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Discrete Flavor Symmetries

Diploma Thesis

by

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Abstract

A discrete flavor symmetry can serve as an explanation for the strong mass hierarchy among quarks and charged leptons, the different mixings in the quark and lepton sector or can be used to obtain a certain mass matrix texture. In this work we introduce in addition to the standard model the dihedral group D_5 and thus obtain certain mass matrices. From this we develop a model which is in agreement with current experimental bounds. Thereby we discuss different Higgs sectors containing three and four Higgs respectively as well as their mass matrices. Finally we show a numerical solution for models containing Dirac and Majorana neutrinos respectively.

Kurzfassung

Diskrete Flavor Symmetrien können als Erklärung dienen, um die große Massen-Hierarchie von Quarks und geladenen Leptonen, sowie die vollkommen unterschiedlichen Mischungen im Quark- und Lepton-Sektor zu erklären. Sie können auch dazu dienen, um gewisse Texturen in Massen-Matrizen zu erhalten. In dieser Diplomarbeit führen wir zusätzlich zum Standard Modell die diedrische Gruppe D_5 ein und erhalten dadurch bestimmte Massen-Matrizen. Daraus entwickeln wir dann ein Modell, das mit den experimentellen Ergebnissen in Übereinstimmung ist. Dazu diskutieren wir verschiedene Higgs-Sektoren mit 3 bzw. 4 Higgs Bosonen, sowie deren Massen-Matrizen. Abschließend geben wir zwei mögliche numerische Lösungen an, wobei Neutrinos entweder Dirac oder Majorana Teilchen sind.

Contents

Chapter 1

Introduction

In the area of high energy physics symmetries turn out to be the fundamental concept to explain the properties and behavior of the particles. This is formulated in the standard model (SM). But it also leaves some open questions like why do we have three generations of particles with such a strong hierarchy in the quark sector, why is the mixing in the quark and lepton sector so different or what is the structure of the Higgs sector.

In order to fix the open questions of the standard model, different approaches will be considered as illustrated in figure 1.1 (taken from [1]). They lead to extensions of the standard model such as supersymmetry, extra dimensions and $SO(10)$ grand unified theories.

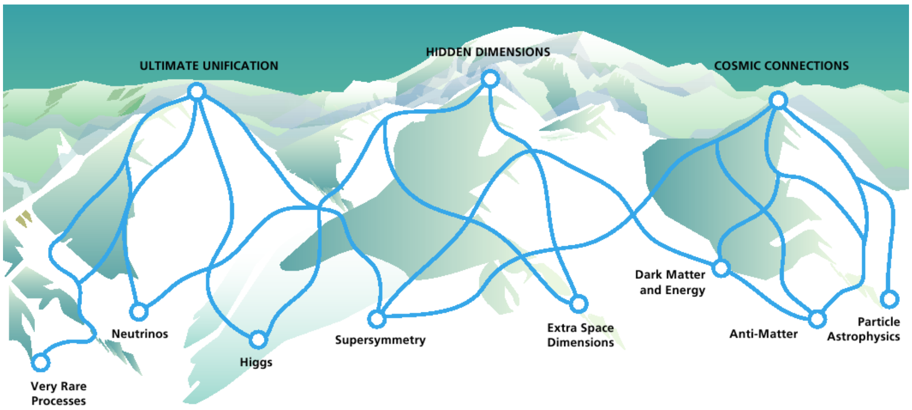

Figure 1.1: The Different Ways to Grand Unification.

In this thesis we will introduce a discrete flavor symmetry besides the known gauge groups of the standard model.

Our specific choice is D_5 which is founded in the general properties of a non-Abelian discrete symmetry and in particular in its interesting product structure which allows the product of two doublets to be decomposed into two singlets and one doublet as well as into two doublets. Since the product structure effects for example the mass matrices we hope to get new results. In addition, in the context of flavor symmetries, D_5 is until now only used to realize a certain texture for the neutrino mass matrix containing $SU(2)_L$ Higgs triplets [2].

In general, such extensions are multi-Higgs models. We therefore encounter a more complicated Higgs spectrum, accompanied by possible flavor changing neutral currents (FCNCs) and lepton flavor violation (LFV).

This work is structured as follows: In chapter 2 we shortly describe the standard model. Then in chapter 3 we want to provide an overview of discrete symmetries and in particular about discrete flavor symmetries. There we will also motivate our choice of the group D_5 and show how such a flavor symmetry affects the Higgs sector and consequently the mass matrices.

After this we begin to construct and discuss various Higgs sectors which are of phenomenological interest. This we will do in chapter 4–6.

In chapter 7 we summarize and give a short outlook for possible future work. The experimentally measured values of the standard model are shown in appendix A. A short introduction to group theory where we also explain basic notations can be found in appendix B. Properties of the chosen dihedral group D_5 are listed in appendix C. This includes the Clebsch-Gordan coefficients as well as certain mass matrices coming from D_5 invariant Yukawa couplings. The appendices D and E contain the calculated Higgs masses for the three and four Higgs models.

Chapter 2

The Standard Model and Beyond

2.1 The Standard Model of Particle Physics

Today we have, over some "detours" [3], the so-called standard model (SM) of particle physics. This model is based on continuous groups, the so-called Lie groups [4]. It is already very good in explaining the world and phenomena. But it also goes beyond this level and had given predictions we were (e.g. $W^{\pm}$ and Z boson) and (hopefully) will be (e.g. Higgs boson) able to measure.

For this work we are using without loss of generality the usual convention $\hbar = c = 1$.

2.2 Theory of the Standard Model

The SM (described in [5] and its quantum field theoretical basics in [6]) is a renormalizable [7] gauge theory [8] of the symmetry group

$$SU(3)_C \times SU(2)_L \times U(1)_Y \ , \tag{2.1}$$

where $SU(3)_C$ describes the strong interaction, the quantum chromodynamics (QCD), and $SU(2)_L \times U(1)_Y$ the Glashow-Weinberg-Salam model of the electroweak interaction. A more detailed description and derivation can be found e.g. in [3].

The electroweak theory $SU(2)_L \times U(1)_Y$ is spontaneously broken (SSB) to $U(1)_{em}$, the symmetry group of quantum electrodynamics (QED), by the Higgs mechanism described in 2.3.1:

$$SU(2)_L \times U(1)_Y \xrightarrow{SSB} U(1)_{em} \ . \tag{2.2}$$

An important property of this model is parity violation which is caused by the dependence of the transformation behavior of chirality on the fermions (for reviews on flavor physics and CP violation see for example [9, 10]). Any Dirac spinor Ψ can be decomposed in its right- and left-handed components, so-called Weyl spinors

$$\begin{aligned} \Psi &= \Psi_R + \Psi_L = R + L \ , \\ \Psi_{L\ ,R} &= \tfrac{1}{2}(1 \mp \gamma_5)\Psi \ . \end{aligned} \tag{2.3}$$

In the SM, left-handed (LH) leptons and quarks belong to $SU(2)_L$ doublets and the right-handed (RH) ones are singlets.

The electric charge Q of the particles arises from the hypercharge Y and the component T_3 of the weak isospin ($T_i = \frac{1}{2}\tau_i$, where i=1,2,3 are the generators of the $SU(2)_L$ Lie-algebra and τ_i the Pauli matrices) as follows

$$Q = T_3 + \frac{Y}{2} \; . \tag{2.4}$$

This relations is called the Gell-Mann-Nishijima relation.

For extensions of the SM it is common to take Weyl- instead of Dirac-spinors because this simplifies the structure of Yukawa couplings to a certain degree. Therefore we are using instead of a right-handed particle the left-handed component of the anti-particle. To get the anti-particle we have to use the charge-conjugation $\mathcal{C}$

$$\widehat{C}\,\Psi\,\widehat{C}^{-1} \equiv \Psi^c = \mathcal{C}\,\bar{\Psi}^T \quad \text{with } \mathcal{C} = i\,\gamma_0\,\gamma_2 \;, \quad (\Psi^c)_{L[R]} \equiv \left(\Psi_{R[L]}\right)^c \; . \tag{2.5}$$

Then we get the identities

$$\Psi_{R[L]} = \mathcal{C}\,(\bar{\Psi}^c_{L[R]})^T \quad \text{and} \quad \bar{\Psi}_{R[L]} = (\Psi^c_{L[R]})^T\,\mathcal{C} \; . \tag{2.6}$$

The fermions in the SM appear in three families each with the same quantum numbers. Every family transforms under the symmetry-group of the SM (G_{SM}), i.e.

$$\underbrace{(\mathbf{1},\mathbf{2},-1)}_{\text{LH Leptons}} \oplus \underbrace{(\mathbf{1},\mathbf{1},+2)}_{\text{RH Leptons}} \oplus \underbrace{(\mathbf{3},\mathbf{2},+1/3)}_{\text{LH Quarks}} \oplus \underbrace{(\bar{\mathbf{3}},\mathbf{1},-4/3)}_{\text{RH Up-Quarks}} \oplus \underbrace{(\bar{\mathbf{3}},\mathbf{1},+2/3)}_{\text{RH Down-Quarks}} \;, \tag{2.7}$$

where the first component denotes the representation under $SU(3)_C$, the second under $SU(2)_L$, the last component the hypercharge and the bar above the representation their adjoint. It

	Fermions			Quantum Numbers				
	1. Family	2. Family	3. Family	T	T_3	Y	Q	G_{SM}
Leptons	$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L$	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ $-\frac{1}{2}$	-1 -1	0 -1	$(\mathbf{1},\mathbf{2},-1)$
	e^c_L	μ^c_L	τ^c_L	0	0	2	1	$(\mathbf{1},\mathbf{1},+2)$
Quarks	$\begin{pmatrix} u \\ d \end{pmatrix}_L$	$\begin{pmatrix} c \\ s \end{pmatrix}_L$	$\begin{pmatrix} t \\ b \end{pmatrix}_L$	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ $-\frac{1}{2}$	$\frac{1}{3}$ $\frac{1}{3}$	$\frac{2}{3}$ $-\frac{1}{3}$	$(\mathbf{3},\mathbf{2},+\frac{1}{3})$
	u^c_L	c^c_L	t^c_L	0	0	$-\frac{4}{3}$	$-\frac{2}{3}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{4}{3})$
	d^c_L	s^c_L	b^c_L	0	0	$\frac{2}{3}$	$\frac{1}{3}$	$(\bar{\mathbf{3}},\mathbf{1},+\frac{2}{3})$

Table 2.1: Fermions in the Standard Model.

should be noted that in the SM no right-handed neutrinos are contained and that they transform as $(\mathbf{1},\mathbf{1},0)$. Therefore they would not take part at any SM gauge interaction. That is in agreement with experiments in which no right-handed neutrinos and left-handed anti-neutrinos are yet observed.

The gauge bosons, the exchange particles of the interactions are always in the adjoint representation of the symmetry-group and so for the SM in $(\mathbf{8},\mathbf{1},0) \oplus (\mathbf{1},\mathbf{3},0) \oplus (\mathbf{1},\mathbf{1},0)$.

The coupling strength of the gauge bosons with the fermionic currents $j^\mu = \bar{\Psi}\gamma^\mu\Psi$ are denoted by g', g_2 and g_3. In models with non-Abelian symmetry-groups like electroweak interactions, the gauge bosons have self-coupling. In extensions of the SM, like in grand unified theories (GUT), g' will normally replaced by $g_1 = \sqrt{\frac{5}{3}}g'$ which has then the correct normalization.

