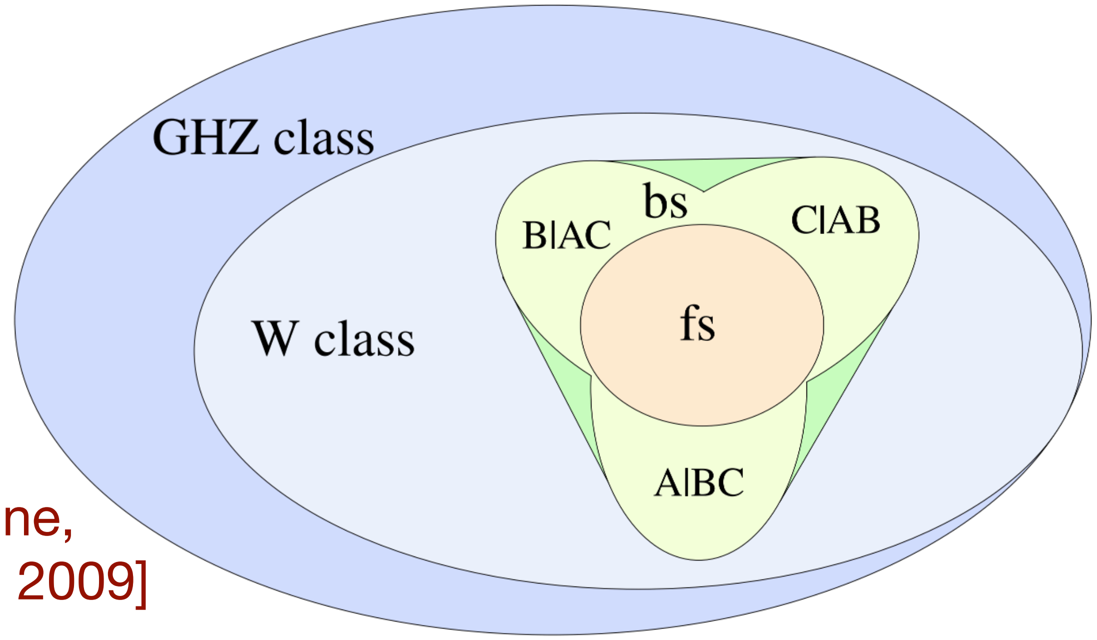
maximally GME

- All GME states can be classified either ***GHZ*** or ***W*** classes! [Dur, Vidal, Cirac 2000]

$$|\phi^{\rm GME}\rangle \longrightarrow \hat{A}\otimes\hat{B}\otimes\hat{C}\,|\phi^{\rm GME}\rangle = \begin{cases} |GHZ_3\rangle \\ |W_3\rangle \end{cases}$$

$\hat{A}\in I(\mathcal{H}_A), \hat{B}\in I(\mathcal{H}_B), \hat{C}\in I(\mathcal{H}_C)$

$I(\mathcal{H})$: set of intertible operators in $\mathcal{H}$



[Guhne, Toch 2009]

Monogamy

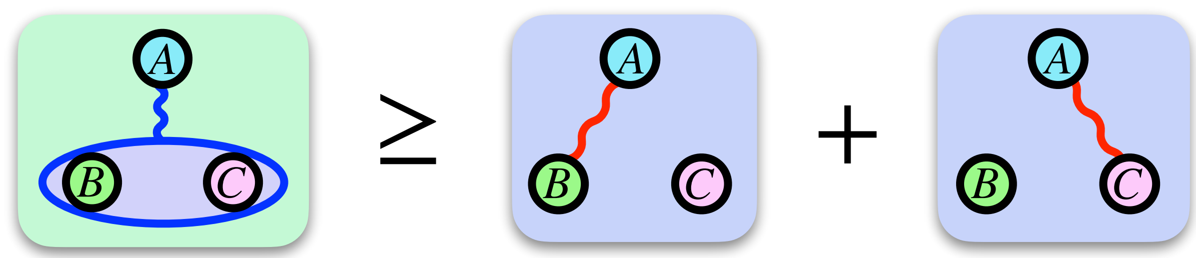
- All 3-qubit pure states can be transformed by a local unitary to

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \qquad \lambda_i \geq 0, \sum_i \lambda_i^2 = 1 \text{ and } \theta \in [0;\pi]$$

- **1-2** and **1-1** entanglements are related to by the **monogamy** relations:

[Coffman, Kundu, Wootters '99]

$$\mathcal{C}^2_{A(BC)} = \mathcal{C}^2_{AB} + \mathcal{C}^2_{AC} + \tau \qquad \tau = 4\lambda_0^2\lambda_4^2 \geq 0 \longleftarrow$$ **3-tangle**



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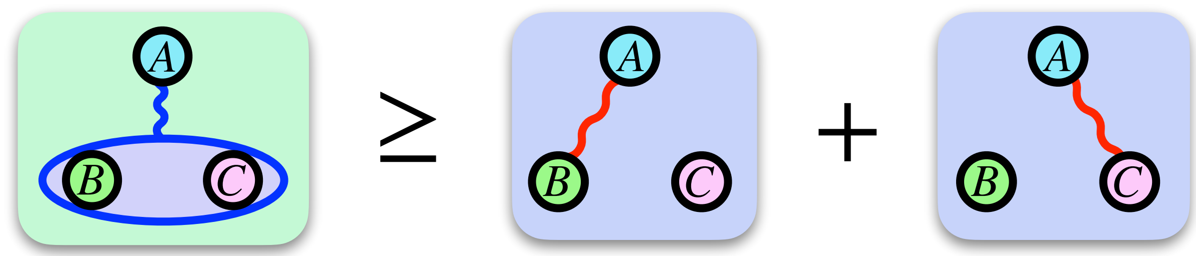
$$\mathcal{C}^2_{B(AC)} = \mathcal{C}^2_{AB} + \mathcal{C}^2_{BC} + \tau$$

$$\mathcal{C}^2_{C(AB)} = \mathcal{C}^2_{AC} + \mathcal{C}^2_{BC} + \tau$$



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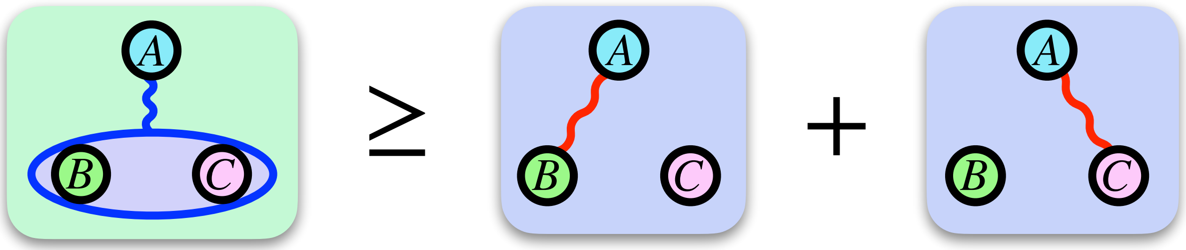

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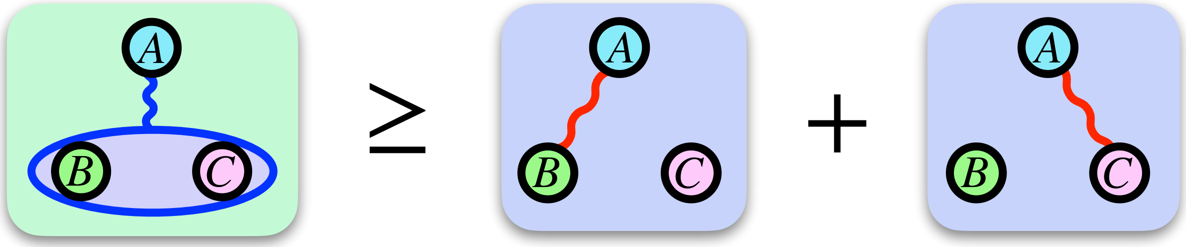


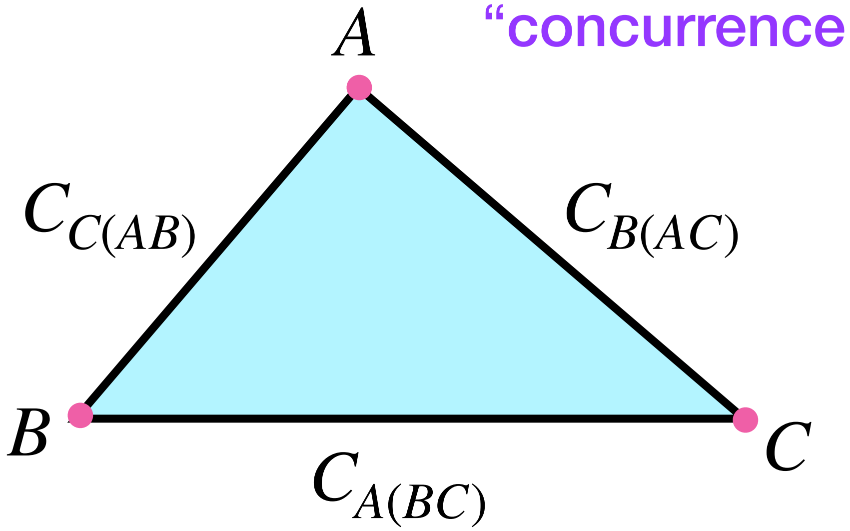
- 1-2 concurrence inequalities

$$\mathcal{C}^2_{A(BC)} + \mathcal{C}^2_{B(AC)} \geq \mathcal{C}^2_{C(AB)}$$

$$\downarrow$$

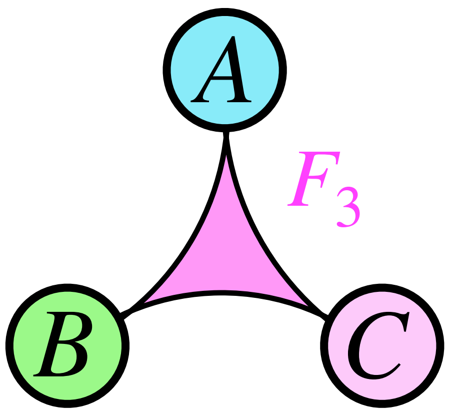
$$\mathcal{C}_{A(BC)} + \mathcal{C}_{B(AC)} \geq \mathcal{C}_{C(AB)} \qquad \Rightarrow$$

"concurrence triangle"



GME measure

Genuine Multi-particle Entanglement (GME) measure: [Dur, Vidal, Cirac ’00, Ma, Chen, Chen, Spengler, Gabriel, Huber ’11, Xie, Eberly ’21]



The measure should satisfy:

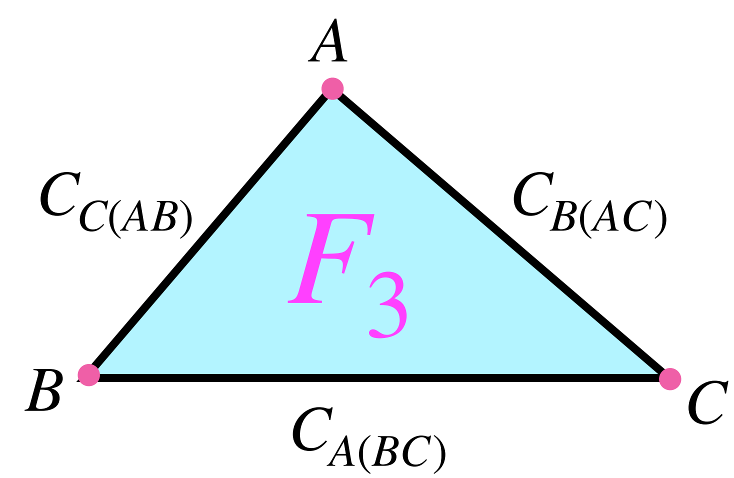
(1) vanishes for all fully- and bi-separable states

(2) positive for all GME states

(3) non-increasing under LOCC

- The **area** of the “**concurrence triangle**” satisfies (1), (2), (3) **!**

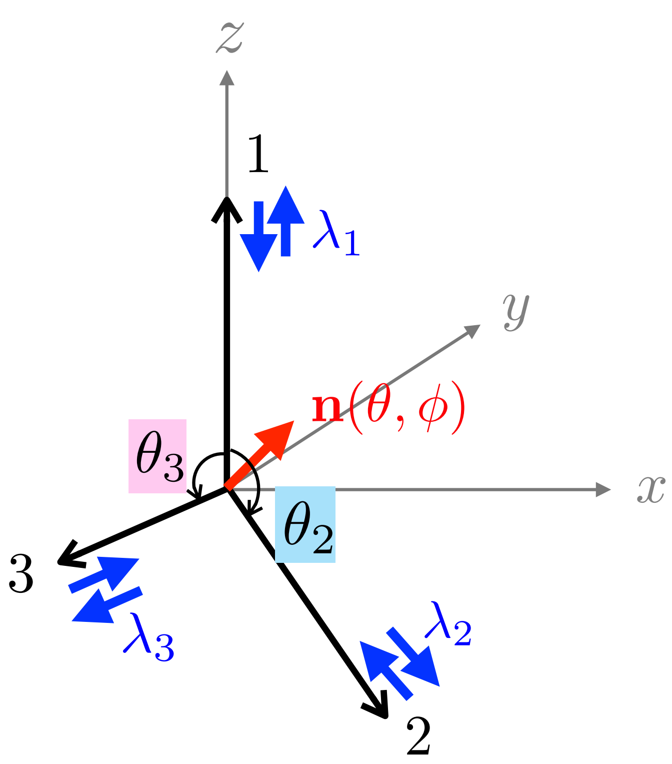
[Jin, Tao, Gui, Fei, Li-Jost, Qiao (2023)]



$$F_3 \equiv \left[\tfrac{16}{3} Q \left(Q - \mathcal{C}_{A(BC)}\right)\left(Q - \mathcal{C}_{B(AC)}\right)\left(Q - \mathcal{C}_{C(AB)}\right)\right]^{\frac{1}{2}} \in [0, 1]$$

$$Q \equiv \frac{1}{2}\left[\mathcal{C}_{A(BC)} + \mathcal{C}_{B(AC)} + \mathcal{C}_{C(AB)}\right]$$