Bosons	G_{SM}	Spin	coupling
g $[SU(3)_C]$	$(\mathbf{8},\mathbf{1},0)$	1	g_3
W $[SU(2)_L]$	$(\mathbf{1},\mathbf{3},0)$	1	g_2
B $[U(1)_Y]$	$(\mathbf{1},\mathbf{1},0)$	1	$g_1 = \sqrt{\frac{5}{3}}g$'

Table 2.2: Gauge Bosons of the Standard Model .

2.3 Particles Acquire Mass

2.3.1 The Higgs Mechanism in the Standard Model

Let us now continue with the Higgs mechanism in the SM. For this we introduce a complex scalar doublet

$$\mathbf{\Phi} \equiv \begin{pmatrix} \Phi^+ \\ \Phi^0 \end{pmatrix} \sim (\mathbf{1},\mathbf{2},+1)\ , \tag{2.8}$$

corresponding to 4 real scalar fields and its Lagrangian

$$\mathscr{L}_{scalar} = (\partial_\mu \mathbf{\Phi}^\dagger)\,(\partial^\mu \mathbf{\Phi}) - V(\mathbf{\Phi}^\dagger\mathbf{\Phi})\ , \tag{2.9}$$

with the potential

$$V(\mathbf{\Phi}^\dagger\mathbf{\Phi}) = -\mu^2(\mathbf{\Phi}^\dagger\mathbf{\Phi}) + \lambda(\mathbf{\Phi}^\dagger\mathbf{\Phi})^2\ . \tag{2.10}$$

Analogous to quantum electrodynamics we introduce the covariant derivative to obtain the gauge invariance under $SU(2)_L \times U(1)_Y$, i.e.

$$\partial_\mu \to D_\mu = \partial_\mu + ig\frac{\tau^i}{2}W^i_\mu + i\frac{g'}{2}YB_\mu\ . \tag{2.11}$$

Now we choose the vacuum expectation value of the Higgs field to be

$$\langle\mathbf{\Phi}\rangle_0 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}v \end{pmatrix} \quad \text{, with } v = \sqrt{\frac{\mu^2}{\lambda}} \text{ and} \tag{2.12}$$

where we used the gauge freedom to rotate the VEV in a way that it gets real[1].

Through this spontaneous symmetry breaking only $U(1)_{em}$ is left as a symmetry of the vacuum:

$$SU(2)_L \times U(1)_Y \xrightarrow{SSB} U(1)_{em}\ .$$

In figure 2.1 we show the Higgs potential before and after the spontaneous symmetry breaking. If we now take into account that

[1] In models which contain more than one Higgs field this freedom can only be used to rotate one of the VEVs to be real. The others are in general complex.

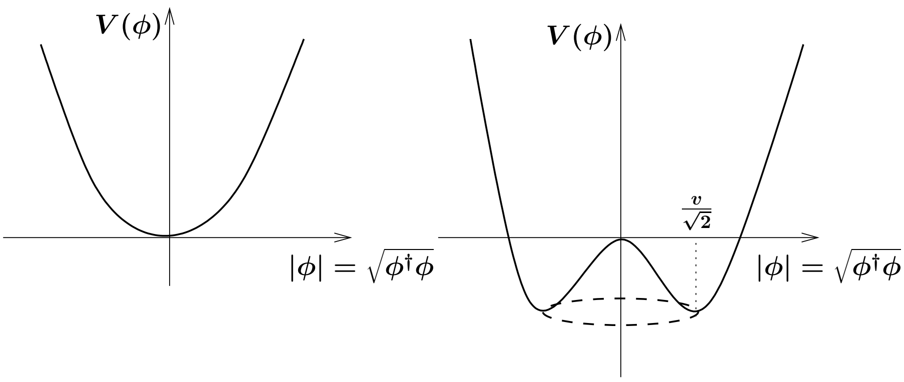

Figure 2.1: Higgs Potential of a Complex Scalar Field.

$$\frac{g}{2\sqrt{2}} = \left(\frac{m_W^2 G_F}{\sqrt{2}}\right)^{\frac{1}{2}} , \tag{2.13}$$

we get for the VEV

$$v = \left(\sqrt{2}G_F\right)^{\frac{1}{2}} \simeq 246\,\text{GeV} , \tag{2.14}$$

where G_F is the Fermi coupling constant. If we insert this parametrization in the first term of (2.9) with (2.11) we get from SSB that the photon γ remains massless while the $W^\pm$ and Z bosons acquire the following masses,

$$m_W = \tfrac{gv}{2} \sim 80\,\text{GeV} \quad \text{and} \quad m_Z = \tfrac{gv}{2\cos\theta_W} = \tfrac{m_W}{\cos\theta_W} \sim 90\,\text{GeV} \tag{2.15}$$

and a massive Higgs boson with Spin 0 appears with

$$m_H = \sqrt{2\mu^2} . \tag{2.16}$$

But since the value of μ^2 is a priori not determined by theory we do not have a prediction for the Higgs boson mass. However, experiments can give a upper and lower bound which are shown in (A.5).

2.3.2 Mass Terms

Massive Gauge Bosons

In this section we want to concentrate in more detail on mass terms. Before we have already seen how the gauge bosons (except the photon) acquire mass through the Higgs mechanism. Let us now again have a look at them from a different point of view:

$$\begin{aligned} \mathscr{L}_{scalar}^{W^3-B} &= \tfrac{v^2}{2}\left|\left(g\tfrac{\tau^3}{2}W_\mu^3 + \tfrac{g'}{2}YB_\mu\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right|^2 \\ &= \tfrac{v^2}{8}\left[\begin{pmatrix} B_\mu & W_\mu^3 \end{pmatrix}\begin{pmatrix} g'^2 & -gg' \\ -gg' & g^2 \end{pmatrix}\begin{pmatrix} B^\mu \\ W^{3\mu} \end{pmatrix}\right] \end{aligned} \tag{2.17}$$

The eigenvalues of this mass matrix are

$$0 \quad \text{and} \quad \tfrac{1}{2}\tfrac{\left(g^2+g'^2\right)}{4}v^2 = \tfrac{1}{2}m_Z , \tag{2.18}$$

which corresponds to the photon and to the Z boson mass. This diagonalization can be achieved through a transformation by the rotation about the Weinberg angle.

The underlying reason why some gauge bosons become massive is formulated in the

- *Goldstone theorem:*
  For every broken generator of a continuous, global symmetry appears a massless particle, a Goldstone boson.

But in the SM are the three GB which come from the SSB "eaten" by the gauge fields which in return become massive.

Now let us have a closer look at these Goldstone bosons as well as the massive Higgs boson. To do this we expand the Higgs potential which can consist of more then one Higgs field, about its minimum at $\phi_i = v_i$

$$V(\phi_i) = V(v_i) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_k}\right|_{\phi_i=v_i}(\phi_i - v_i)(\phi_k - v_k) + \dots \ . \tag{2.19}$$

As we can see, the mass matrix is

$$(M^2)_{ik} = \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_k}\right|_{\phi_i=v_i} \ . \tag{2.20}$$

By calculating now this mass matrix, we get a number of Goldstone bosons as well as massive Higgs bosons.

Nota bene:
In the SM, the VEVs and the other parameters in the Higgs potential are real. Then, the mass matrix for the neutral component will break-up into two matrices, one for the real part and one for the imaginary part.

If the VEVs (or some of the other parameters) are complex, like they are in general because we can choose through the gauge freedom only one VEV to be real, the real and imaginary part mix and we get therefore off-diagonal elements which disallows to split the mass matrix for the charged-neutral Higgs bosons into two separated matrices.

Massive Leptons

Our picture from the SM is nearly complete but we still have a problem: Until here the fermions are massless because the term

$$m_l \bar{l}l = m_l(\overline{R}L + \overline{L}R) \ , \tag{2.21}$$

where $L \equiv \begin{pmatrix} \nu \\ l \end{pmatrix}_L$ and R equivalent, would break gauge invariance and is therefore forbidden. But we can change this by keeping our gauge invariance through the couplings of leptons with the Higgs field, so-called Yukawa couplings Y_l:

$$\begin{aligned}
\mathscr{L}^l_{yuk} &= -Y_l\left[\overline{R}\left(\Phi^\dagger L\right) + \left(\overline{L}\Phi\right)R\right] \\
&= -Y_l\tfrac{v+H}{\sqrt{2}}\left[\overline{R}\begin{pmatrix} 0 & 1 \end{pmatrix}L + \overline{L}\begin{pmatrix} 0 \\ 1 \end{pmatrix}R\right] \\
&= -\tfrac{Y_l v}{\sqrt{2}}\bar{l}l - \tfrac{Y_l}{\sqrt{2}}\bar{l}lH \ ,
\end{aligned} \tag{2.22}$$

where H is the Higgs boson and L and R the left and right-handed fermions. So we see that the fermions acquire the mass, i.e.

$$m_l = \frac{Y_l v}{\sqrt{2}} \tag{2.23}$$

and their coupling strength to the Higgs is

$$Y_{\bar{l}Hl} = \frac{m_l}{v} \ . \tag{2.24}$$

But because v is a priori not known, this value must be measured by experiments and can then be used to test these predictions.

2.4 The Mixing Matrices

The basis we begin with is the most general renormalizable gauge-invariant coupling, i.e.

$$\mathscr{L}_m = -\lambda^d_{ij}\, Q_i{}^T\, \phi\, d^c_{Lj} - \lambda^u_{ij}\, \epsilon_{ab}\, Q_i^{aT}\, \phi^\dagger_b\, u^c_{Lj} \ , \tag{2.25}$$

where $Q_i = \begin{pmatrix} u^i \\ d^i \end{pmatrix}_L$ and $\lambda^{ij}_{u[d]}$ are general complex-valued matrices which do not have to be symmetric or Hermitian.

If we now apply a CP transformation on (2.25) we have to replace the operators by their Hermitian conjugate but let the coefficients be invariant. This means that

$$\lambda^{ij}_{u[d]} \to \left(\lambda^{ij}_{u[d]}\right)^* \tag{2.26}$$

would be a symmetry of (2.25) if the matrices $\left(\lambda^{ij}_{u[d]}\right)$ would be real valued. If this is not the case, we will have explicit CP violation.

However, the next step for finding the mixing matrix, i.e. for quarks the so-called CKM Matrix, is to define the unitary matrices U_u and W_u by

$$\lambda_u \lambda_u^\dagger = U_u D_u^2 U_u^\dagger \ , \qquad \lambda_u^\dagger \lambda_u = W_u D_u^2 W_u^\dagger \ , \tag{2.27}$$

where D_u^2 is a diagonal matrix with positive eigenvalues and equivalently we do it for the down-type quarks. From this it follows that

$$\lambda_u = U_u D_u W_u^\dagger \ , \tag{2.28}$$

where D_u is the diagonal matrix whose diagonal elements are the positive roots of the eigenvalues of (2.27). Thus if we associate them with the quark masses and combine them with the VEV, we get

$$m_u^i = \frac{1}{\sqrt{2}} D_u^{ii} v \ , \qquad m_d^i = \frac{1}{\sqrt{2}} D_d^{ii} v \ . \tag{2.29}$$

Now, through this we can construct the CKM mixing matrix as following:

$$V_{CKM} = U_u^T U_d^* \ . \tag{2.30}$$

(The usual parametrization for the CKM and PMNS matrices is given by (A.2) and (A.3).)

The strategy to get the PMNS matrix is similar to the one for the CKM matrix we have shown before. Thereby correspond λ_ν to λ_u and λ_e to λ_d respectively. There exist also some efforts to make a correlation between V_{CKM} and U_{PMNS}. One motivation is the relation $\theta_{sol} + \theta_C \simeq \frac{\pi}{4}$ which can be interpreted as an hint to quark-lepton complementary (see for example [11–14]).

For the PMNS matrix we want to mention here two popular forms because we will encounter them later in section 6.4.3 which can result from a $\mu\tau$–symmetry [15–18] in the mixing matrices, i.e. invariance under a permutation symmetry S_2 for the last two components ($\mu\tau$ generation). These are the so-called bimaximal and tri-bimaximal mixing patterns. They lead to a mixing angle $\theta_{13} = 0$ and $\theta_{23} = \frac{\pi}{4}$ and, in addition, for the tri-bimaximal mixing to $\sin^2\theta_{12} = \frac{1}{3}$ as we will see below. In addition, both do not lead to CP violation in neutrino oscillations because of $\theta_{13} = 0$.

- Bimaximal Mixing Pattern:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} , \quad \begin{array}{l} \theta_{13} = 0 \\ \theta_{23} = \frac{\pi}{4} \\ \theta_{12} = \frac{\pi}{4} \end{array} , \quad \text{no CP violation} \tag{2.31}$$

- Tri-bimaximal Mixing Pattern:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} , \quad \begin{array}{l} \theta_{13} = 0 \\ \theta_{23} = \frac{\pi}{4} \\ s_{12} = \frac{1}{\sqrt{3}} \end{array} , \quad \text{no CP violation} \tag{2.32}$$

The tri-bimaximal mixing pattern is difficult to understand only by Abelian discrete symmetries because unbroken lepton family symmetries only produce mixing matrices which are experimentally ruled out [19] and all Abelian representations are equivalent to diagonal representations which fixes the mixing angles to be zero or maximal, otherwise the angle depends on the free parameters of the model and can therefore take any value [20]. Other references are [21–24] and, for the tri-bimaximal mixing, [25–28].

Nota bene:
Mostly, it is used a base where the diagonalizing matrix for the charged-leptons is the unity matrix and so the matrix diagonalizing the neutrino mass matrix having a 2-3 symmetry is identical to the PMNS matrix.

2.5 Masses and Mass Matrices

To determine the parameters of the chosen mass matrices we have to fit them on the measured masses [29]. But the severity of this undertaking depends on the number of variables and texture zeros. And if some of the parameters are complex we have to consider $M_u M_u^\dagger$ (or $M_u^\dagger M_u$) whose eigenvalues are the squared quark masses. Therefore we need some constraints on our mass matrix. At best we use for this the strong mass hierarchy in combination with the trace of the mass matrix, i.e. for example for the up quarks

$$tr(M_u M_u^\dagger) \approx m_t^2 \ . \tag{2.33}$$

The trace is one of the so-called invariants so independent if we transform the mass matrix. Other invariants are the determinant so the product of the squared masses, e.g. for the up quarks

$$det\left(M_u M_u^\dagger\right) = \prod_i m_i^2 \tag{2.34}$$

with $i \in \{u, c, t\}$) and the Jarlskog invariant denoted as $\mathcal{J}$ or $\mathcal{J}_{CP}$ which is related to CP violation as we will see in the next section. For a good book about CP violation see [9].

2.6 Jarlskog Invariant & Co.

The Jarlskog invariant $\mathcal{J}$ describing the CP breaking observables is defined as follows [29–32]:

$$\mathcal{J} = \Im\left(V_{11} V_{22} V_{12}^* V_{21}^*\right) \;, \tag{2.35}$$

where V_{ij} is an element of the CKM or PMNS matrix. But we want to restrict us here on the quark sector because their CP violating phase is the one which is restricted by experiments ($\delta_{13} = 60° \pm 14°$). We also introduce the abbreviations $M_u M_u^\dagger \equiv H_u$ and $M_d M_d^\dagger \equiv H_d$. If we now build the commutator of these Hermitian matrices

$$[H_u, H_d] \equiv i\,\mathcal{C} \tag{2.36}$$

we will be able to relate it to the Jarlskog invariant in the following way [29].

$$det\,\mathcal{C} = -2\mathcal{J} \prod_{i<j}\left(m_i^2 - m_j^2\right) \prod_{k<l}\left(m_k^2 - m_l^2\right) \;, \tag{2.37}$$

where $i,\, j \in \{u,\, c,\, t\}$ and $k,\, l \in \{d,\, s,\, b\}$.

In this way we have connected our CP violating phases contained in $\mathcal{C}$ through our mass matrices with the measured values contained in the CKM matrix and therefore in $\mathcal{J}$.

2.7 Beyond the Standard Model

2.7.1 Open Questions

As we saw is the Standard Model (SM) already very good to explain the world of particles we are living in but there are still some questions remaining, e.g.:

- Why do we have three families of fermions and such a strong mass hierarchy?
- If the Higgs particle exists, what will be its mass?

These questions show us that the SM is good but not perfect. More about phenomenology beyond the SM can be found in [33]. There are a lot of ansätze trying to explain some or all of these questions, see for example [34, 35] or the grand unified theories (GUTs) like $SO(10)$ (for $SO(10)$ group theory needed for model building, see for instance [36]).

Our approach within this work will be the introduction of a discrete flavor symmetry or more precisely, of the dihedral group D_5. But this choice as well as other discrete symmetries we will elucidate in more detail in the chapter 3.

2.7.2 Right-Handed Neutrinos and the Seesaw Mechanism

As we already have pointed out, in the SM no right-handed neutrinos N_R are contained and consequently no mixings of left-handed neutrinos possible because they are assumed to be massless in the SM. A resume of neutrino theory and mixing is given in [37–40]. But if right-handed neutrinos exist, they can help us to understand why the left-handed neutrinos are so much lighter compared to the other particles. The keyword here is the so-called seesaw mechanism [41, 42] which exist in various forms: type-I, type-II or a mixture of both. Also double seesaw frameworks are considered [43].

This mechanism is related to the possible Majorana nature of neutrinos which implies that neutrinos are their own anti-particles. A Majorana mass term would have the form

$$-\frac{1}{2}m_M^L\left(\overline{\nu_L}\,(\nu_L)^c+\overline{(\nu_L)^c}\,(\nu_L)\right)-\frac{1}{2}m_M^R\left(\overline{N_R}\,(N_R)^c+\overline{(N_R)^c}\,(N_R)\right) \tag{2.38}$$

and therefore violates lepton number[2].

In addition, for the type-II seesaw a $SU(2)_L$ triplet Δ_L will be introduced which has the following interaction:

$$f_\Delta\,L\,L^T\,\Delta_L+h.c. \tag{2.39}$$

If we now combine Majorana and the usual Dirac masses we get the following Lagrangian written in matrix form:

$$\mathscr{L}_{mass}=-\frac{1}{2}\left(\overline{\nu_L}\,\overline{(N_R)^c}\right)\mathscr{M}\begin{pmatrix}(\nu_L)^c\\ N_R\end{pmatrix}+h.c.\;, \tag{2.40}$$

with the "seesaw" matrix

$$\mathscr{M}=\begin{pmatrix}m_M^L & m_D\\ m_D^T & m_M^R\end{pmatrix} \tag{2.41}$$

If we assume a hierarchy in the values of the elements, i.e.

$$m_M^R=M\gg m_D\gg m_M^L\;, \tag{2.42}$$

where m_M^L is negligible or zero in the type-I seesaw, then for the example of one generation one particle becomes light while the other one becomes heavy. This is illustrated in figure 2.2 and comes from the following equation:

$$m_\nu=m_M^L-m_D\,M^{-1}\,m_D^T\;. \tag{2.43}$$

So for the simple seesaw model containing three light and the same number of heavy neutrinos

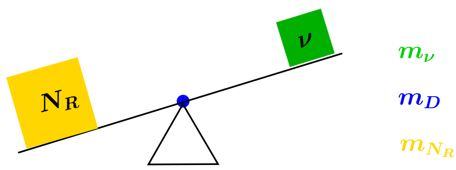

Figure 2.2: Seesaw Mechanism

[2] N_R exists in the most GUTs, as well as lepton number violation.

which are not massless the masses are (if all matrices would be diagonal)

$$\frac{m_{D1}^2}{M_1},\ \frac{m_{D2}^2}{M_2},\ \frac{m_{D3}^2}{M_3},\ M_1,\ M_2,\ M_3\ , \tag{2.44}$$

where m_i and M_i are the eigenvalues of the matrix M and m_D respectively. The heavy masses M_i can be degenerate or follow the same hierarchy as the Dirac masses.

A viable model for neutrino mixings and masses for a type-I and type-II seesaw scenario is given e.g. by [44] in a minimal formulation of an $SO(10)$ GUT and the running of the neutrino masses, leptonic mixings and CP violating phase in [45].

But aside from this also other models are conceivable. A classification is for instance given in [46].

Chapter 3

Discrete Flavor Symmetries

3.1 Discrete Groups

3.1.1 General Remarks on Discrete Groups

One of the main fields in which discrete groups are important is solid state physics, where the crystallographic point groups are used to describe phenomena related to crystals and in chemistry to describe the symmetries of atoms and molecules. There exist 32 crystallographic point groups characterized in table 3.1, where the so-called Schönflies symbols are used and the improper rotations have the same notation as the symmetric groups S_n, whereas they are not equivalent but we used it because this is the usual convention. A short introduction about grouptheory and concepts we are using is provided in appendix B. More about crystallographic point groups and Schönflies symbols can be found in every solid state book.

Type	Point Groups
Nonaxial	C_i , C_s
Cyclic	C_1 , C_2 , C_3 , C_4 , C_6
Cyclic with Horizontal Planes	C_{2h} , C_{3h} , C_{4h} , C_{6h}
Cyclic with Vertical Planes	C_{2v} , C_{3v} , C_{4v} , C_{6v}
Dihedral	D_2 , D_3 , D_4 , D_6
Dihedral with Horizontal Planes	D_{2v} , D_{3v} , D_{4v} , D_{6v}
Dihedral with Planes between Axes	D_{2d} , D_{3d}
Improper Rotation	S_4 , S_6
Cubic Groups	T , T_h , T_d , O , O_h

Table 3.1: Crystallographic Point Groups

A point group is a symmetry group which leaves at least one point unmoved. The requirement in crystallography that this symmetry is present on a lattice requires that only 1, 2, 3, 4 and 6-fold symmetry axes are possible. This restriction is the explanation why there exists 32 crystallographic point groups.

But in general, a discrete group is a topological group with a discrete topology. In practice, discrete groups often arise as discrete subgroups of continuous Lie groups acting on a geometric space but also appear naturally as symmetries of discrete structures (e.g. graphs, tilings, lattices), fundamental groups of topological spaces and so on.

In the following sections we want to introduce some of the discrete group which can be (and have already been) used as flavor symmetries as well as in other contexts.

3.1.2 Symmetric Groups S_n

The symmetric group S_n is the group of all permutations of n symbols. As an example we show a permutation of degree $n = 8$, namely

$$\begin{pmatrix} 12345678 \\ 23154768 \end{pmatrix} . \tag{3.1}$$

In this notation are the upper components in original order and the lower ones after the permutation, i.e. the permutation takes 1 into 2, 2 into 3, 3 into 1 and so on. Only the element 8 is unchanged.