3-body decay: $\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$



Assumptions:

[KS, M.Spannowsky 2310.01477]

- all particles have spin 1/2
- all final particles 1,2,3 are massless

Kinematics:

- rest frame of the initial particle 0
- p_1 is in the z-axis
- decay is in the x-z plane

$$p_1^\mu = p_1(1, 0, 0, 1)$$
$$p_2^\mu = p_2(1, \sin\theta_2, 0, \cos\theta_2)$$
$$p_3^\mu = p_3(1, -\sin\theta_3, 0, \cos\theta_3)$$

$\mathbf{n}(\theta, \phi)$: polarisation of initial spin

$\lambda_1, \lambda_2, \lambda_3 \in (+, -)$: helicities of 1,2,3

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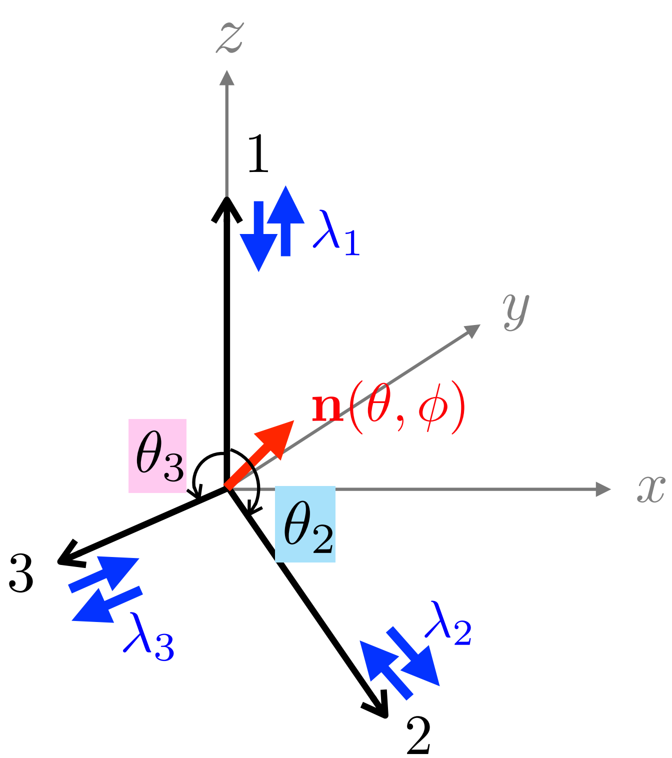

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$\mathbf{n}(\theta, \phi)$: polarisation of initial spin

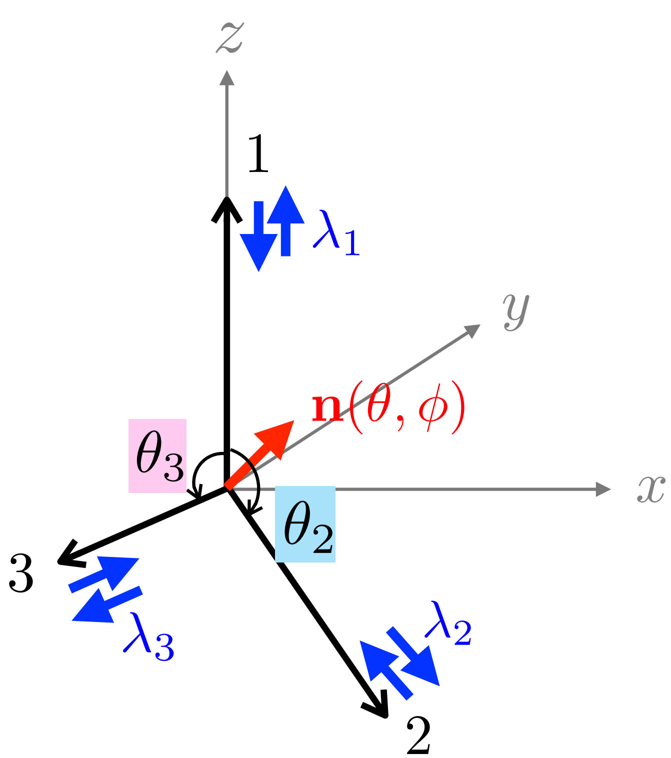
$\lambda_1, \lambda_2, \lambda_3 \in (+, -)$: helicities of 1,2,3

initial state

$$|\mathbf{n}(\theta, \phi)\rangle$$

3-body decay: $\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$

[KS, M.Spannowsky 2310.01477]



Assumptions:

- all particles have spin 1/2
- all final particles 1,2,3 are massless

Kinematics:

- rest frame of the initial particle 0
- p_1 is in the z-axis
- decay is in the x-z plane

$$p_1^\mu = p_1(1, 0, 0, 1)$$
$$p_2^\mu = p_2(1, \sin\theta_2, 0, \cos\theta_2)$$
$$p_3^\mu = p_3(1, -\sin\theta_3, 0, \cos\theta_3)$$

$\mathbf{n}(\theta, \phi)$: polarisation of initial spin

$\lambda_1, \lambda_2, \lambda_3 \in (+, -)$: helicities of 1,2,3

amplitude

$$\hat{\mathbf{1}} = \sum_{\lambda_1, \lambda_2, \lambda_3} |\lambda_1, \lambda_2, \lambda_3\rangle\langle\lambda_1, \lambda_2, \lambda_3| \qquad \mathcal{M}^{\mathbf{n}}_{\lambda_1, \lambda_2, \lambda_3} = \langle\lambda_1, \lambda_2, \lambda_3|\mathbf{n}(\theta, \phi)\rangle$$

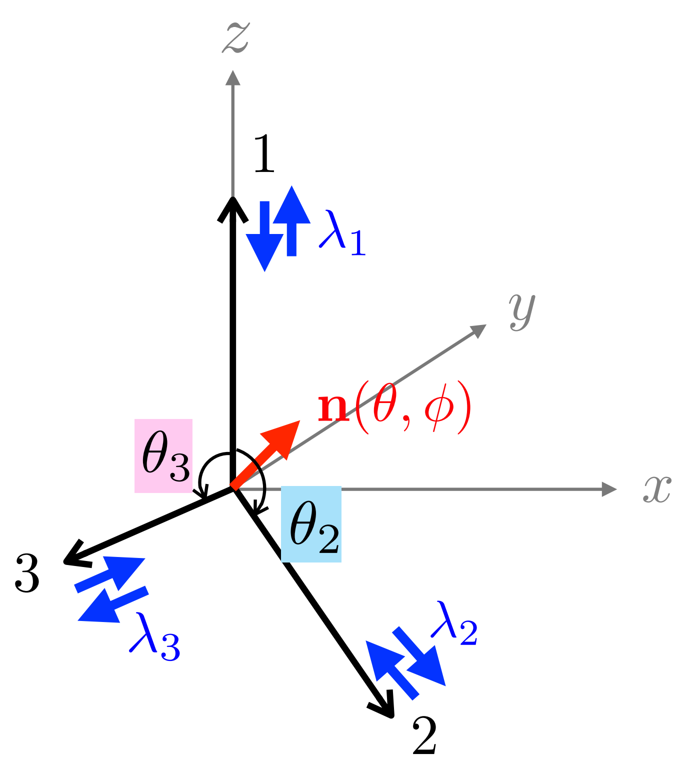
initial state

$$|\mathbf{n}(\theta, \phi)\rangle = \sum_{\lambda_1, \lambda_2, \lambda_3} \mathcal{M}^{\mathbf{n}}_{\lambda_1, \lambda_2, \lambda_3} |\lambda_1, \lambda_2, \lambda_3\rangle + \cdots$$

final state

3-body decay: $\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$

[KS, M.Spannowsky 2310.01477]



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initial state

$$|\mathbf{n}(\theta, \phi)\rangle = \boxed{\sum_{\lambda_1,\lambda_2,\lambda_3} \mathcal{M}^{\mathbf{n}}_{\lambda_1,\lambda_2,\lambda_3} |\lambda_1, \lambda_2, \lambda_3\rangle} = |\Psi\rangle \leftarrow$$

pure (entangled)
3-spin state

Interaction

- Consider **most general** Lorentz invariant 4-fermion interactions

$$\mathcal{L}_{\rm int} = (\bar{\psi}_1 \Gamma_A \psi_0)(\bar{\psi}_3 \Gamma_B \psi_2)$$

$$\Gamma_{A/B} \in \{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$$

$$\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$$

❖ **Scalar-type**

$$[\bar{\psi}_1(c_S + i c_A \gamma_5)\psi_0][\bar{\psi}_3(d_S + i d_A \gamma_5)\psi_2]$$

$$c \equiv c_S + i c_A = e^{i\delta_1}$$
$$d \equiv d_S + i d_A = e^{i\delta_2}$$

❖ **Vector-type**

$$[\bar{\psi}_1 \gamma_\mu (c_L P_L + c_R P_R)\psi_0][\bar{\psi}_3 \gamma^\mu (d_L P_L + d_R P_R)\psi_2]$$

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$
$$c_L, c_R, d_L, d_R \in \mathbb{R}$$

❖ **Tensor-type**

$$[\bar{\psi}_1(c_M + i c_E \gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + i d_E \gamma_5)\sigma_{\mu\nu}\psi_2]$$

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[KS, M.Spannowsky 2310.01477]

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$$\mathcal{L}_{\rm int} = \; [\bar{\psi}_1(c_S + ic_A\gamma_5)\psi_0][\bar{\psi}_3(d_S + id_A\gamma_5)\psi_2]$$

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[KS, M.Spannowsky 2310.01477]

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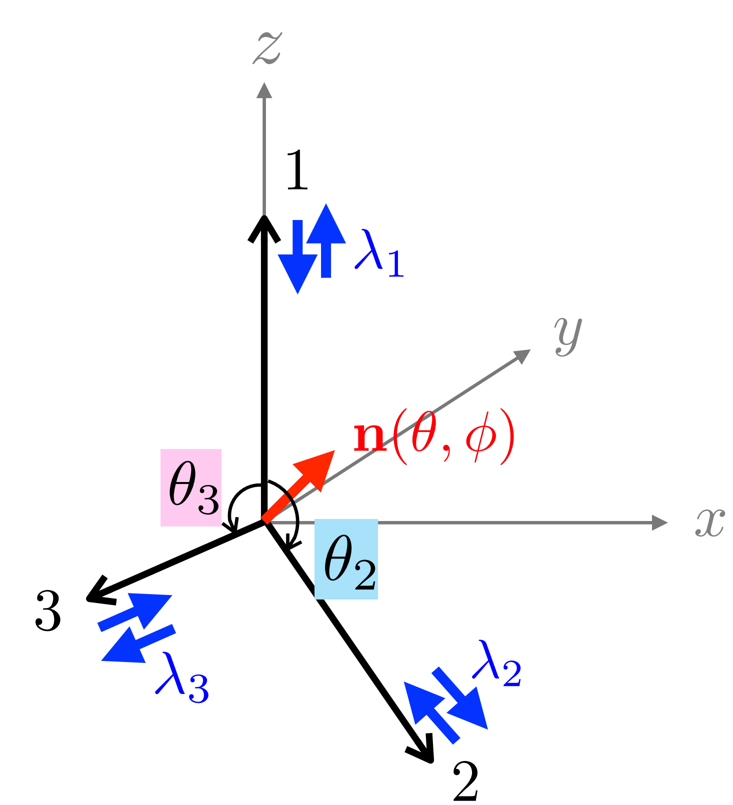
independent of final state momenta θ_2, θ_3



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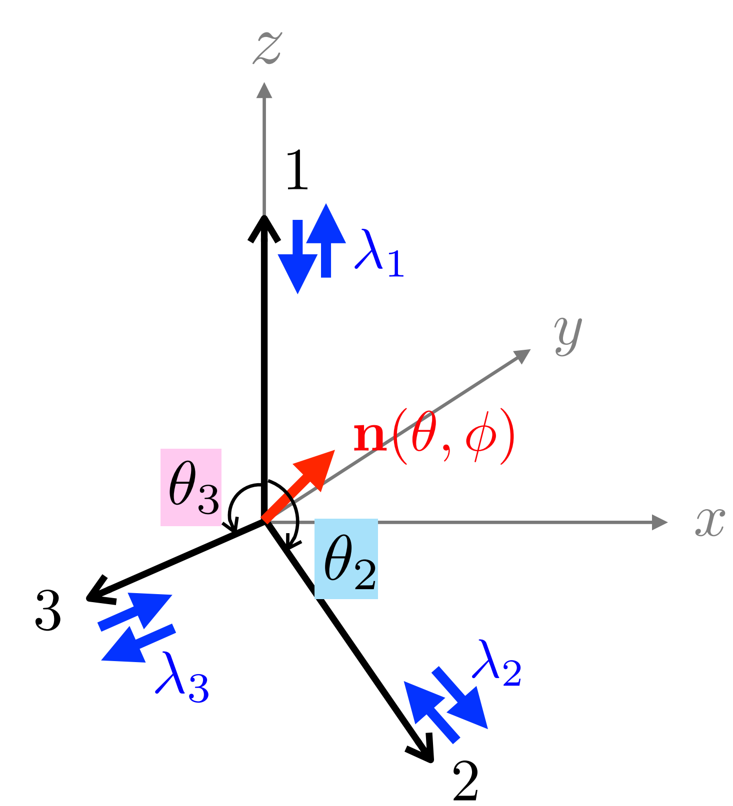
bi-separable