This group is of great importance for mathematics as well as for physics. E.g. if we have a set of n identical particles the Hamiltonian will contain the group S_n. Another example is that it is easy to classify tensors into irreducible sets with respect to any group of linear transformations in n dimensions if the representations of the symmetric groups are known. The order of S_n is $n!$.

3.1.3 Alternating Groups A_n

The alternating group A_n contain all even[1] permutation of n elements, where even means that the number of performed permutation is even and is a subgroup of the symmetric group S_n. As an example we show an element of A_4, which is constructed through two permutation, e.g.

$$\left[\begin{pmatrix} 1234 \\ 1234 \end{pmatrix} \xrightarrow{1\to 3} \begin{pmatrix} 1234 \\ 3214 \end{pmatrix} \xrightarrow{2\to 3}\right] \begin{pmatrix} 1234 \\ 2314 \end{pmatrix} . \tag{3.2}$$

It therefore has $\frac{n!}{2}$ elements.

3.1.4 Cyclic Groups C_n

A cyclic group of finite group order n denoted as C_n is a group generated by a single element, the group generator, and it is Abelian. The generator A satisfies the relation

$$A^n = E , \tag{3.3}$$

where E is the identity. If we take again example (3.1) we can see that $1 \to 2$, $2 \to 3$ and 3 goes again into 1, so they are forming a "cycle". If we now write this as $(123) \equiv$ "$(1 \to 2 \to 3 \to 1)$", we can rewrite (3.1) as

$$(123)(45)(67)(8) \tag{3.4}$$

and see that it consist of four cycles, whereas the cycle (8) is trivial because it only consist of one element.

There exists a unique cyclic group of every order $n \geq 2$. Therefore cyclic groups of the same order are always isomorphic. Furthermore, subgroups of cyclic groups are again cyclic and all groups of prime group order too. In fact, the cyclic groups of order one or a prime

[1] Since the product of two even (or odd) permutations is even as well as the product of an even and an odd one, the odd permutation of degree n cannot form a group unlike even permutations.

are the only simple Abelian groups. Every Abelian group can be written as a direct product of cyclic subgroups, by computing the characteristic factors.

In addition the structure of the multiplication table of all cyclic groups is the same.

3.1.5 Icosahedral group I

The icosahedral group I is the point group of an icosahedron and dodecahedron and has order 120. This group is equivalent to $A_5 \times \mathbb{Z}_2$ and a subgroup of $SO(3)$. Its pure rotational subgroup I_h is of order 60 and isomorphic to A_5. The icosahedral group is in contrast to the following octahedral and tetrahedral group not observed in solid states.

3.1.6 Octahedral Group O

The symmetry group of the octahedron, cube, cuboctahedron and truncated octahedron is O which has the pure rotation subgroup O_h and has order 48. O_h has order 24 and is isomorphic to T.

3.1.7 Tetrahedral Group T

The tetrahedral groups T is the symmetry group of the tetrahedron plus the inversion operation and has order 24. Its isomorphic to the group $A_4 \times C_2$ and is therefore one of the 12 non-Abelian groups of order 24. The symmetry operations for this group are illustrated for a tetrahedron in figure 3.1.

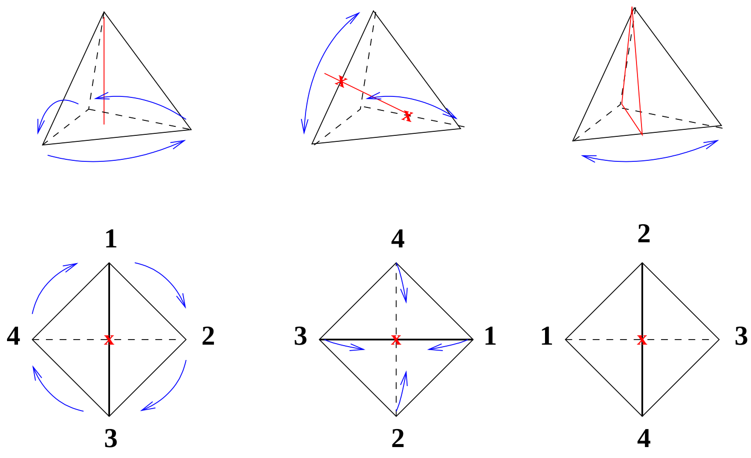

Figure 3.1: T_d, the Symmetry Group of a Tetrahedron

Thereby the first picture shows a rotation about the axis going through one corner and the center of the opposite side (3-fold). In the second one goes the axis through the midpoints of two edges (2-fold). The third figure shows the symmetry on a mirror plane going through two corners and the center of their opposite edge. The second row illustrate the improper rotations, resulting through a quarter rotation about the 2-fold axis described before and an inversion afterwards in order to get again a tetrahedron.

The pure rotational subgroup of T is denoted as T_d and is isomorphic to A_4. Its order is 12.

3.1.8 Dihedral Groups D_n

The dihedral groups D_n is the symmetry group of an n-sided regular polygon for $n > 1$. The group order is $2n$. They are non-Abelian rotation groups for $n > 2$. A two-dimensional reducible representation of D_n consisting of real matrices has a generator which is a rotation by π radiants around an axis passing through the center of the regular n-sided polygon and one of its vertices and another generator which is a rotation by $\frac{2\pi}{n}$ about the center. A visualization of the group D_5 which we will use as a flavor symmetry to build a model is shown in figure 3.2.

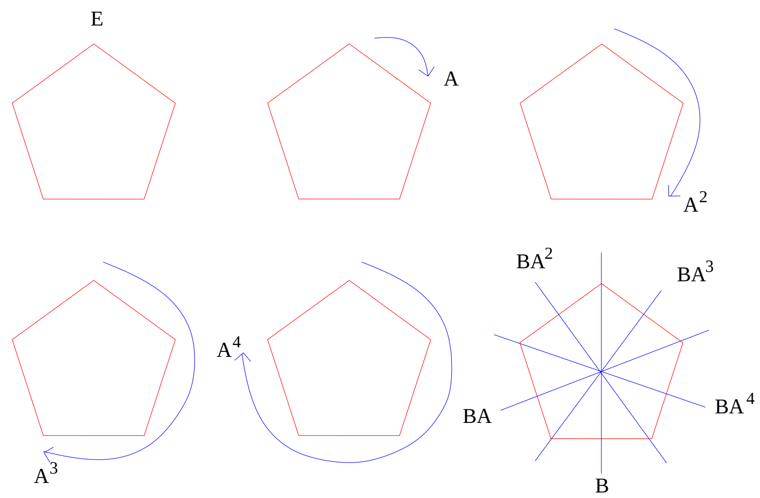

Figure 3.2: D_5, the Symmetry Group of a Regular Pentagon

3.1.9 Double Groups

The double groups descend from the point groups by adding the operation R which has the matrix representation $\pm\mathbb{1}_{n\times n}$ for a n dimensional representation. Their elements are called single-valued if the matrix representation of R is $+\mathbb{1}_{n\times n}$ otherwise double-valued. The order of the double group is twice the one of the originate group and the groups are denoted with a '. So the double group of D_n is D'_n and the one of T_n is T'_n. Another notation for D'_2 is Q and Q_{2n} for D'_n with $n \geq 3$. Since the single-valued representations are the same as the ones for single groups, the double-valued representations are new. For Abelian single groups the corresponding double group can (for example D_2) but do not have to be non-Abelian like it is the case for the cyclic groups C_n which double groups are isomorphic to C_{2n} and so Abelian.

3.2 Discrete Flavor Symmetries

A flavor-, family-, generation- or also called horizontal-symmetry is a symmetry which acts at the generation space and is in general broken at low energies.

But what is the maximal possible flavor symmetry compatible with the SM gauge groups? Because there are five particle assignments in the SM or six if we add right-handed neutrinos, i.e.

$$\begin{pmatrix} u_i \\ d_i \end{pmatrix}_L , \qquad u^c_{i\,L} , \qquad d^c_{i\,L} , \\ \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L , \qquad e^c_{i\,L} , \qquad N^c_{i\,L} , \tag{3.5}$$

where i denotes the flavor, the maximal possible flavor symmetry is $U(3)^6$ or $U(3)^5$ without $N^c_{i\,L}$. The reason is that there are three flavors (i=1,2 or 3) and that gauge interactions are flavor-blind. Therefore are all particles invariant under an $U(3)$ flavor symmetry. This independence of the flavor index can also be interpreted as the invariance of a permutation among the families, i.e. invariance under a S_3 symmetry. Some of the first papers that consider permutation symmetries are [47–54] or for a review [29]. An outdated model can be found e.g. in [55]. For extentsions of the SM we can get more restrictions like in a LR-symmetric model where left- and right-handed particles in a $SU(2)_{L[R]}$ doublet arranged and consequently the maximal flavor symmetry[2] is $U(3)^4$ or in a $SO(10)$ GUT are all particles assigned in a 16 dimensional representation and hence $G_f = U(3)$. A look at the Higgs sector in LR models is e.g. given by [56, 57].

Therefore the maximal possible flavor symmetry which we want to classify (and motivate with our choice of a D_5 symmetry) in the next sections is an $U(3)^6$ (if we assume the existence of right-handed neutrinos and no other extensions of the SM as LR-symmetric models etc.) or subgroups[3] of it.

The three generations of quarks and leptons form a three dimensional representation which can be reducible or irreducible. In general, in the SM left- and right-handed particles as well as quark and lepton generations can transform in different or in the same way. Furthermore, it is not necessary that all generations unify in one three dimensional representation but from the point of grand unification it would be desirable. An older example for such a model would be [58]. But also a combination of a singlet and a doublet is possible. If for example the top quark transforms as a singlet under G_f and the doublet contains the up- and charm-quark then this separation can serve as an explanation why the top-quark is so much heavier than the others. Also in supersymmetric models the unification of the first two generations is used, namely to suppress FCNC in the sfermion sector of these generations. An introductory literature to supersymmetry (SUSY) is provided by [59–63] and an examples to solve the SUSY flavor problem by an horizontal symmetry is [64].

In this thesis we consider an extension of the SM based on the discrete, non-Abelian flavor symmetry D_5. This symmetry as well as other discrete flavor symmetries can be used to favor a certain mass-matrix texture [29, 65–69] which is useful to describe the mixings or rather the mass hierarchy and/or make predictions for one or the other. Besides that, a flavor symmetry can help us to understand the existence of three generations. A possible argument for this is that they contain a finite number of irreducible representation with dimension usually smaller than four. It also commutes with the gauge groups in every model we know. Therefore the transformation properties of fermions under a flavor symmetry are equal under the gauge groups and the gauge bosons transform only trivially and it can be assumed in addition to a GUT, e.g. $SO(10) \times D_5$.

In general, a flavor symmetry can be continuous or discrete, Abelian or non-Abelian, global or local and can be broken in different ways. But other possibilities are also supposable like

[2] If the LR symmetry should not be broken by G_f.

[3] To avoid the breaking of G_f by gravitational quantum corrections for theories valid up to the Planck-scale, is the origin of it assumed to be a continuous gauge symmetry.

product groups. But here we want to restrict on the more apparent categories mentioned before. A very good and systematic classification of flavor symmetries is done in [70], where our flavor symmetry D_5 is denoted as the way mathematicians do it, i.e. $D_5 \Leftrightarrow$ type 10/2, where the first number denotes the group order and the second one is just a counter for the case that more than one group of the same order exits.

3.2.1 Continuous or Discrete Flavor Symmetries?

The first question arising with the search of a suitable flavor symmetry is if it should be a continuous one like the other groups of the SM or a discrete one like for crystals. The advantage of a discrete flavor symmetry is the already mentioned: They have a finite number of representations whose dimension usually is smaller than four because we only expect to unify three generations. In addition no further GB or gauge bosons arise what is not the case for continuous symmetries according to the Goldstone theorem. So our choice drops to a discrete symmetry.

3.2.2 Abelian or non-Abelian Flavor Symmetries?

For the question if we favor an Abelian or a non-Abelian flavor symmetry we have to think in the direction of GUTs. This means, that if we want (and we do) to unify at least two of the generations, we will favor a non-Abelian symmetry because it contains representations whose dimension is larger than one.

From the point of model building, discrete non-Abelian flavor symmetries have a further advantage over continuous ones: Most of them have several two or three dimensional representations which leaves more freedom for the particle assignment and therefore to embed experimental results like masses and mixings.

3.2.3 Local or Global Flavor Symmetries?

Now we have to decide if a flavor symmetry should be a local or a global one or in other words: Should G_f be gauged or not? To answer this question we first of all have to explain how a discrete flavor symmetry can be gauged.

For this we assume at high energies (e.g. GUT scale) a continuous group which is free of anomalies and which will be broken spontaneously to our residual discrete flavor symmetry. This continuous group can now be gauged in the conventional way. The problem with this is just that this can lead to heavy particles which have to be introduced to cancel possible anomalies coming from the known particles or strictly speaking their representation.

A model based on the double tetrahedral flavor symmetry T' and explaining in more detail what a gauged discrete symmetry is, is shown in [71].

3.2.4 Breaking a Flavor Symmetry

Before we numerate the different breaking mechanism of a flavor symmetry: Why do we have to break it at all? The reason is simply that we did not observe further symmetries than the ones in the SM at low energies so we have to break it before.

In principle we have two possibilities at our disposal, namely spontaneous (SSB) and explicit symmetry breaking.

A precondition for SSB is the existence of at least one Higgs boson transforming non-trivial under G_f and acquiring a VEV. Models using this mechanism (like ours) are in general multi-Higgs models and therefore have the problems associated with it like flavor changing neutral currents (FCNCs) and lepton flavor violation (LFV) which are strongly bounded by experiments.

Another important aspect is that the demand for invariance of the Higgs potential under G_f often yields a further symmetry of the Higgs potential, a so-called accidental symmetry and so to additional GBs which are not found in experiments. A solution for this problem is given in chapter 5, the so-called soft-breaking (SB).

3.2.5 Why exactly D_5 as flavor symmetry?

In the sections before we already narrowed our choice of a possible flavor symmetry down, but: Why D_5 and not for example D_3 or D_4?

To answer this we have to deal in more detail with the dihedral group. One aspect is that D_n for $n \in \{1, 2\}$ are Abelian groups. Another one is the number of different representations since this is important from the point of model building. Because if we have more than one possibility to accommodate our generations this will enhance our chances to build a model which is in agreement with measured values. It is also important if the representations include only singlets what will eliminate the possibility to unify at least two generations. For D_n we have to distinguish between even and odd n. If n is even, D_n has four singlets and $\frac{n}{2} - 1$ doublets. For odd n it has two one-dimensional and $\frac{n-1}{2}$ two-dimensional representations. For illustration we show in table 3.2 some dihedral groups and their representations.

Group	Singlets	Doublets
D_3	$\mathbf{1}_1$, $\mathbf{1}_2$	$\mathbf{2}$
D_4	$\mathbf{1}_1$, $\mathbf{1}_2$, $\mathbf{1}_3$, $\mathbf{1}_4$	$\mathbf{2}$
D_5	$\mathbf{1}_1$, $\mathbf{1}_2$	$\mathbf{2}_1$, $\mathbf{2}_2$
D_6	$\mathbf{1}_1$, $\mathbf{1}_2$, $\mathbf{1}_3$, $\mathbf{1}_4$	$\mathbf{2}_1$, $\mathbf{2}_2$
D_7	$\mathbf{1}_1$, $\mathbf{1}_2$	$\mathbf{2}_1$, $\mathbf{2}_2$, $\mathbf{2}_3$
D_8	$\mathbf{1}_1$, $\mathbf{1}_2$, $\mathbf{1}_3$, $\mathbf{1}_4$	$\mathbf{2}_1$, $\mathbf{2}_2$, $\mathbf{2}_3$

Table 3.2: Dihedral Groups and Their Representations

Here we can see that the groups D_5 and D_8 are the most promising ones since they contain more than one doublet. But since D_3 is isomorphic to S_3 this group is anyway discussed as shown in table 3.3. The same is true for D_4. The "missing" group D_6 is isomorphic to $S_3 \times C_2 = D_3 \times C_2$ and D_7 to type 14/2.

But the most important aspect for the choice of the group is the product structure of the groups which will affect the structure of the mass matrices. For the product of two doublets exist in general the following three possibilities:

$$\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1} + \mathbf{1} + \mathbf{1} \tag{3.6}$$
$$\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1} + \mathbf{2} \tag{3.7}$$
$$\mathbf{2} \times \mathbf{2} = \mathbf{2} + \mathbf{2} \tag{3.8}$$

Thereby D_3 possess the product structure 3.7, D_4 the one of 3.6 and D_5 3.7 and 3.8. The smallest group where all of the structures shown before appear is D_8.

Therefore we have the choice of the symmetries D_5 and D_8. Both containing as favored more than one doublet and possess further product structures which consequently affects the structure of the mass matrices we want to learn from. But our choice is D_5 as flavor symmetry since it has a manageable number of representations and product structures. In addition this group is used until now only once in the context of flavor symmetries [2]. Thereby the D_5 symmetry is used to realize a certain texture for the neutrino mass matrix containing $SU(2)_L$ Higgs triplets.

3.3 Models with Discrete Flavor Symmetries

In the last section we have shown some categories which can be used to distinguish models by means of their underlying flavor symmetry. But often not a whole model is proposed but only a part of it, e.g. to explain the quark or neutrino mixings. In the following we want to give a short list of some currently available models.

Models separated by their discrete flavor symmetry can be found in table 3.3. Another possibility is to classify them in the context they are used, e.g. SUSY, $SO(10)$ etc. This is done in [70]. But beside of taking only one symmetry also products of e.g. Abelian discrete symmetries are used to perform viable models [72–76]. And aside from this [77–81].

Discrete Symmetry	References
S_3	[55, 82–91]
A_4	[91–100]
D_4	[101, 102]
D_7	[103]
Q_8	[104]
T'	[58, 71]

Table 3.3: Models with Discrete Flavor Symmetries

An important aspect in a viable model is as we will see the Higgs potential. Therefore we want to give some references in which the Higgs potential of the probably most used flavor symmetry S_3 is discussed. These are for instance [47, 82, 88]. Similar potentials with non-Abelian discrete symmetries and their discussion can be found in [48, 49, 101, 105].

Another aspect is the usage of a discrete flavor symmetry in the context of extra dimensions (for introductory literature see for instance [106]). An example for this is [28], where a A_4 flavor symmetry is used.

3.4 D_5 Invariant Masses and Higgs Potential

3.4.1 D_5 Invariant Masses

Now that we have an idea about the SM and group theory, we can go on with our actual topic, the extension of the SM by the discrete flavor symmetry D_5. The content of this section is also shown in [107]. For this we denote $L = \{L_1, L_2, L_3\}$, where L_i is the i^{th} left-handed generation and L^C similar.

We will now show now how to calculate the general form of fermion mass matrices, where we use Clebsch-Gordan coefficients for complex generators:

1. As we already know, a mass term for fermions arises from Yukawa couplings. For Dirac particles they have the form
$$\lambda_{ij} L_i^T \epsilon \phi L_j^c \tag{3.9}$$
for charged leptons and down-type quarks, where
$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ in } SU(2)_L . \tag{3.10}$$
For neutrinos and up-type quarks we have to replace ϕ by its conjugate $\phi^c = \epsilon\phi^*$. The equivalent Yukawa term for Majorana particles is
$$\lambda_{ij} L_i^T \Delta_L L_j \ , \quad \text{with } \ \Delta_L = \begin{pmatrix} \xi^0 & -\frac{\xi^+}{\sqrt{2}} \\ -\frac{\xi^+}{\sqrt{2}} & \xi^{++} \end{pmatrix} \sim (\mathbf{1}, \mathbf{3}, +2) \tag{3.11}$$
and the mass term for the neutrinos
$$m_{ij}^R \nu_{iL}^c \nu_{jL}^c \ , \tag{3.12}$$
where m_{ij}^R is symmetric and M the matrix consisting of the elements m_{ij}.

2. The maybe best way to start this is to calculate all possible combinations[4] of
$$L^T \times L^c \ . \tag{3.13}$$

3. Then we take these terms and calculate every possible combination with ϕ which gives the singlet[5] $\mathbf{1}_1$ at the end:
$$\left(L^T \times L^c\right) \times \phi \ . \tag{3.14}$$

4. In the final step we arrange our results in a matrix form and we are done.

In the following we want to illustrate this proceeding on an example:

1. We are searching for the Dirac term of charged leptons and down-type quarks, i.e. (3.9) transforming e.g. as
$$\begin{aligned} &L \sim (\mathbf{1}_2, \mathbf{1}_1, \mathbf{1}_1) \ \text{ and} \\ &L^C \sim (\mathbf{2}_1, \mathbf{1}_1) \ . \end{aligned} \tag{3.15}$$

2. For this we search for the combinations[6] $L^T \times L^C$:
$$\begin{aligned} \mathbf{1}_1 \times \mathbf{1}_1 = \mathbf{1}_1 &\sim L_{1[2]}^T \times L_3^c \\ \mathbf{1}_1 \times \mathbf{2}_1 = \mathbf{2}_1 &\sim L_{1[2]}^T \times L_{1-2}^c \\ \mathbf{1}_2 \times \mathbf{1}_1 = \mathbf{1}_2 &\sim L_1^T \times L_3^c \\ \mathbf{1}_2 \times \mathbf{2}_1 = \mathbf{2}_1 &\sim L_1^T \times L_{1-2}^c \end{aligned} \tag{3.16}$$

[4] To break the flavor symmetry at weak scale as well as to constrain our arbitrary $\lambda_{ij}(m_{ij})$, the transformation behavior of L, L^c, ϕ and Ξ is assumed to be non-trivial.

[5] Each term has to transform as the trivial singlet 1_1 to ensure invariance under our D_5 flavor symmetry.

[6] All Kronecker products of D_5 are shown in table C.3.