[KS, M.Spannowsky 2310.01477]

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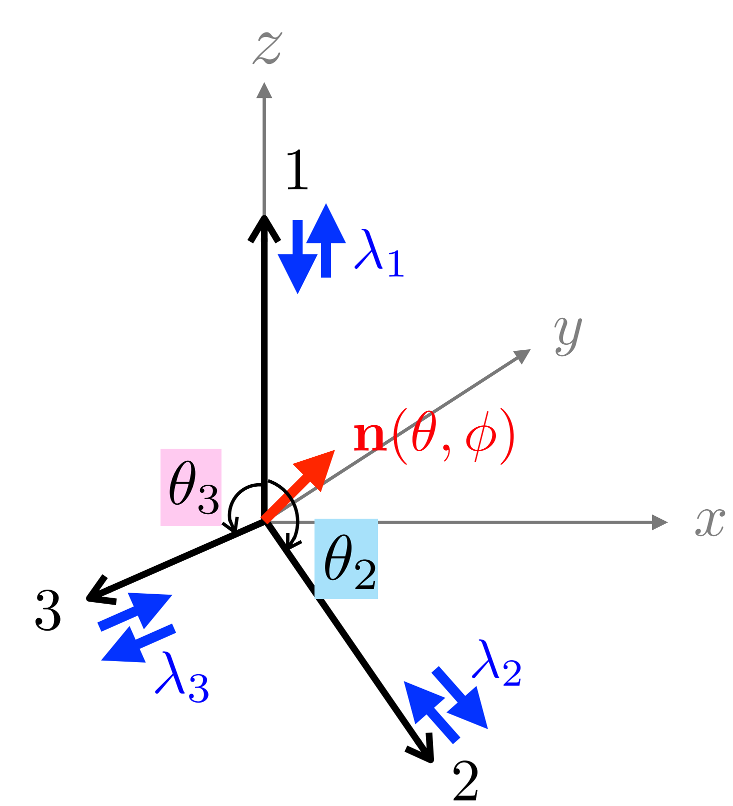
$$\Rightarrow F_3 = 0$$



[KS, M.Spannowsky 2310.01477]

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✤ **1** is **not entangled** with **2** and **3** in any way: $\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{1(23)} = 0$

[KS, M.Spannowsky 2310.01477]

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independent of final state momenta θ_2, θ_3

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[KS, M.Spannowsky 2310.01477]

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independent of final state momenta θ_2, θ_3

$$= \left[ce^{i\phi} s_{\frac{\theta}{2}} |-\rangle_1 + c^* c_{\frac{\theta}{2}} |+\rangle_1\right] \otimes \frac{1}{\sqrt{2}}\left[d|--\rangle_{23} - d^*|++\rangle_{23}\right]$$

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- **1** is **not entangled** with **2** and **3** in any way: $\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{1(23)} = 0$
- **2** and **3** are **maximally entangled**: $\mathcal{C}_{23} = 1$
- Due to **monogamy**, **2** and **3** must be **maximally entangled** with the rest:

$$\mathcal{C}^2_{2(13)} = \underset{\substack{\| \\ 0}}{\mathcal{C}^2_{12}} + \underset{\substack{\| \\ 1}}{\mathcal{C}^2_{23}} + \tau \qquad \mathcal{C}^2_{3(12)} = \underset{\substack{\| \\ 0}}{\mathcal{C}^2_{13}} + \underset{\substack{\| \\ 1}}{\mathcal{C}^2_{23}} + \tau$$

[KS, M.Spannowsky 2310.01477]

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[KS, M.Spannowsky
2310.01477]

Vector

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1 \gamma_\mu (c_L P_L + c_R P_R) \psi_0][\bar{\psi}_3 \gamma^\mu (d_L P_L + d_R P_R) \psi_2] \qquad c_L, c_R, d_L, d_R \in \mathbb{R}$$

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[KS, M.Spannowsky 2310.01477]

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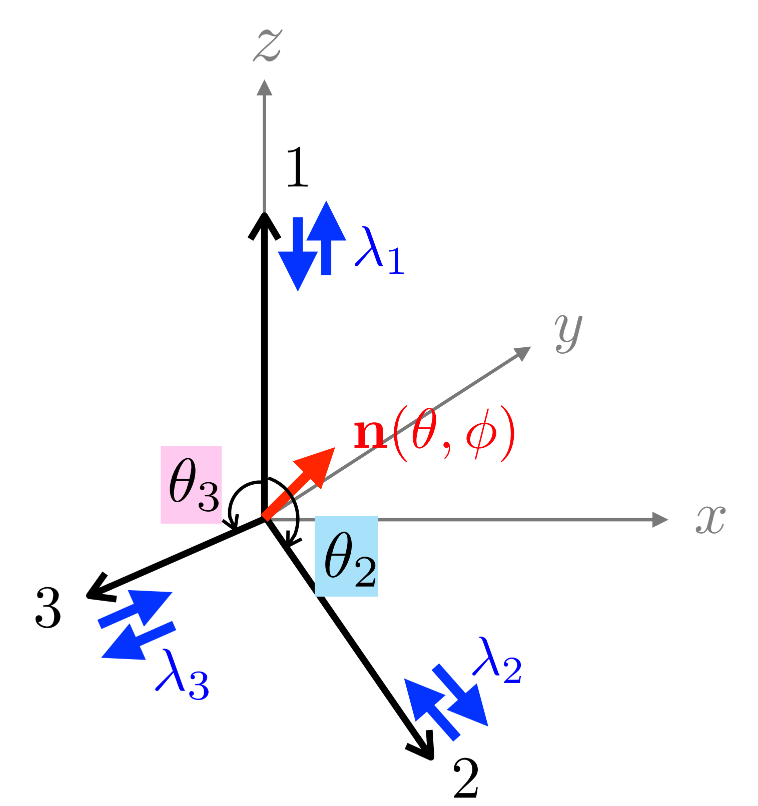
$$+ \; c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}}\right] |++-\rangle \; + \; c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}}\right] |+-+\rangle$$



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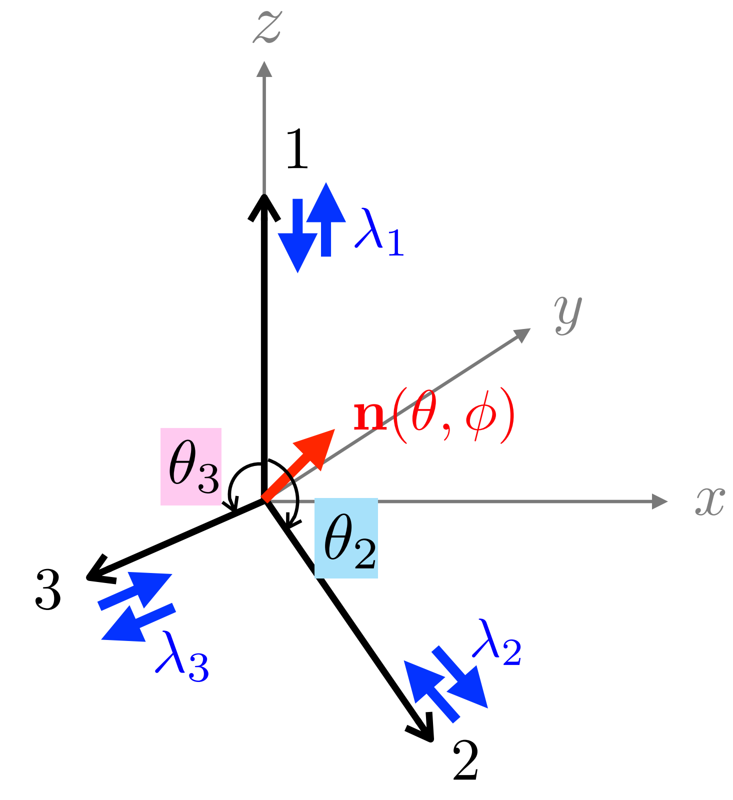
✤ Individual 2-party entanglement:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = 0\,, \quad \mathcal{C}_{23} = 2|M_{LL}M_{LR}^* + M_{RL}M_{RR}^*|$$



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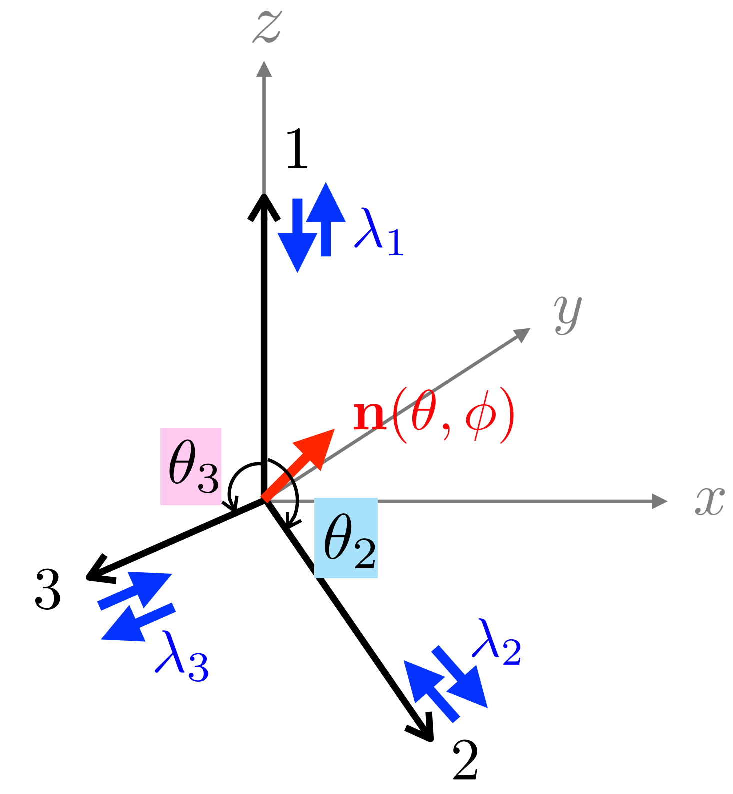
✤ one-to-other entanglement:

$$\mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2\sqrt{(|M_{LL}|^2 + |M_{RL}|^2)\,(|M_{LR}|^2 + |M_{RR}|^2)}$$

$$\mathcal{C}_{1(23)} = 2\big|M_{RR}M_{LL} - M_{LR}M_{RL}\big|$$



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$$+ \; c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}}\right] |++-\rangle \; + \; c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}}\right] |+-+\rangle$$

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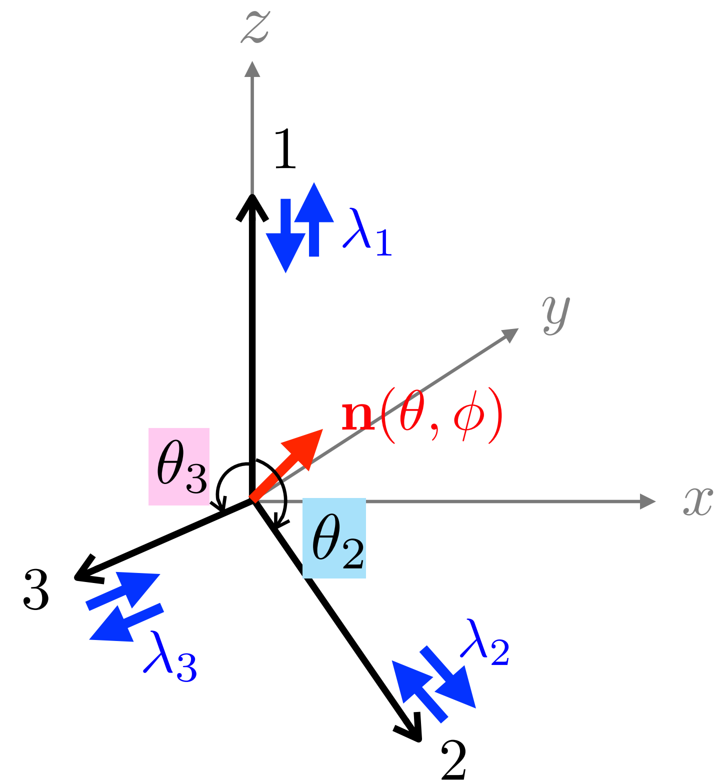
✤ 3-tangle

$$\tau = \mathcal{C}^2_{1(23)} - [\underbrace{\mathcal{C}^2_{12}}_{=0} + \underbrace{\mathcal{C}^2_{13}}_{=0}] = \mathcal{C}^2_{1(23)}$$



[KS, M.Spannowsky 2310.01477]

Vector

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1 \gamma_\mu (c_L P_L + c_R P_R)\psi_0][\bar{\psi}_3 \gamma^\mu (d_L P_L + d_R P_R)\psi_2] \qquad c_L, c_R, d_L, d_R \in \mathbb{R}$$

$$\Rightarrow \quad |\Psi\rangle = M_{LL}|-+-\rangle + M_{LR}|--+\rangle + M_{RL}|++-\rangle + M_{RR}|+-+\rangle$$

$$\propto \; c_L d_L s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}}\right] |-+-\rangle \; + \; c_L d_R s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}} + e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}}\right] |--+\rangle$$

$$+ \; c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}}\right] |++-\rangle \; + \; c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}}\right] |+-+\rangle$$