3. Now we "add" one of the Higgs bosons transforming under D_5 as $\phi_{1[2]} \sim \mathbf{1}_{1[2]}$ or $\psi_{1[2]} \sim \mathbf{2}_{1[2]}$. But we only take terms having the trivial singlet $\mathbf{1}_1$ at the end.

$$\begin{aligned} \underbrace{\mathbf{1}_1}_{=\mathbf{1}_1\times\mathbf{1}_1} \times \underbrace{\mathbf{1}_1}_{\sim\phi_1} &= \mathbf{1}_1 \;, \\ \underbrace{\mathbf{1}_2}_{=\mathbf{1}_2\times\mathbf{1}_1} \times \underbrace{\mathbf{1}_2}_{\sim\phi_2} &= \mathbf{1}_1 \;, \\ \underbrace{\mathbf{2}_1}_{=\mathbf{1}_{1[2]}\times\mathbf{2}_1} \times \underbrace{\mathbf{2}_1}_{\sim\psi_1} &= \mathbf{1}_1\,(+\mathbf{1}_2+\mathbf{2}_2) \;, \end{aligned} \tag{3.17}$$

4. In the last step we now arrange the last results in matrix from. For this we have to exchange each "×" by the corresponding Clebsch-Gordan coefficient[7] in table C.5 and get as the final result

L	L^C	Mass Matrix
$(\mathbf{1}_2, \mathbf{1}_1, \mathbf{1}_1)$	$(\mathbf{2}_1, \mathbf{1}_1)$	$\begin{pmatrix} \kappa_1\psi_2^1 & -\kappa_1\psi_1^1 & \kappa_4\phi^2 \\ \kappa_2\psi_2^1 & \kappa_2\psi_1^1 & \kappa_5\phi^1 \\ \kappa_3\psi_2^1 & \kappa_3\psi_1^1 & \kappa_6\phi^1 \end{pmatrix}$

where we have absorbed the numerical prefactors we got from the CGs in the Yukawa couplings.

But the in this way obtained mass matrices shown in table C.6 are not specified in the way that we can interchange the generation assignment and so have to modify the mass matrices. This can be achieved by the following transformation depending if we switch the last two, the first and third or the first two generations, i.e.

$$M_{new} = Q\,M\,Q^T \;, \tag{3.18}$$

where the matrix Q is one of the following permutation matrices:

$$\begin{aligned} &\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \quad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \\ &\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \quad \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \quad \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}. \end{aligned} \tag{3.19}$$

[7] By using CGs for complex generators the multiplication is in general not commutative, e.g. $\mathbf{2}_1 \times \mathbf{2}_2 \neq \mathbf{2}_2 \times \mathbf{2}_1$, contrary by using CGs for real generators. Because for example $D(\mathbf{2}_1^*) = D(\mathbf{2}_1)^* \neq D(\mathbf{2}_1)$

3.4.2 D_5 Invariant Higgs Potentials

The procedure to get a D_5 invariant Higgs potential is very similar to the one we used to get invariant masses:

1. First we have to decide how our Higgs bosons transform under D_5.
2. Then we calculate all possible combinations of $\phi^\dagger \times \phi$ and separate those which have already lead to the trivial singlets because they are the terms in our Higgs potential of order two, the mass terms.
3. The next step is to find for each term from before all possible representations which can be added in order to obtain the trivial singlet.
4. And last but not least we have to identify the in this way received representations with all feasible products we got in step 2.
   Then we only have to join it together and our D_5 invariant potential is done.

And again an example:

1. For simplicity we assume that we only have two Higgs transforming as $\phi_1 \sim \mathbf{1}_1$ and $\phi_2 \sim \mathbf{1}_2$.
2. Then we build all combinations with them:
$$\begin{array}{c} \mathbf{1}_1 \times \mathbf{1}_1 = \mathbf{1}_1 \sim \phi_1 \times \phi_1 \\ \mathbf{1}_2 \times \mathbf{1}_2 = \mathbf{1}_1 \sim \phi_2 \times \phi_2 \\ \mathbf{1}_1 \times \mathbf{1}_2 = \mathbf{1}_2 \sim \phi_1 \times \phi_2 \\ \mathbf{1}_2 \times \mathbf{1}_1 = \mathbf{1}_2 \sim \phi_2 \times \phi_1 \end{array} \tag{3.20}$$
   and write the first two terms as the μ_1 and μ_2 term of our potential, where we use the CG from table C.4 to execute the group multiplication.
3. Now we search for factors we have to apply to the combinations from above to get $\mathbf{1}_1$.
$$\begin{array}{l} \overbrace{\begin{array}{c} \mathbf{1}_1 \times \mathbf{1}_1 \sim \phi_1 \times \phi_1 \\ \mathbf{1}_2 \times \mathbf{1}_2 \sim \phi_2 \times \phi_2 \end{array}}^{\mathbf{1}_1} \times \mathbf{1}_1 = \mathbf{1}_1 \\ \\ \overbrace{\begin{array}{c} \mathbf{1}_1 \times \mathbf{1}_2 \sim \phi_1 \times \phi_2 \\ \mathbf{1}_2 \times \mathbf{1}_1 \sim \phi_2 \times \phi_1 \end{array}}^{\mathbf{1}_2} \times \mathbf{1}_2 = \mathbf{1}_1 \end{array} \tag{3.21}$$
4. And for the last step we have to search which combinations give us $\mathbf{1}_1$ and $\mathbf{1}_2$. But this we can already see in step 2.
   Therefore our Higgs potential with two Higgs, transforming as $\phi_1 \sim \mathbf{1}_1$ and $\phi_2 \sim \mathbf{1}_2$, is:
$$\begin{aligned} V = &-\mu_1^2 \phi_1^\dagger \phi_1 - \mu_2^2 \phi_2^\dagger \phi_2 \\ &\lambda_1 (\phi_1^\dagger \phi_1)^2 + \lambda_2 (\phi_2^\dagger \phi_2)^2 + \lambda_3 (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + \\ &\left[\lambda_4 (\phi_1^\dagger \phi_2)^2 + h.c. \right] + \lambda_5 (\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_1) \ , \end{aligned} \tag{3.22}$$

where λ_4 is in general complex and for the multiplication of terms $(\phi_i^\dagger \phi_j)^T \times (\phi_k^\dagger \phi_l)$ we used the CGs in table C.5.

With the two Higgs potential (3.22) we now could start to construct a model. But before we get enthusiastic, let us have a look at the previously calculated mass matrices in table C.6. There we can see that by assuming only one or two Higgs bosons we are not able to explain the particle masses at tree-level and/or their mass hierarchy. Thus the Higgs potential (3.22) is too simple and we have to fix this what we will do in the chapter 4.

The most considered multi-Higgs model is the two Higgs doublet model (THDM or 2HDM) whose Higgs potential is not to mix up with the one we got in 3.22 from the D_5 symmetry. But with an additional $\mathbb{Z}_2$ symmetry, so $\phi_1 \to -\phi_1$ and $\phi_2 \to +\phi_2$ the general THDM pass into 3.22. For the THDM aspects like CP violation [108,109], symmetries [109], phenomenology [110] as well as other properties or perspectives [111–113] were considered.

Annotation:
A problem arising in multi-Higgs models are so-called flavor changing neutral currents (FCNC) [114]. In the SM, where only one Higgs doublet is contained, the Yukawa couplings are proportional to the fermion mass matrix. Therefore, the transformation diagonalizing the mass matrix also diagonalizes the Yukawa couplings. But in multi-Higgs models the Yukawa couplings are in general flavor changing [115], which is strongly bounded by experiments. One possibility to suppress this FCNC is to make the additional Higgs bosons sufficiently heavy ($\gtrsim$ 10 TeV).

Chapter 4

The Three Higgs Model

4.1 Three Higgs Potential

In the chapters before we have provided the tools which we will need to build a model. And we already derived the two Higgs potential but have to find out that this is phenomenologically not viable. Therefore we go to an improved Higgs potential which allows us describe the mass hierarchy at tree-level, i.e. we add a third Higgs boson.

In the model to be constructed we want that every single Higgs we are considering is a copy of the SM Higgs and therefore a $SU(2)_L$ doublet. Hence when we say that a Higgs transforms as a singlet or doublet we mean it with respect to our flavor symmetry D_5 in case nothing else is mentioned.

For three $SU(2)_L$ Higgs doublets from which one transforms[1] as a singlet (e.g. $\phi \sim 1_1$) and the other two as a doublet (e.g. $\psi = (\psi_1, \psi_2)^T \sim 2_1$) under D_5, where we used the CG from table C.5, the Higgs potential has the form.

$$\begin{aligned}
V_{3H} = \; & -\mu_1^2(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2) - \mu_2^2(\phi^\dagger\phi) \\
& +\lambda_1(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2)^2 + \lambda_2(\psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2)^2 \\
& +\lambda_3(\psi_1^\dagger\psi_2)(\psi_2^\dagger\psi_1) + \lambda_4(\phi^\dagger\phi)^2 \\
& +\sigma_1(\phi^\dagger\phi)(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2) + \sigma_2[(\phi^\dagger\psi_1)(\phi^\dagger\psi_2) + h.c.] \\
& +\sigma_3[(\phi^\dagger\psi_1)(\psi_1^\dagger\phi) + (\phi^\dagger\psi_2)(\psi_2^\dagger\phi)] \;,
\end{aligned} \tag{4.1}$$

in which we absorbed the phase ζ of σ_2 by a redefinition of ϕ with the phase $-\frac{\zeta}{2}$ without changing other parameters. Since every term containing ϕ also contain $\phi^\dagger$ they are invariant under a phase transformation of ϕ.

4.2 Minimization Conditions

Now we parametrize our Higgs doublets as

[1] The form of the Higgs potential is invariant by the exchange of $\phi \sim 1_1 \leftrightarrow 1_2$ and $\psi \sim 2_1 \leftrightarrow 2_2$. Therefore our choice of the singlet or the doublet is unimportant and consequently the Higgs potential for three Higgs is unique.

$$\phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}\left(\phi^r + i\phi^i\right) \end{pmatrix} , \quad \psi_1 = \begin{pmatrix} \psi_1^+ \\ \frac{1}{\sqrt{2}}\left(\psi_1^r + i\psi_1^i\right) \end{pmatrix} , \quad \psi_2 = \begin{pmatrix} \psi_2^+ \\ \frac{1}{\sqrt{2}}\left(\psi_2^r + i\psi_2^i\right) \end{pmatrix} \tag{4.2}$$

and insert this parametrization in V. There we define

$$\begin{array}{c} < \phi^r >= w \ , \\ < \psi_1 >= v_+ e^{i\alpha} = v_+ \cos\alpha + i\, v_+ \sin\alpha = \langle \psi_1^r \rangle + i \left\langle \psi_1^i \right\rangle \ , \\ < \psi_2 >= v_- e^{i\beta} = v_- \cos\beta + i\, v_- \sin\beta = \langle \psi_2^r \rangle + i \left\langle \psi_2^i \right\rangle \ , \end{array} \tag{4.3}$$

with real[2] w, v_+, v_-, α and β. For the case that $w = 0$ we choose the "next" VEV$\neq 0$ to be real, so α or β equal zero.

The resulting minimization conditions for the Higgs potential are

$$0 = \left.\frac{\partial V}{\partial \phi_r^0}\right|_{min} = [\lambda_4 w^2 + \tfrac{1}{2}(\sigma_1 + \sigma_3)(\, {v_+}^2 + \, {v_-}^2) + \sigma_2 \cos(\alpha+\beta)\, v_-\, v_+ - \mu_2^2] w \tag{4.4}$$

$$0 = \left.\frac{\partial V}{\partial \phi_i^0}\right|_{min} = \sigma_2\, v_+\, v_- w \sin(\alpha+\beta) \tag{4.5}$$

$$\begin{aligned} 0 = \left.\frac{\partial V}{\partial \psi_1^r}\right|_{min} = \ & [\tfrac{1}{2}(\sigma_1+\sigma_3)\cos\alpha\, v_+ + \tfrac{1}{2}\sigma_2\, v_- \cos\beta] w^2 + (\lambda_1+\lambda_2)\cos\alpha\, {v_+}^3 + \\ & ((\lambda_1 - \lambda_2 + \tfrac{1}{2}\lambda_3)\, {v_-}^2 - \mu_1^2)\, v_+ \cos\alpha \end{aligned} \tag{4.6}$$

$$\begin{aligned} 0 = \left.\frac{\partial V}{\partial \psi_1^i}\right|_{min} = \ & \tfrac{1}{2}[(\sigma_1+\sigma_3)\sin\alpha\, v_+ - \sigma_2\, v_- \sin\beta] w^2 + (\lambda_1+\lambda_2)\sin\alpha\ v_+^3 \\ & +[(\lambda_1 - \lambda_2 + \tfrac{1}{2}\lambda_3)\, {v_-}^2 - \mu_1^2] \sin\alpha\, v_+ \end{aligned} \tag{4.7}$$

$$\begin{aligned} 0 = \left.\frac{\partial V}{\partial \psi_2^r}\right|_{min} = \ & \left[(\lambda_1 - \lambda_2 + \tfrac{1}{2}\lambda_3)\, {v_+}^2 (\lambda_1+\lambda_2)\, {v_-}^2 - \mu_1^2\right]\ v_- \cos\beta \\ & +(\tfrac{1}{2}\sigma_2\, v_+ \cos\alpha + \tfrac{1}{2}(\sigma_1+\sigma_3)\cos\beta\, v_-) w^2 \end{aligned} \tag{4.8}$$

$$\begin{aligned} 0 = \left.\frac{\partial V}{\partial \psi_2^i}\right|_{min} = \ & \left[(\lambda_1 - \lambda_2 + \tfrac{1}{2}\lambda_3)\, {v_+}^2 + (\lambda_1+\lambda_2)\, {v_-}^2 - \mu_1^2\right] \sin\beta\, v_- \\ & +\tfrac{1}{2}(-\sigma_2 \sin\alpha\, v_+ + (\sigma_3+\sigma_1)\sin\beta\, v_-) w^2 \end{aligned} \tag{4.9}$$

By having a closer look at these equations we can see that (4.5) provides the most constraints for the classification of solutions since all equations have to be zero. E.g. if we demand that all Higgs acquire a non-vanishing VEV a necessary condition fulfill (4.5) is that $\sigma_2 = 0$ and/or $\sin(\alpha+\beta) = 0$. This we will see in the next section, where all solutions of the minimization conditions are listed.

[2]Because we can always rotate our system in a way that one VEV gets real; here we choose $< \phi >$ as real without loss of generality (w.l.o.g.).

4.3 Suitable VEV Structures

If we solve the minimization conditions, we get the following possibilities, subdivided into the ones in which one, two or all of the three Higgs acquire a VEV:

1. One Higgs acquire a VEV:

   (a) For $w \neq 0$:

   $w^2 = \frac{\mu_2^2}{\lambda_4}$

   (b) For $v_\pm \neq 0$:

   $v_\pm^2 = \frac{\mu_1^2}{\lambda_1+\lambda_2}$

2. Two Higgs acquire a VEV:

   (a) For $w = \alpha = 0$
   (Because of $w = 0$ we are free to choose another VEV, i.e. $\langle\psi_1\rangle$ or $\langle\psi_2\rangle$ to be real. We decided α to be 0 which has no effects or restrictions on β.)

   i. and $\lambda_3 = 4\lambda_2$:

   $v_+^2 + v_-^2 = \frac{\mu_1^2}{\lambda_1+\lambda_2}$

   ii. and $\lambda_3 \neq 4\lambda_2$:

   $v_+^2 = v_-^2 = \frac{2\mu_1^2}{4\lambda_1+\lambda_3}$

   (b) For $v_\pm = 0$ and $\sigma_2 = 0$:

   $w^2 = 2\,\frac{2(\lambda_1+\lambda_2)\mu_2^2-(\sigma_1+\sigma_3)\mu_1^2}{4\lambda_4(\lambda_1+\lambda_2)-(\sigma_1+\sigma_3)^2}$,

   $v_\mp^2 = 2\,\frac{2\lambda_4\mu_1^2-(\sigma_1+\sigma_3)\mu_2^2}{4\lambda_4(\lambda_1+\lambda_2)-(\sigma_1+\sigma_3)^2}$

3. All three Higgs acquire a VEV:
   (The following three cases $\sigma_2 = 0$, $\alpha = -\beta$ and $\alpha = \pi - \beta$ coming from minimization condition (4.5).)

   (a) For $\sigma_2 = 0$

   i. and $\lambda_3 = 4\lambda_2$:

   $w^2 = \frac{2\mu_1^2(\sigma_1+\sigma_3)-4\mu_2^2(\lambda_1+\lambda_2)}{(\sigma_1+\sigma_3)^2-4\lambda_4(\lambda_1+\lambda_2)}$,

   $v_+^2 + v_-^2 = \frac{-4\mu_1^2\lambda_4+2\mu_2^2(\sigma_1+\sigma_3)}{(\sigma_1+\sigma_3)^2-4\lambda_4(\lambda_1+\lambda_2)}$

ii. and $v_+^2 = v_-^2$:

$$w^2 = \frac{2\mu_1^2(\sigma_1\sigma_3)-\mu_2^2(4\lambda_1+\lambda_3)}{(\sigma_1+\sigma_3)^2-\lambda_4(4\lambda_1+\lambda_3)} \; ,$$

$$v_+^2 = v_-^2 = \frac{-2\mu_1^2\lambda_4+\mu_2^2(\sigma_1+\sigma_3)}{(\sigma_1+\sigma_3)^2-\lambda_4(4\lambda_1+\lambda_3)}$$

(b) For $\alpha = -\beta$

i. and $v_+^2 \neq v_-^2$:
(Here we give for simplicity the minimization conditions. But this case is anyway of minor interest because, as we will see from (4.4.2), we get more than the three wanted GBs.)

$$0 = \left.\frac{\partial V}{\partial \phi_r^0}\right|_{min} = \lambda_4\, w^2 + \tfrac{1}{2}(\sigma_1+\sigma_3)(v_+^2+v_-^2) + \sigma_2\, v_+ v_- - \mu_2^2 \; ,$$

$$0 = \left.\frac{\partial V}{\partial \psi_1^r}\right|_{min} = \left.\frac{\partial V}{\partial \psi_1^i}\right|_{min} =$$

$$[(\sigma_1+\sigma_3)\, v_+ + \sigma_2 v_-]\, w^2 + 2\,(\lambda_1+\lambda_2)\, {v_+}^3 + \left[(2\lambda_1 - 2\lambda_2 + \lambda_3)\, {v_-}^2 - 2\mu_1^2\right] v_+ \; ,$$

$$0 = \left.\frac{\partial V}{\partial \psi_2^r}\right|_{min} = \left.\frac{\partial V}{\partial \psi_2^i}\right|_{min} =$$

$$[(\sigma_1+\sigma_3)\, v_- + \sigma_2\, v_+]\, w^2 + 2\,(\lambda_1+\lambda_2)\, {v_-}^3 + \left[(2\,\lambda_1 - 2\,\lambda_2 + \lambda_3)\, {v_+}^2 - 2\mu_1^2\right] v_-$$

ii. and $v_+^2 = v_-^2$:

$$w^2 = -\frac{2(\sigma_1+\sigma_2+\sigma_3)\mu_1^2-(4\lambda_1+\lambda_3)\mu_2^2}{\lambda_4(4\lambda_1+\lambda_3)-(\sigma_1+\sigma_2+\sigma_3)^2} \; ,$$

$$v_+^2 = v_-^2 = -\frac{(\sigma_1+\sigma_2+\sigma_3)\mu_2^2-2\lambda_4\mu_1^2}{\lambda_4(4\lambda_1+\lambda_3)-(\sigma_1+\sigma_2+\sigma_3)^2}$$

(c) For $\alpha = \pi - \beta$:
This leads to case 3b with $\sigma_2 \to -\sigma_2$.

Apart from the restrictions we get from the minimization conditions we also have to consider the compulsory constraint that our potential has to be bounded from below. The conditions on the parameters of the Higgs potential which we get from this have to be calculated for each case. But we renounce on this now and come back to it later if we will need it.

4.4 Higgs Masses and Goldstone Bosons

4.4.1 General Considerations of Higgs Mass Matrices

Now we begin to calculate the masses for the Higgs bosons that we get from the 2^{nd} derivative of V, evaluated at the minimum, i.e. the squared mass matrices for the neutral component are

$$\left(M_{ri}^2\right)_{kl} = \left.\frac{\partial^2 V}{\partial\tilde{\phi}_k \partial\tilde{\phi}_l}\right|_{min} = 6\times 6\, matrix \; , \tag{4.10}$$

with $\tilde{\phi} = (\phi_r^0, \psi_{+r}^0, \psi_{-r}^0, \phi_i^0, \psi_{+i}^0, \psi_{-i}^0)^T$, according to the notation of (4.2). And equivalent for the charged components

$$\left(M^2_{pm}\right)_{kl} = \left.\frac{\partial^2 V}{\partial\acute{\phi}^+\partial\acute{\phi}^-}\right|_{min} = 3\times 3\, matrix\ , \tag{4.11}$$

with $\acute{\phi}^\pm = (\phi^\pm, \psi_1^\pm, \psi_2^\pm)^T$.

The mass eigenvalues of these matrices including the corresponding eigenvectors are shown in appendix D.

Nota bene:
In a models in which all VEVs and Yukawa couplings are real, like in the SM, the matrix M^2_{ri} is block diagonal. This fact makes it possible to calculate the masses for the scalars and pseudo-scalars separately, thus in our case with 3 Higgs we have two 3×3 matrices instead of one 6×6. The reason is that a complex parameter mixes these parts and we therefore get off-diagonal elements.

4.4.2 Symmetries and Goldstone-Bosons

Like we already mentioned, every Higgs in this model is a copy of the standard model Higgs, i.e.

$$\phi_{sm} = \begin{pmatrix} \phi^+_{sm} \\ \frac{1}{\sqrt{2}}\left(\phi_{sm}{}^0_r + i\phi_{sm}{}^0_i\right) \end{pmatrix} \tag{4.12}$$

in which the zero eigenvalue in the $\phi_{sm}{}^0_i$ base is "responsible" for the Z boson mass and the two eigenvalues in the $\phi^\pm$ base for the one of $W^\pm$. Thus we need 3 Goldstone-bosons, not more or less [3].

Because of this fact and the Goldstone-theorem 2.3.2, symmetries of a Higgs-potential are very important to get the "correct" number of Goldstone-bosons (one neutral and two charged ones) and not more. Therefore we analyze the symmetries of our Higgs potential in the following.

The maximal symmetry of our Higgs potential is

$$SU(2)_L \times U(1)_Y \times X \xrightarrow{VEV} U(1)_{em} \times Y\ , \tag{4.13}$$

where X is a further symmetry[4], broken by a VEV to Y ($X \to Y$). This means, that if at least one Higgs acquires a VEV which breaks X, we will get $3 + Z$ Goldstone-bosons, where Z is the number of broken generators of $X \to Y$ according to the Goldstone-theorem and therefore the number of additional, unwanted Goldstone-bosons.