✤ Individual 2-party entanglement:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = 0\,, \quad \mathcal{C}_{23} = 2|M_{LL}M^*_{LR} + M_{RL}M^*_{RR}| \quad \longleftarrow \; \text{vanish if } \; d_L d_R = 0$$

✤ one-to-other entanglement:

$$\mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2\sqrt{\left(|M_{LL}|^2 + |M_{RL}|^2\right)\left(|M_{LR}|^2 + |M_{RR}|^2\right)} \quad \longleftarrow \; \text{vanish if } \; c_L c_R = d_L d_R = 0$$

$$\mathcal{C}_{1(23)} = 2\big|M_{RR}M_{LL} - M_{LR}M_{RL}\big| \quad \longleftarrow \; \text{vanish if } \; c_L c_R d_L d_R = 0$$

✤ 3-tangle

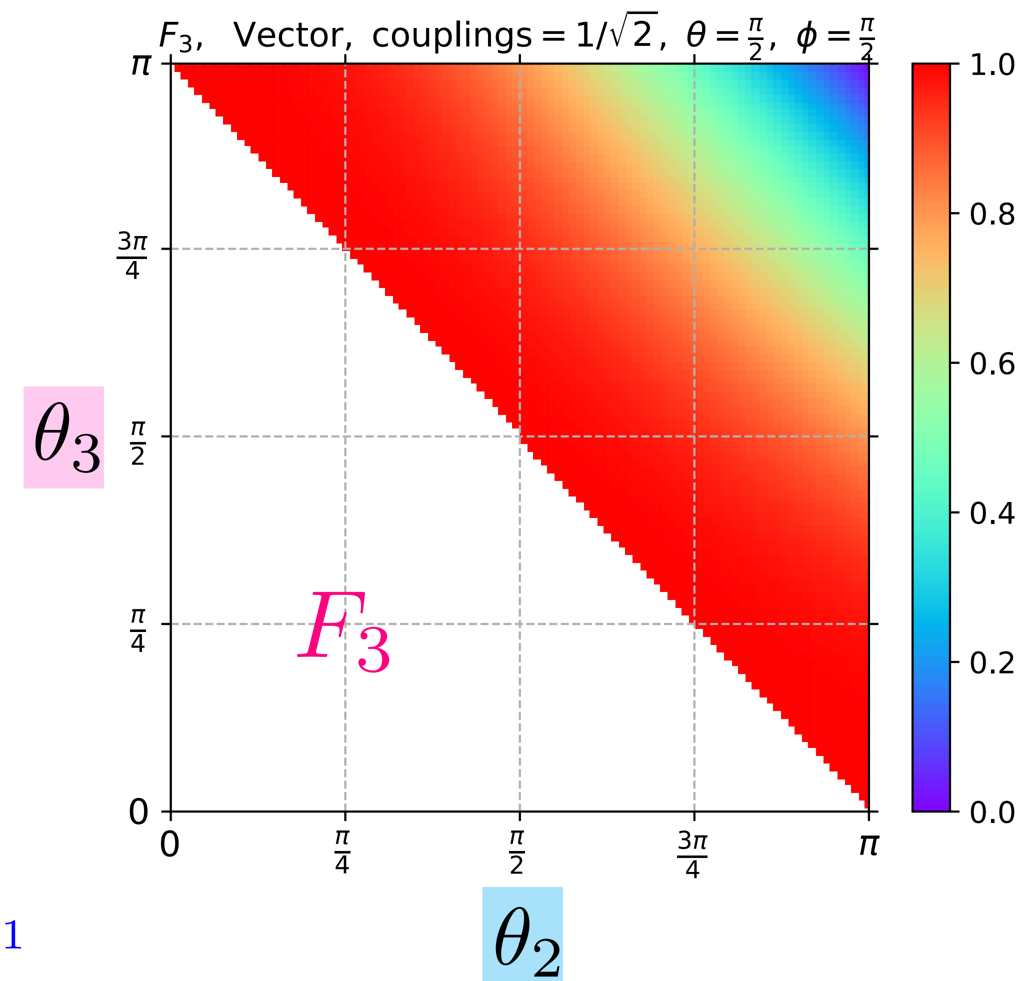
$$\tau = \mathcal{C}^2_{1(23)} - [\overset{0}{\overset{\|}{\mathcal{C}^2_{12}}} + \overset{0}{\overset{\|}{\mathcal{C}^2_{13}}}] = \mathcal{C}^2_{1(23)}$$

➡ **All** entanglements **vanish** for **weak decays**

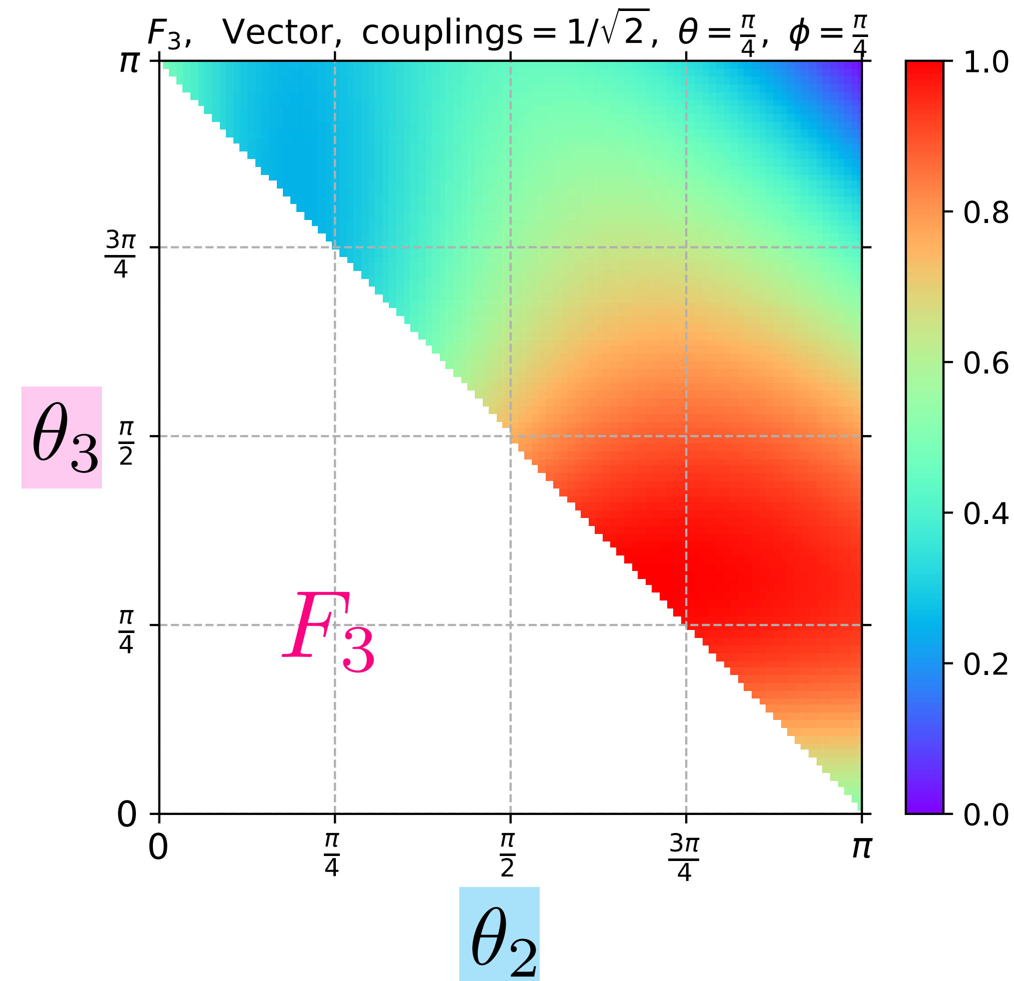
$$c_R = d_R = 0$$

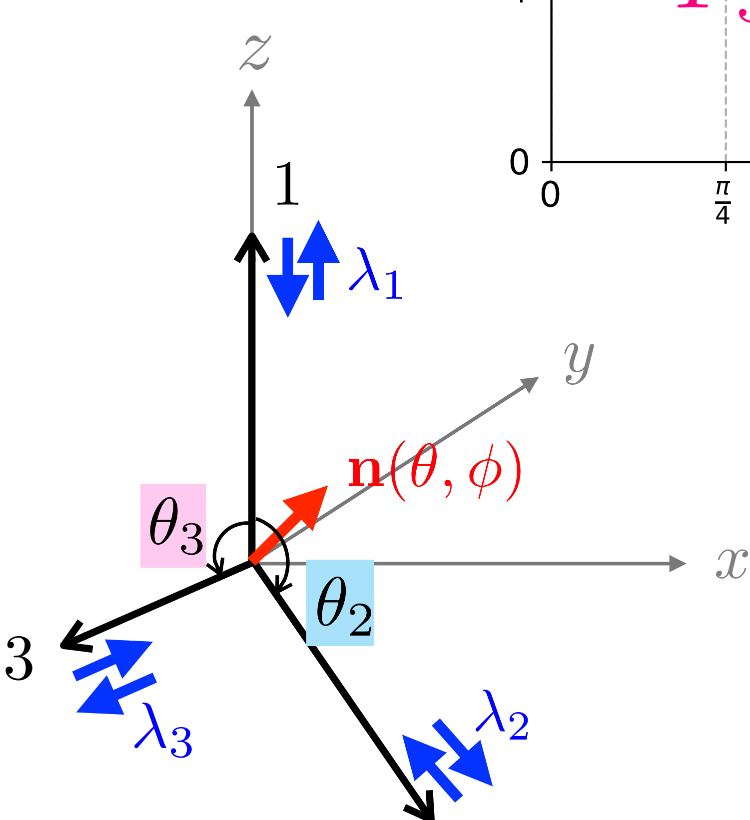
F_3 for Vector

$$\mathbf{n} = \mathbf{e}_y$$

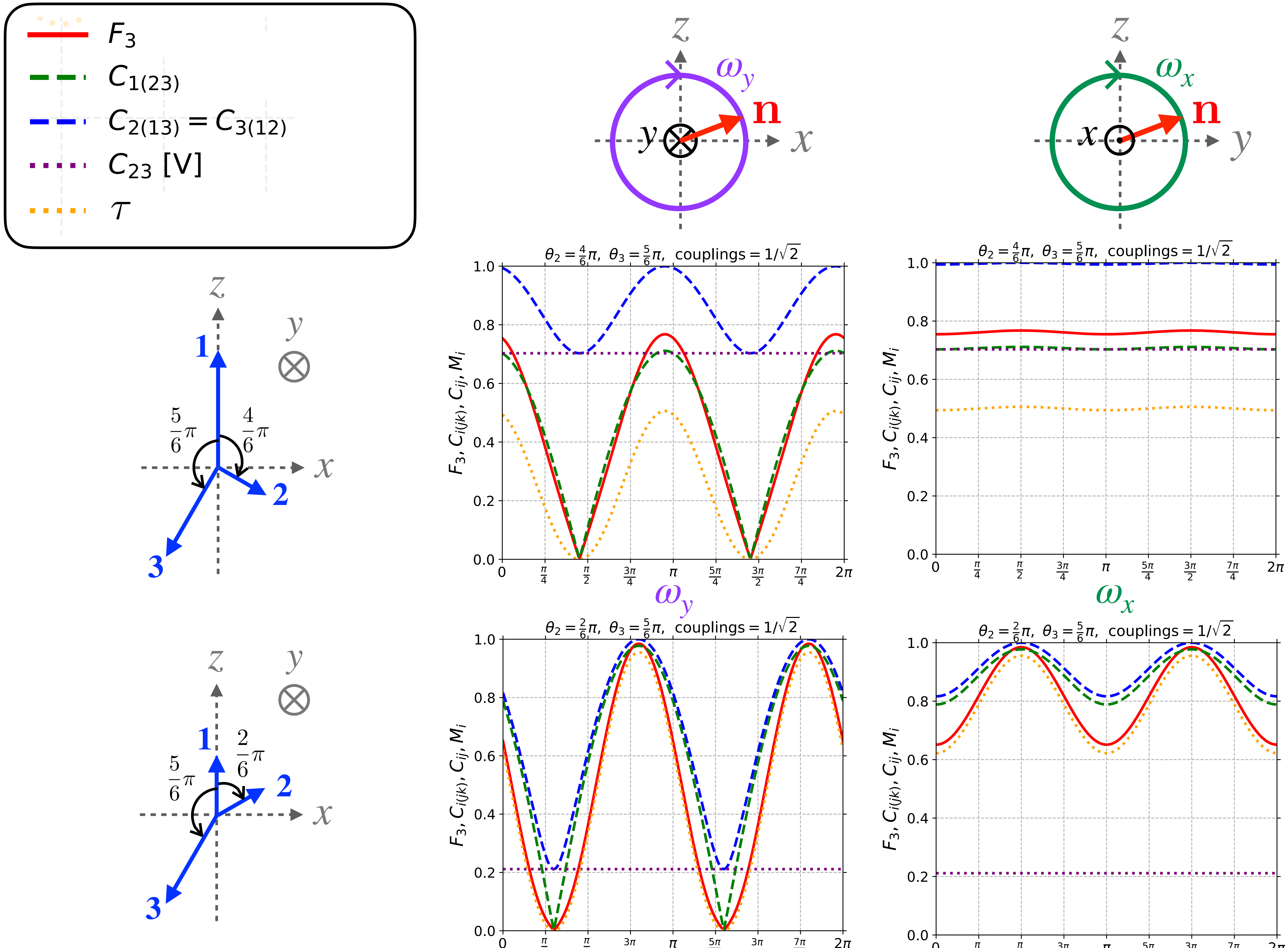


$$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)$$





[KS, M.Spannowsky 2310.01477]



[KS, M.Spannowsky 2310.01477]

[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\text{int}} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + ic_E = e^{i\omega_1}$$
$$d \equiv d_M + id_E = e^{i\omega_2}$$

[KS, M.Spannowsky 2310.01477]

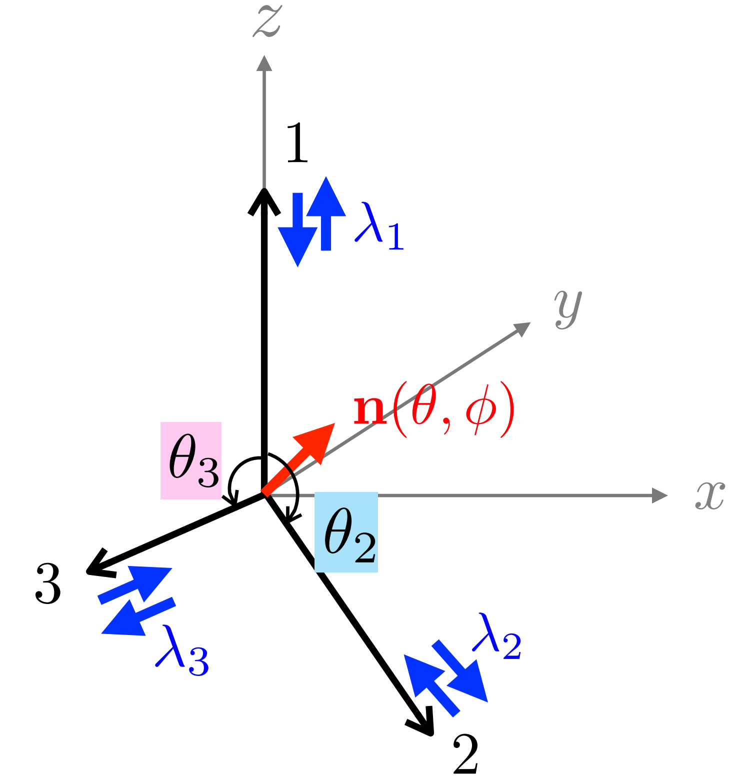
Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + ic_E = e^{i\omega_1}$$
$$d \equiv d_M + id_E = e^{i\omega_2}$$

$$\Rightarrow \quad |\Psi\rangle = M_R|+++\rangle + M_L|---\rangle$$

$$\propto c^* d^* \left[2e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}} + c_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}}\right] |+++\rangle + cd\left[-e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}} + 2c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}}\right]|---\rangle$$



[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

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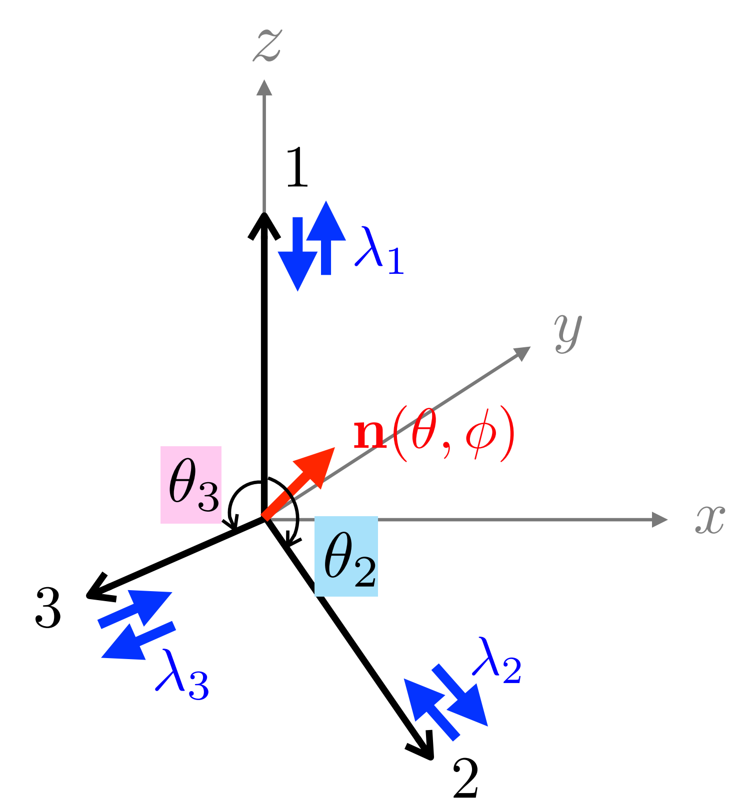
$$\propto \; c^* d^* \, [2e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}} + c_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}}]\,|+++\rangle + cd\,[-e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}} + 2c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}}]\,|---\rangle$$

✤ $|\Psi\rangle$ *interpolates* **product states** and the **maximally entangled** state:

$$(M_R M_L = 0) \qquad |\pm\pm\pm\rangle \longleftrightarrow |{\rm GHZ}\rangle = \frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle) \qquad (M_R = M_L)$$



[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar\psi_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar\psi_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + ic_E = e^{i\omega_1}$$
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✤ **No** individual 2-party entanglements:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 0$$

[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + i c_E \gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + i d_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + i c_E = e^{i\omega_1}$$
$$d \equiv d_M + i d_E = e^{i\omega_2}$$

$$\Rightarrow \quad |\Psi\rangle = M_R|+++\rangle + M_L|---\rangle$$

$$\propto \; c^* d^* \left[2 e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}} + c_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}}\right]|+++\rangle + cd\left[-e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}} + 2 c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}}\right]|---\rangle$$

✤ $|\Psi\rangle$ *interpolates* **product states** and the **maximally entangled** state:

$$(M_R M_L = 0) \qquad |\pm\pm\pm\rangle \longleftrightarrow |{\rm GHZ}\rangle = \frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle) \qquad (M_R = M_L)$$

✤ **No** individual 2-party entanglements:

3-tangle

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 0 \longrightarrow \tau = \mathcal{C}^2_{i(jk)} - [\underbrace{\mathcal{C}^2_{ij}}_{=0} + \underbrace{\mathcal{C}^2_{ik}}_{=0}] = \mathcal{C}^2_{i(jk)}$$

[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

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$$\propto \ c^* d^* \left[2e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}} + c_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}}\right] |+++\rangle + cd \left[-e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}} + 2c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}}\right] |---\rangle$$

✤ $|\Psi\rangle$ *interpolates* **product states** and the **maximally entangled** state:

$$(M_R M_L = 0) \qquad |\pm\pm\pm\rangle \longleftrightarrow |\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle) \qquad (M_R = M_L)$$

✤ **No** individual 2-party entanglements:

3-tangle

$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 0 \longrightarrow \tau = \mathcal{C}^2_{i(jk)} - [\underbrace{\mathcal{C}^2_{ij}}_{=0} + \underbrace{\mathcal{C}^2_{ik}}_{=0}] = \mathcal{C}^2_{i(jk)}$$

✤ one-to-other entanglements are **universal**:

$$\mathcal{C}_{1(23)} = \mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2|M_R M_L|$$

$$F_3 = 4|M_R M_L|^2$$

[KS, M.Spannowsky 2310.01477]

Tensor

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

$$c \equiv c_M + ic_E = \boxed{e^{i\omega_1}}$$
$$d \equiv d_M + id_E = \boxed{e^{i\omega_2}}$$

$$\Rightarrow \quad |\Psi\rangle = M_R|+++\rangle + M_L|---\rangle$$

$$\propto \; c^* d^* \left[2e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}} + c_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}}\right] |+++\rangle + cd \left[-e^{i\phi} s_{\frac{\theta}{2}} s_{\frac{\theta_3-\theta_2}{2}} + 2c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} s_{\frac{\theta_3}{2}}\right] |---\rangle$$

✤ $|\Psi\rangle$ *interpolates* **product states** and the **maximally entangled** state:

$(M_R M_L = 0)$ $\quad |\pm\pm\pm\rangle \longleftrightarrow |{\rm GHZ}\rangle = \frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle)$ $\quad (M_R = M_L)$

✤ **No** individual 2-party entanglements:

3-tangle

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✤ one-to-other entanglements are **universal**:

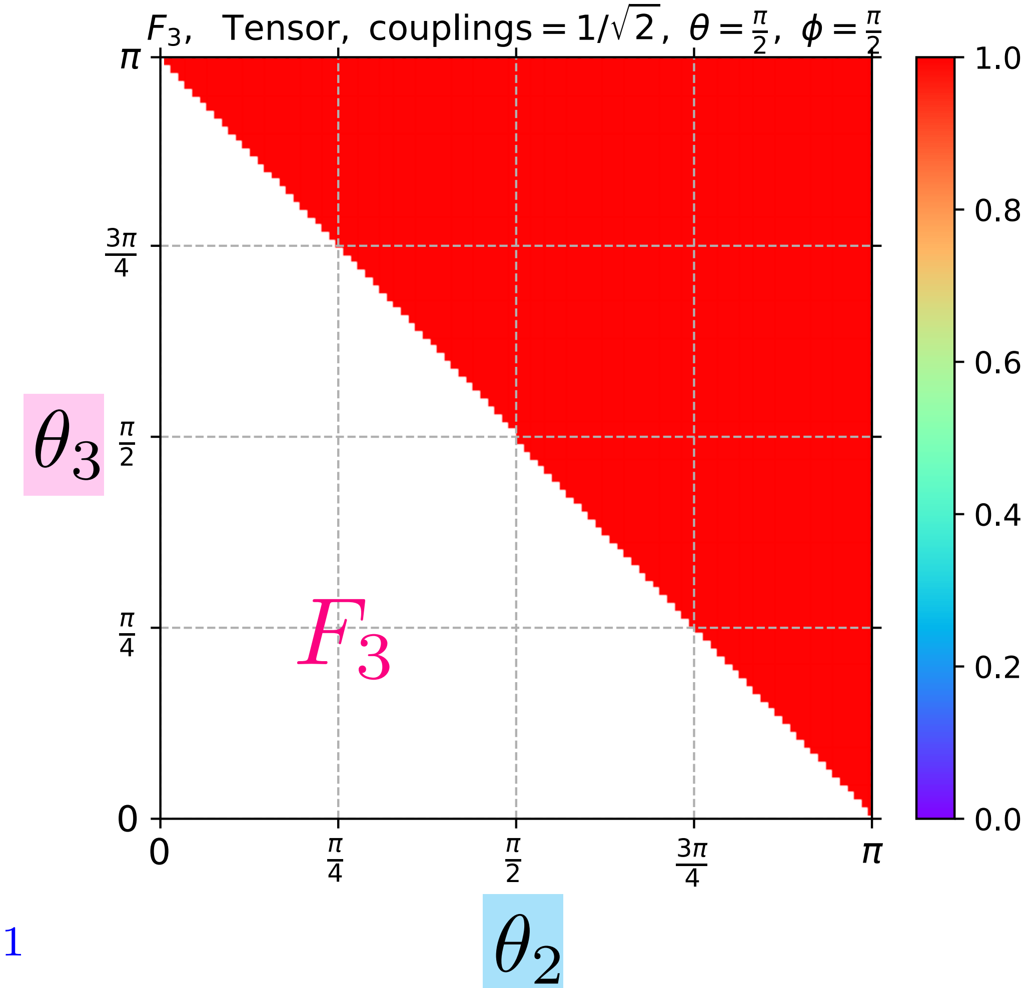
$$\left.\begin{aligned} \mathcal{C}_{1(23)} = \mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2|M_R M_L| \\ F_3 = 4|M_R M_L|^2 \end{aligned}\right\}$$

independent of the coupling structure (CP phases)

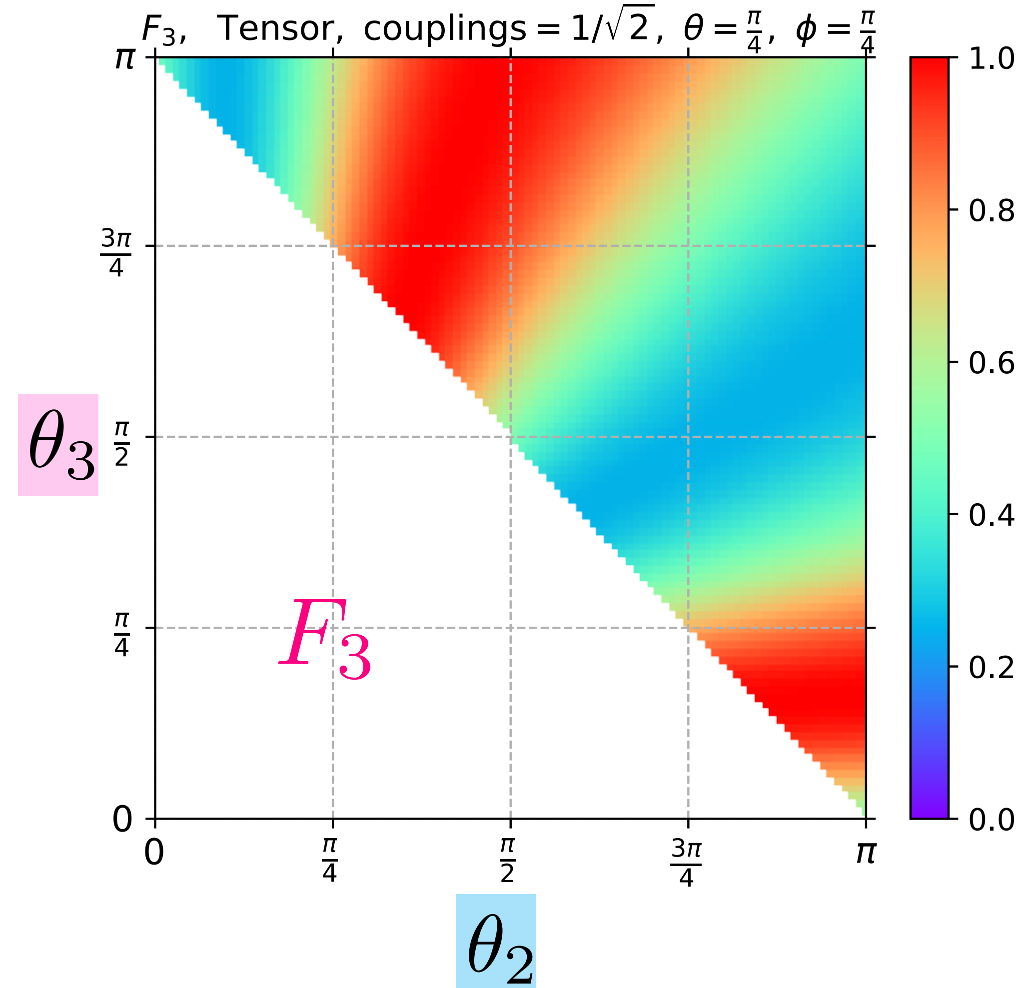
ω_1, ω_2

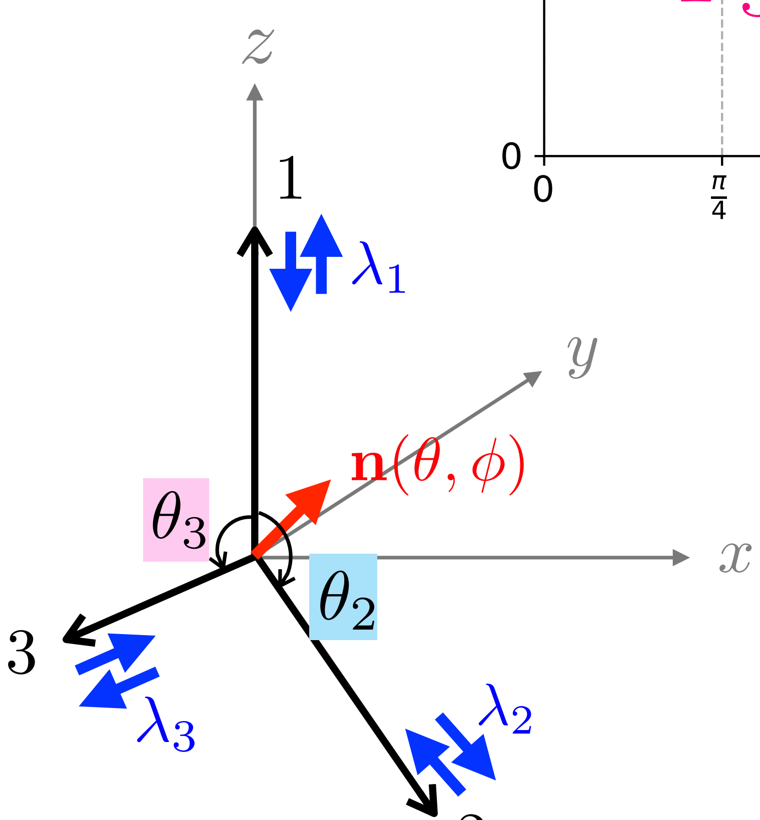
F_3 for Tensor

$$\mathbf{n} = \mathbf{e}_y$$



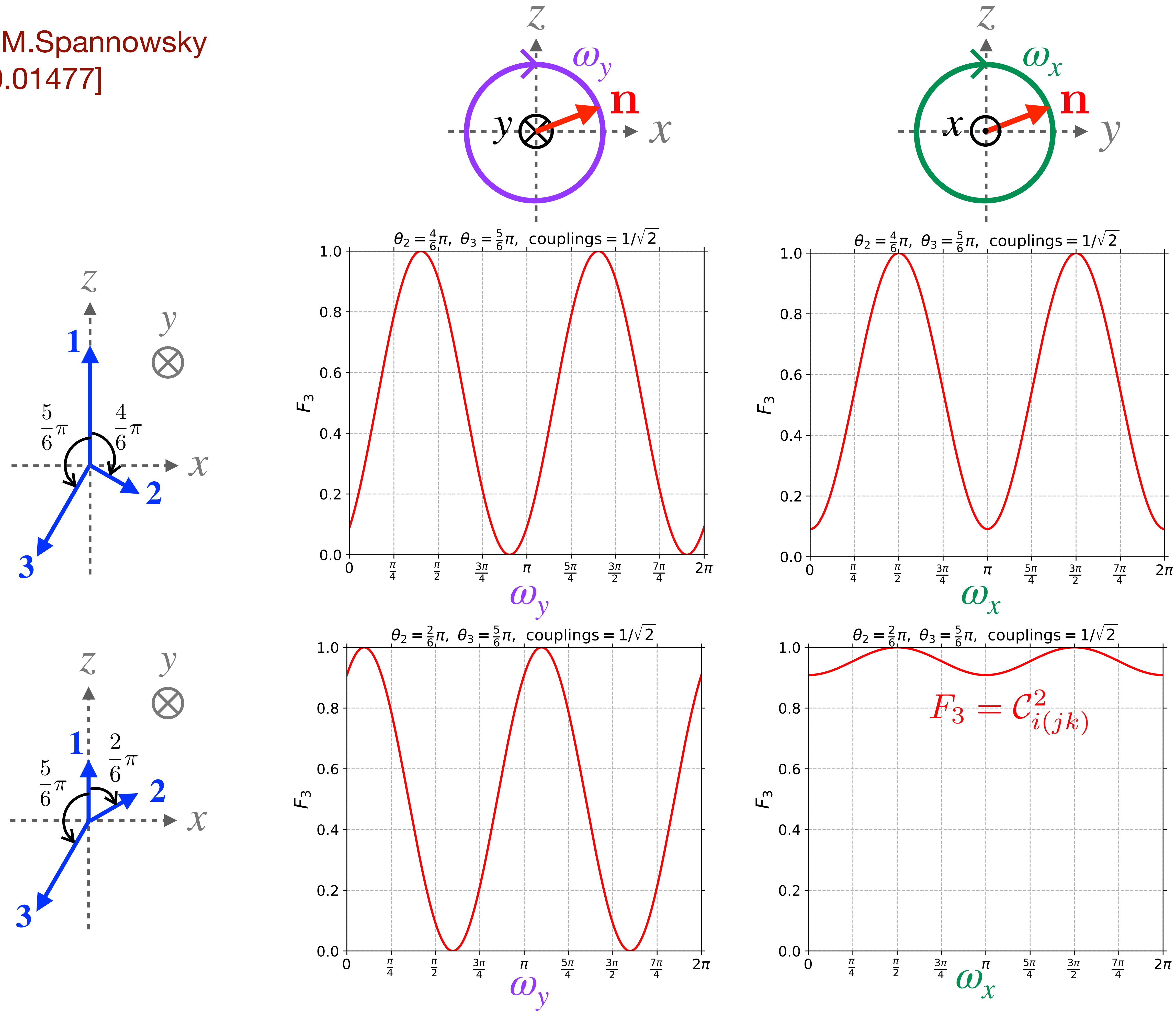
$$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)$$

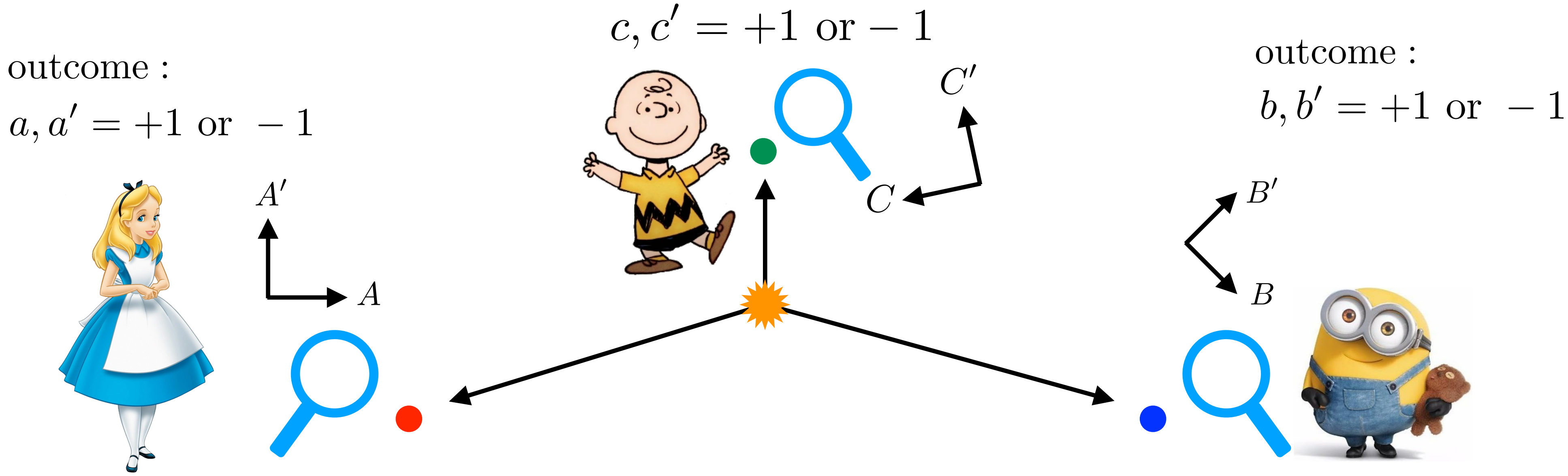




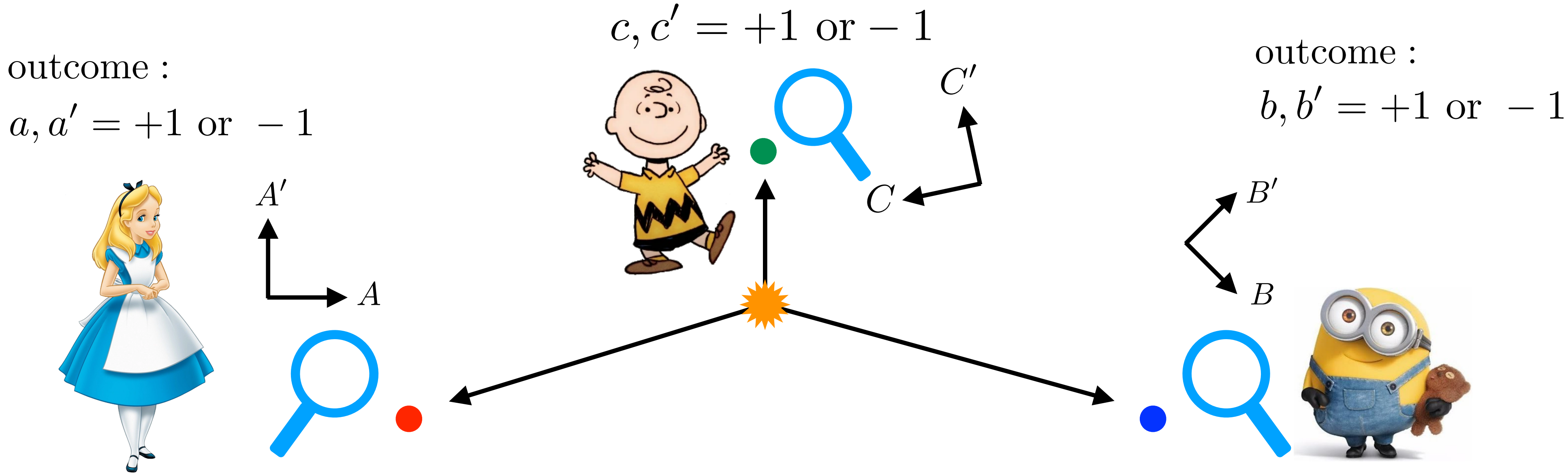
[KS, M.Spannowsky 2310.01477]

[KS, M.Spannowsky
2310.01477]





3-partite Bell inequality



- Completely Local (CL): $\langle ABC \rangle_{\mathrm{CL}} = \int p(\lambda)[a(\lambda)b(\lambda)c(\lambda)]d\lambda$

- Mermin correlation: $\mathcal{B}_{\mathrm{M}} = ABC' + AB'C + A'BC - A'B'C'$

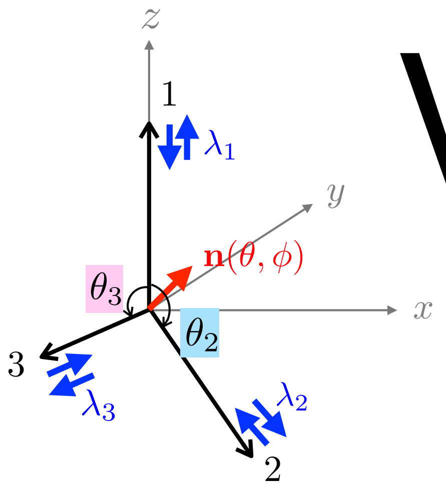
- **Mermin inequalities**: $\langle \mathcal{B}_{\mathrm{M}} \rangle_{\mathrm{CL}} \leq 2 \qquad \langle \mathcal{B}_{\mathrm{M}} \rangle_{\mathrm{QM}} \leq 4$ [Mermin '90]

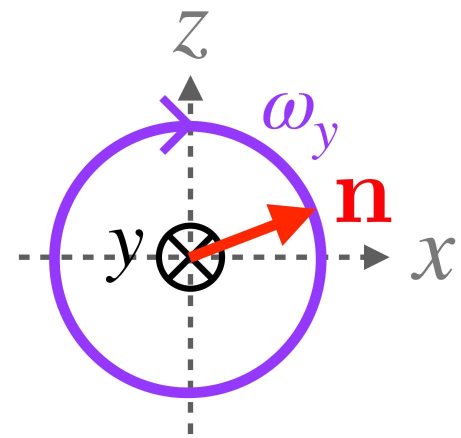
- Partially Local (PL): $\langle ABC \rangle_{\mathrm{PL}} = \int p(\lambda)[ab(\lambda)c(\lambda)]d\lambda, \ \int p(\lambda)[a(\lambda)bc(\lambda)]d\lambda, \cdots$

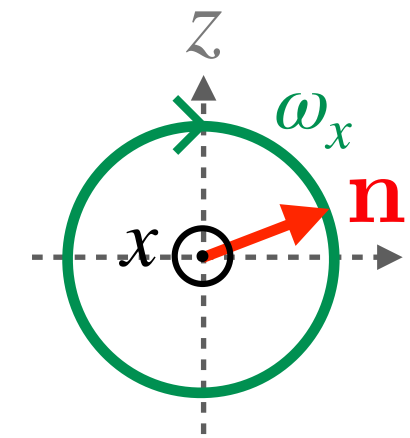
- Svetlichny correlation:
$$\mathcal{B}_{\mathrm{S}} = ABC + ABC' + AB'C + A'BC$$
$$-A'B'C' - A'B'C - A'BC' - AB'C'$$

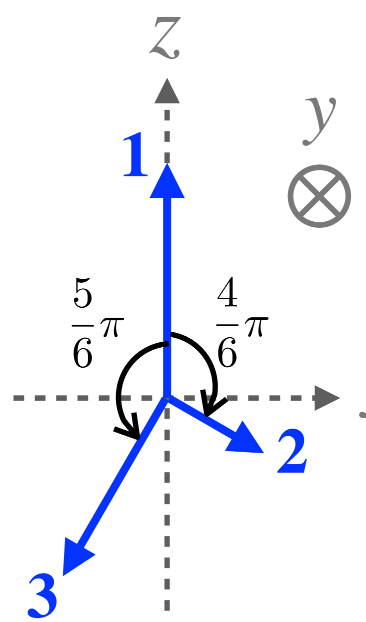
- **Svetlichny inequalities**: $\langle \mathcal{B}_{\mathrm{S}} \rangle_{\mathrm{PL}} \leq 4 \qquad \langle \mathcal{B}_{\mathrm{S}} \rangle_{\mathrm{QM}} \leq 4\sqrt{2}$ [Svetlichny '87]

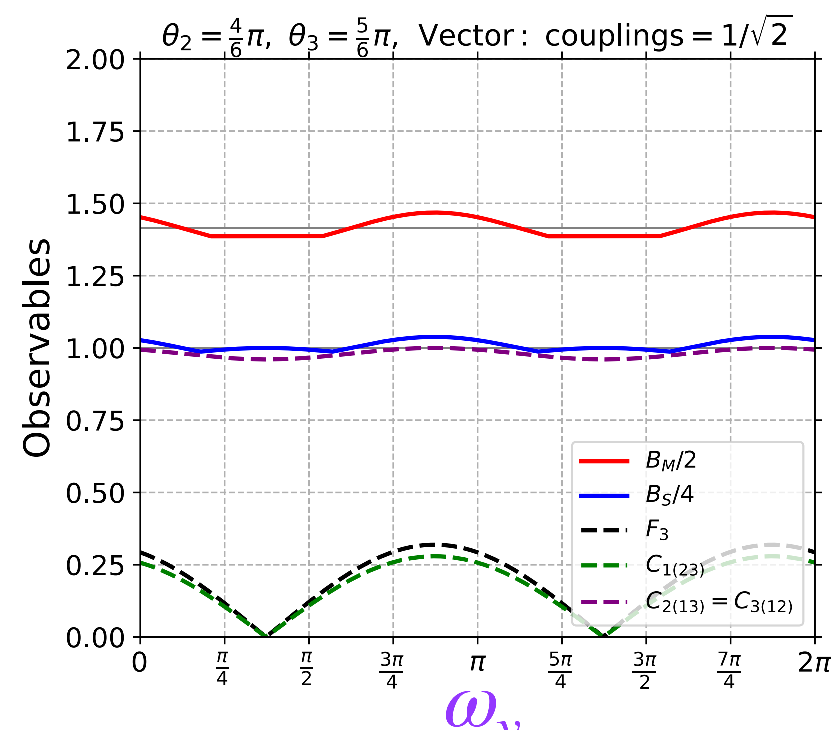
Vector

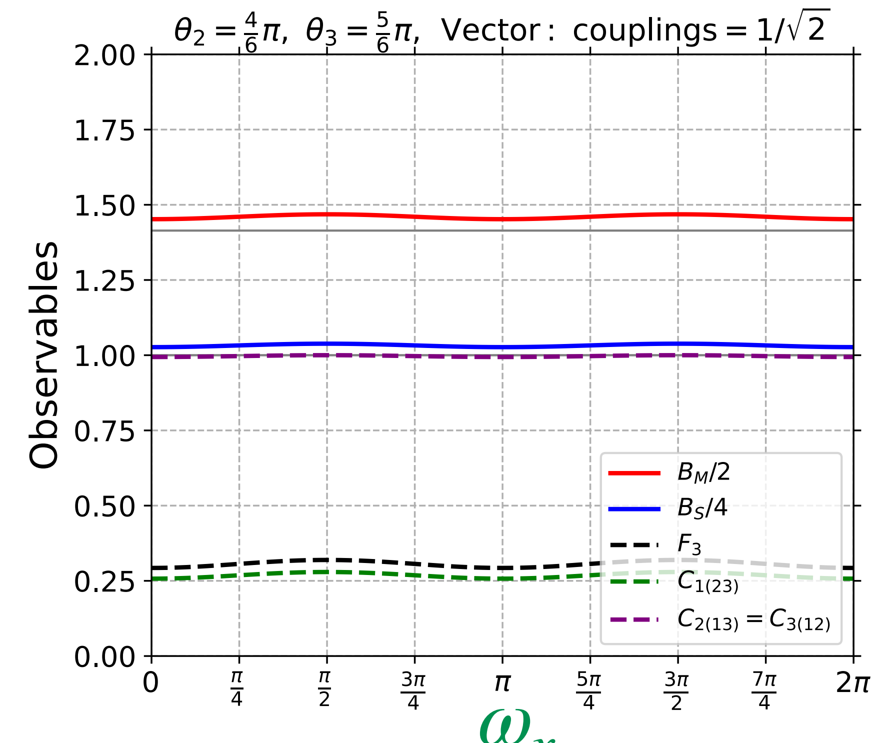

z
1
λ_1
y
$\mathbf{n}(\theta, \phi)$
θ_3
x
θ_2
3
λ_3
λ_2
2

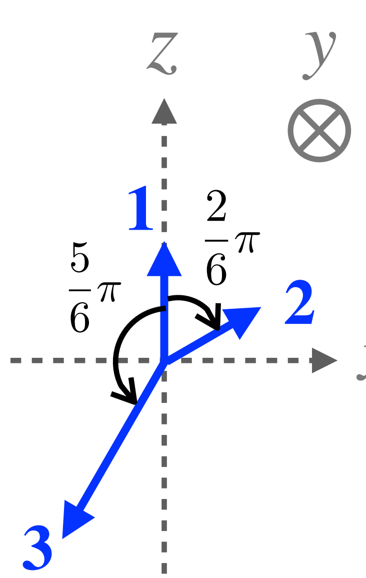

z
ω_y
n
y
x

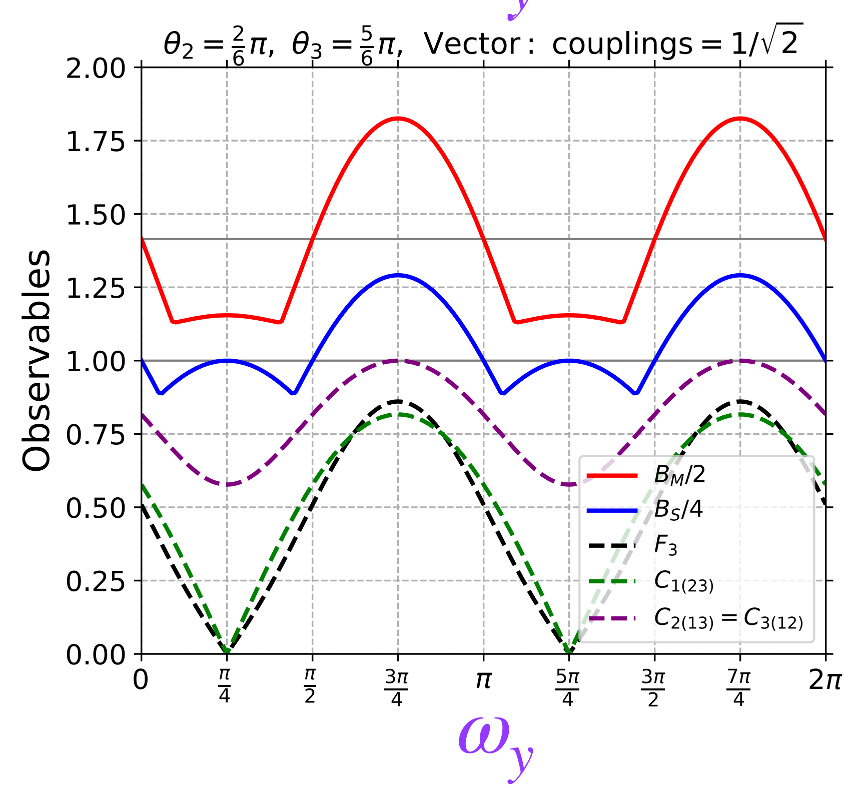

z
ω_x
n
x
y

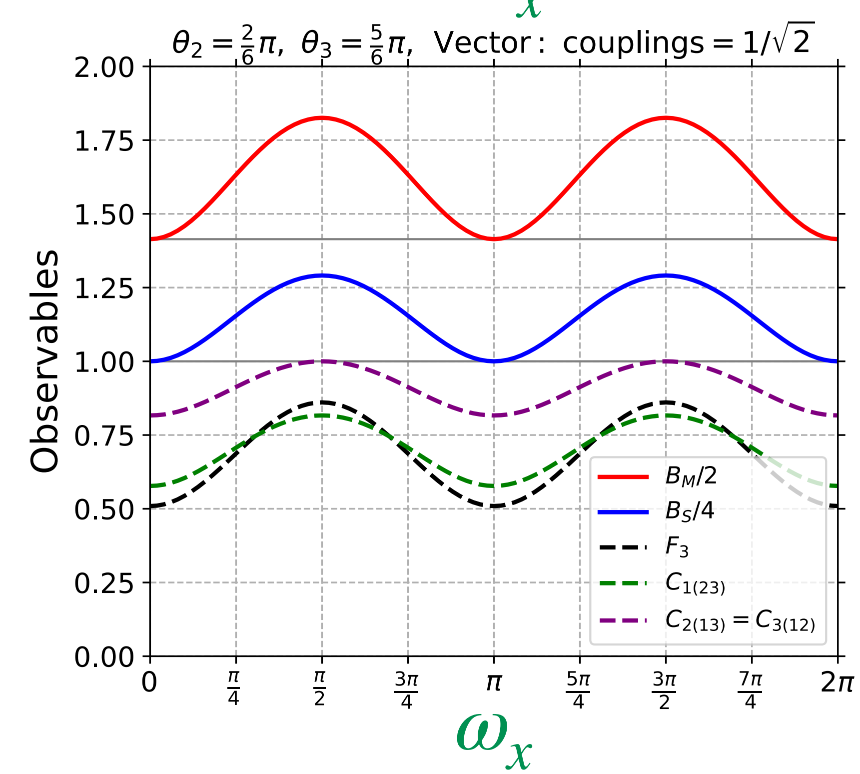

z
y
1
$\frac{5}{6}\pi$
$\frac{4}{6}\pi$
x
2
3


$\theta_2 = \frac{4}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,\ \mathrm{Vector:\ couplings} = 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
$C_{1(23)}$
$C_{2(13)} = C_{3(12)}$
ω_y

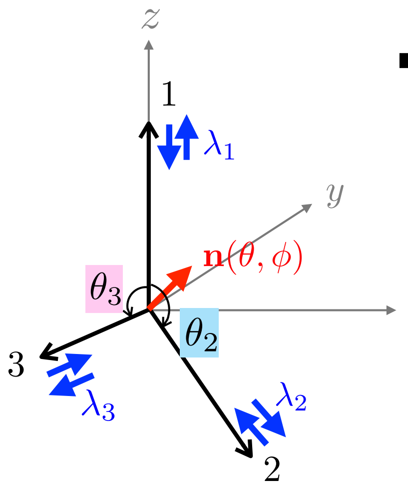

$\theta_2 = \frac{4}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,\ \mathrm{Vector:\ couplings} = 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
$C_{1(23)}$
$C_{2(13)} = C_{3(12)}$
ω_x


z
y
1
$\frac{5}{6}\pi$
$\frac{2}{6}\pi$
2
x
3

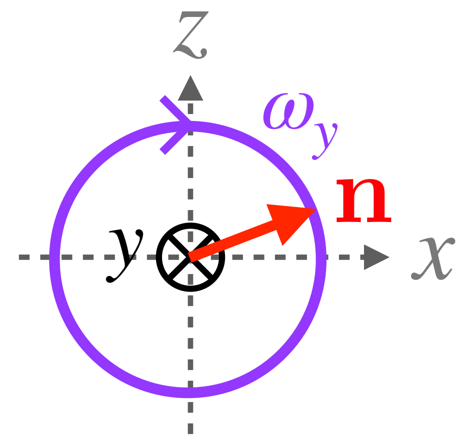

$\theta_2 = \frac{2}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,\ \mathrm{Vector:\ couplings} = 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
$C_{1(23)}$
$C_{2(13)} = C_{3(12)}$
ω_y


$\theta_2 = \frac{2}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,\ \mathrm{Vector:\ couplings} = 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
$C_{1(23)}$
$C_{2(13)} = C_{3(12)}$
ω_x

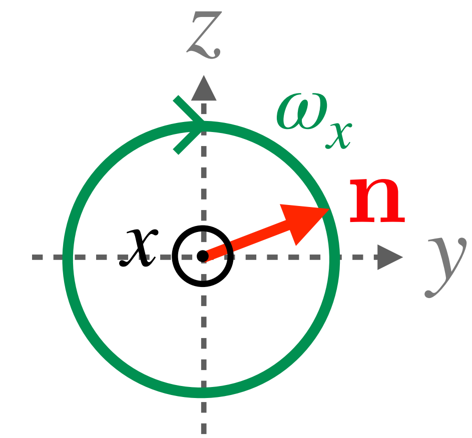
Tensor



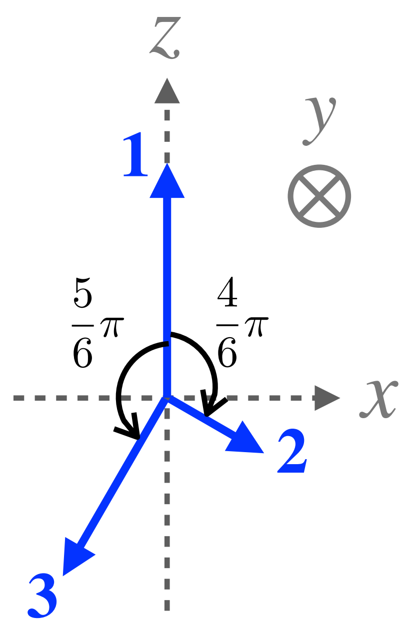
z
x
y
1
2
3
λ_1
λ_2
λ_3
$\mathbf{n}(\theta, \phi)$
θ_3
θ_2



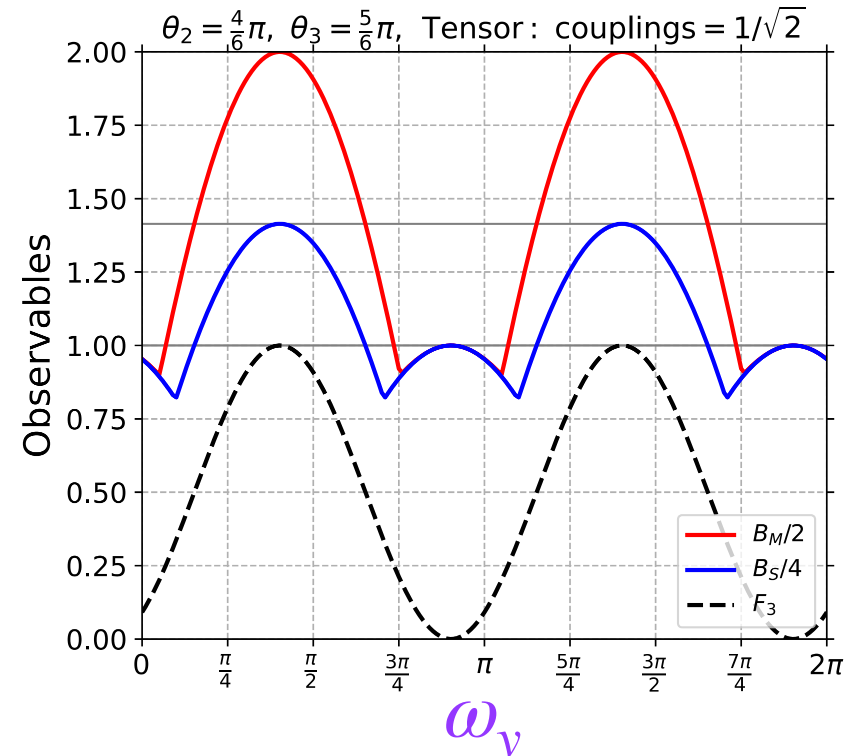
z
ω_y
$\mathbf{n}$
y
x



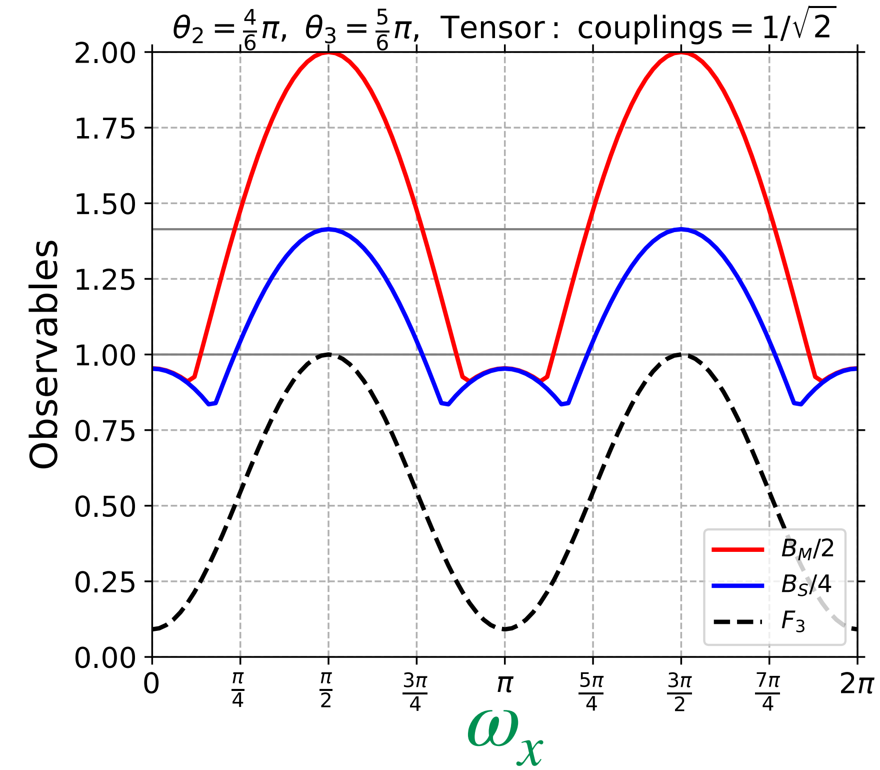
z
ω_x
$\mathbf{n}$
x
y



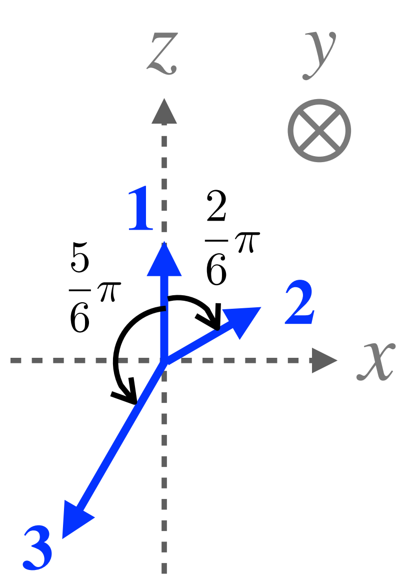
z
y
x
1
2
3
$\frac{5}{6}\pi$
$\frac{4}{6}\pi$



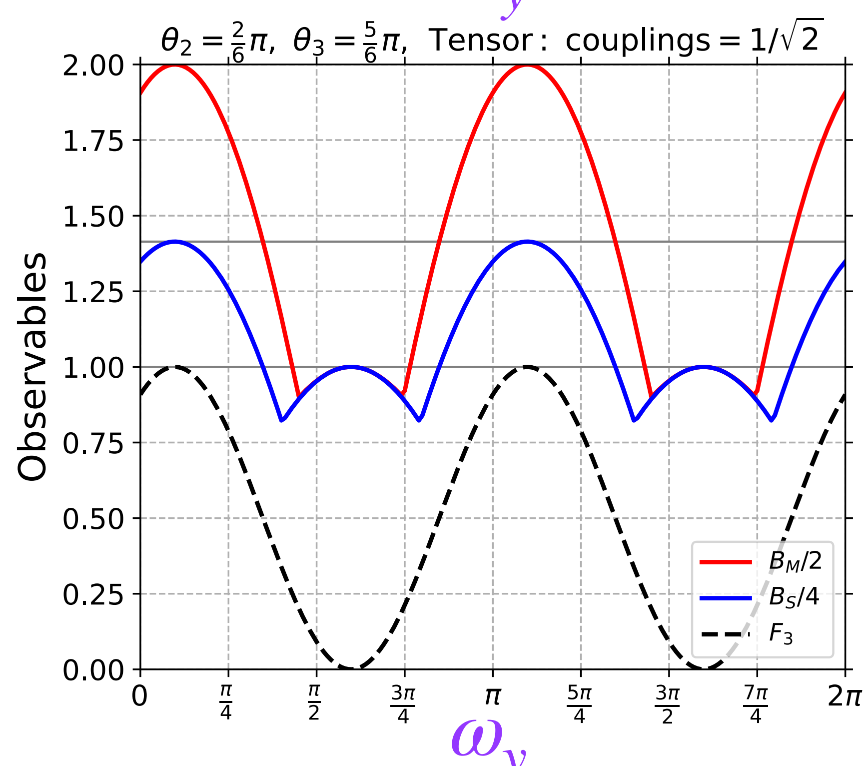
$\theta_2 = \frac{4}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,\ \mathrm{Tensor: couplings} = 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
ω_y



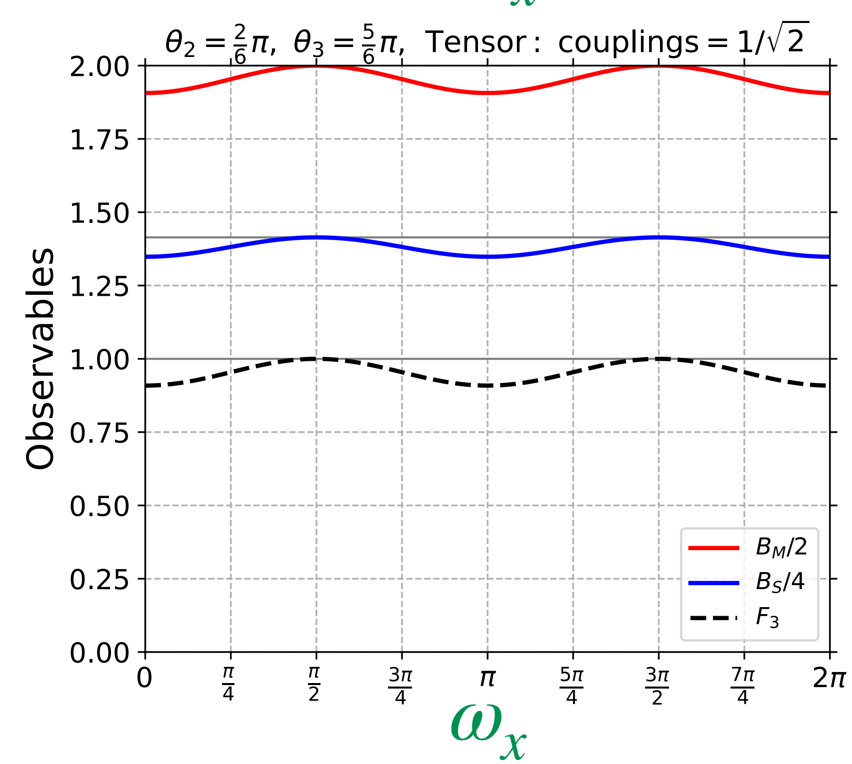
$\theta_2 = \frac{4}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,\ \mathrm{Tensor: couplings} = 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
ω_x



z
y
x
1
2
3
$\frac{5}{6}\pi$
$\frac{2}{6}\pi$



$\theta_2 = \frac{2}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,\ \mathrm{Tensor: couplings} = 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
ω_y



$\theta_2 = \frac{2}{6}\pi,\ \theta_3 = \frac{5}{6}\pi,\ \mathrm{Tensor: couplings} = 1/\sqrt{2}$
Observables
$B_M/2$
$B_S/4$
F_3
ω_x

Conclusions

- High-energy tests of QM are important:
  - Locality may be an emerging property.
  - QM may be modified at high-energy to be reconciled with gravity.
  - sensitive to BSM and non-perturbative effects of SM.
- Observation of the Bell variable at colliders provides a powerful test of QM.
- Observation of entanglement and Bell variable at LHC and lepton colliders has recently been studies in various processes
- The investigation of 3-particle entanglement and non-locality in particle physics has just been started.
- A study on massless 3-qubit entanglement from a massive fermion decay has been studied and presented.

Future works and directions

- Effect of **masses** in the final particles
- More **spin structures**: $SFFV,\ VVFF,\ SFVF_{3/2},\ SVVT\cdots$
- How entanglement is created and evolves in the particle physics interactions/measurements?
- Can we constrain new physics from the general properties of entanglement?

Thank you for listening!