As an example for the procedure how to determine an accidental symmetry and VEV configuration which break it we introduce for the Higgs the following charges ρ_i and do not make any assumption on parameters of the Higgs potential:

$$\begin{aligned} \phi &\to \phi\, e^{i\rho_1}\ , \\ \psi_1 &\to \psi_1\, e^{i\rho_2}\ , \\ \psi_2 &\to \psi_2\, e^{i\rho_3}\ . \end{aligned} \tag{4.14}$$

If we insert this transformation in the Higgs potential 4.1 we can see that every term except $\sigma_2[(\phi^\dagger\psi_1)(\phi^\dagger\psi_2) + h.c.]$ is invariant. If the charges fulfill the relation

$$-2\rho_1 + \rho_2 + \rho_3 = 0 \tag{4.15}$$

[3] But less is anyway not possible because of the breaking of $SU(2)_L \times U(1)_Y \to U(1)_{em}$, we get always at least three Goldstone-bosons according to the Goldstone theorem.

[4] But in general, X can "mix" with $SU(2)_L \times U(1)_Y$.

then also this term will be invariant. This means that all terms have an $U(1)^3$ symmetry except the one including σ_2 which possess an $U(1)^2$ symmetry (thereby is $U(1)_Y$ included). Now we introduce the following notation:

$$Q(\phi)^{\rho_1} \ , \tag{4.16}$$

which denotes the charge of ϕ under ρ_1. So in our case (where $\rho_3 = 2\rho_1 - \rho_2$) holds

$$\begin{array}{lcl lcl} Q(\phi)^{\rho_1} & = & 1\ , & Q(\phi)^{\rho_2} & = & 0\ , \\ Q(\psi_1)^{\rho_1} & = & 0\ , & Q(\psi_1)^{\rho_2} & = & 1\ , \\ Q(\psi_2)^{\rho_1} & = & 2\ , & Q(\psi_2)^{\rho_2} & = & -1\ . \end{array}$$

And if we define a new charge $\widehat{\rho_2}$ by $\widehat{\rho_2} = \rho_1 + \rho_2$ (and consequently, $\rho_3 = 3\rho_1 - \widehat{\rho_2}$) we can identify the charge ρ_1 as the one of $U(1)_Y$:

$$\begin{array}{lcl lcl} Q(\phi)^{\rho_1} & = & 1\ , & Q(\phi)^{\widehat{\rho_2}} & = & 0\ , \\ Q(\psi_1)^{\rho_1} & = & 1\ , & Q(\psi_1)^{\widehat{\rho_2}} & = & 1\ , \\ Q(\psi_2)^{\rho_1} & = & 1\ , & Q(\psi_2)^{\widehat{\rho_2}} & = & -1\ . \end{array} \tag{4.17}$$

There we can see if any of the Higgs acquires a VEV, $U(1)_Y$ will be broken as expected, and that a VEV of ψ_1 or ψ_2 would break the additional $U(1)$ symmetry denoted in the following as $U^{'''}(1)$.

For our three Higgs potential we have in general three possible cases of accidental symmetries (a more specified discussion is done in 4.5) which are shown below and where $Z = 0$ will hold if we have a VEV configuration which lets X unbroken:

- Without restrictions:
  (But none of the following cases)

$$X = U^{'''}(1) \xrightarrow{\langle\psi_{1[2]}\rangle} \emptyset \tag{4.18}$$

Z=0 or 1.

- For $\sigma_2 = 0$ and
  - if only ψ_1 or ψ_2 acquires a VEV:

$$X = U^{'}(1) \times U^{''}(1) \xrightarrow{\langle\psi_{1[2]}\rangle} U^{''[']}(1) \tag{4.19}$$

  - if ψ_1 and ψ_2 acquire a VEV:

$$X = U^{'}(1) \times U^{''}(1) \xrightarrow{\langle\psi_1\rangle,\langle\psi_2\rangle} \emptyset \tag{4.20}$$

Z=0, 1 or 2.

- For $\sigma_2 = 0 \wedge \lambda_3 = 4\,\lambda_2$ and
  - if only ψ_1 or ψ_2 acquires a VEV:

$$X = SU^{'}(2) \xrightarrow{\langle\psi_{1[2]}\rangle} U^{''[']}(1) \tag{4.21}$$

– if ψ_1 and ψ_2 acquire a VEV:

$$X = SU'(2) \xrightarrow{\langle\psi_1\rangle,\langle\psi_2\rangle} \emptyset \tag{4.22}$$

Z=0, 1, 2 or 3,

where "$\langle\psi_{1[2]}\rangle$" means, that the VEV of ψ_1 or ψ_2 breaks the symmetry and "$\langle\psi_1\rangle, \langle\psi_2\rangle$" that ψ_1 and ψ_2 need to acquire a VEV to break the symmetry. Thereby the Higgs doublet $(\psi_1, \psi_2)^T$ transforms under $SU'(2)$, ψ_1 under $U'(1)$, ψ_2 under $U''(1)$ and ψ_1 or ψ_2 under $U'''(1)$.

4.5 Phenomenologically Possible Structures

Now that we know the symmetries of the 3 Higgs potential and how they will be broken as well as the Higgs mass matrices, we can start to separate phenomenologically possible structures from disallowed ones.

To do this and because our model should be valid at tree-level, we will have to check if it is able to explain the mass hierarchy in the fermion sector without loop-corrections[5], i.e. we do not want eigenvalues to be zero or degenerate. If we now consider the mass-matrices for D_5 (table C.6) we will see that if the number of Higgs-bosons acquiring a VEV is less than 3, it will not be auspicious to succeed. That is the reason why we have begun with a three Higgs potential.

The detailed calculation of the eigenvalues and eigenvectors as well as the characteristic polynomials of the Higgs mass matrices are done in appendix D.

Now, let us summarize the results and subdivide them into the different cases, where we understand the symmetries as the ones of the Higgs potential:

Problems / Further Symmetries	**Number of Goldstone-bosons** (green [light gray] if 3 (what we want), red [dark gray] if $\neq 3$ (too many GBs))

1. One Higgs acquire a VEV:

   (a) for $w \neq 0$:

Mass hierarchy problem at tree-level and unbroken $U'''(1)$.	3

   (b) for $v_\pm \neq 0$:

Mass hierarchy problem at tree-level and unbroken $U'''(1)$.	3

2. Two Higgs acquire a VEV:

   (a) For $w = \alpha = 0$, $\sigma_2 = 0$
   (Because of $\langle\phi\rangle = 0$, the vacuum is independent of σ_2 and therefore we are free to choose $\sigma_2 = 0$.)

[5] That our model should be valid at tree-level is an assumption and not a necessity.

i. and $\lambda_3 = 4\lambda_2$:

Mass hierarchy problem at tree-level. 5
$SU'(2) \xrightarrow{\langle\psi_1\rangle,\langle\psi_2\rangle} U'''(1)$

ii. and $v_+^2 = v_-^2$:

Mass hierarchy problem at tree-level. 5
$U'(1) \times U''(1) \xrightarrow{\langle\psi_1\rangle,\langle\psi_2\rangle} \emptyset$

(b) and $v_\pm = 0$, $\sigma_2 = 0$:

Mass hierarchy problem at tree-level. 4
$U'(1) \times U''(1) \xrightarrow{\langle\psi_{1[2]}\rangle} U''^{[\prime]}(1)$

3. All Three Higgs acquire a VEV:

(a) For $\sigma_2 = 0$

i. and $4\,\lambda_2 = \lambda_3$:
$SU'(2) \xrightarrow{\langle\psi_1\rangle,\langle\psi_2\rangle} \emptyset$ 6

ii. and $v_+^2 = v_-^2$:
$U'(1) \times U''(1) \xrightarrow{\langle\psi_1\rangle,\langle\psi_2\rangle} \emptyset$ 5

(b) For $\alpha = -\beta \vee \alpha = \pi - \beta$

i. and $v_+^2 \neq v_-^2$:
$U'''(1) \xrightarrow{\langle\psi_{1[2]}\rangle} \emptyset$ 4

ii. and $v_+^2 = v_-^2$ (if $v_+ = -v_- : \sigma_2 \to -\sigma_2$):
$U'''(1) \xrightarrow{\langle\psi_{1[2]}\rangle} \emptyset$ 4

Here we can see that only one Higgs-boson can acquire a VEV to get the right number of GBs. But in these cases we get at tree-level at least one eigenvalue equal to 0 in the fermion mass sector. Therefore a "normal" three Higgs boson model under D_5 cannot work.

To fix this we have two possibilities which we will discuss in the next chapters: Adding so-called soft-breaking terms to our "normal" three Higgs potential and keeping in this way the number of Higgs bosons or to take a further Higgs boson into account, so consider a four Higgs model.

Chapter 5

Three Higgs Model with Soft-Breaking Terms

5.1 Soft-Breaking Terms

As mentioned before, one possibility to avoid accidental symmetries and consequently unobserved GB is by adding so-called soft-breaking (SB) terms. These terms are of the dimension two and therefore these mass terms are the "softest" operators we can work with in order not to loose the consistency and prediction of our D_5 model.

Sometimes it is quoted that such terms can for example drop out from a break-down of a GUT and then "disturb" our Higgs potential through additional terms in a way that no further accidental symmetry is left and only the $SU(2)_L \times U(1)_Y$ symmetry remains. But other terms can also remain and bother us. To avoid this, a further symmetry can be assumed which then bans these terms.

If we consider soft-breaking terms in our model, we will first of all have to find all possible terms of dimension two, even if they do not give the trivial singlet under D_5 (that is the breaking of our flavor symmetry) and add them to our "normal" three Higgs potential. To do this, the μ_1 term is replaced (because it is already included in these terms) by the following ones:

$$[\mu_3 \psi_1^\dagger \psi_2 + h.c] \ , \ \mu_4 \psi_1^\dagger \psi_1 \ , \ \mu_5 \psi_2^\dagger \psi_2 \ , \ [\mu_6 \phi^\dagger \psi_1 + h.c.] \text{ and } [\mu_7 \phi^\dagger \psi_2 + h.c.] \tag{5.1}$$

These terms now destroy our additional accidental symmetry as we will see below and we therefore expect no further Goldstone bosons than the three we get by $SU(2)_L \times U(1)_Y \rightarrow U(1)_{em}$. By taking the same notation for the charges as in (4.14) we get from the soft-breaking terms the following constraints:

$$\begin{aligned} \mu_3 : &\quad -\rho_2 + \rho_3 = 0 \ , \\ \mu_6 : &\quad -\rho_1 + \rho_2 = 0 \ , \\ \mu_7 : &\quad -\rho_1 + \rho_3 = 0 \ . \end{aligned} \tag{5.2}$$

$$\Rightarrow \rho_1 = \rho_2 = \rho_3$$

and consequently no accidental symmetry.

5.2 Three Higgs-Potential with Soft-Breaking

In the chapter before we have done a full discussion of the Higgs potential. But as we know, we need at least three Higgs with a non-vanishing VEV to explain the fermion masses. Therefore from now on we assume all VEVs to be unequal to zero.

With this assumption we can start discussing the three Higgs potential including soft-breaking terms which now has 18 real parameters and the following form:

$$\begin{aligned} V = &\ \cancel{-\mu_1^2(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2)} - \mu_2^2(\phi^\dagger\phi) \\ &+[\mu_3\psi_1^\dagger\psi_2 + h.c] + \mu_4\psi_1^\dagger\psi_1 + \mu_5\psi_2^\dagger\psi_2 + [\mu_6\phi^\dagger\psi_1 + h.c.] + [\mu_7\phi^\dagger\psi_2 + h.c.] \\ &+\lambda_1(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2)^2 + \lambda_2(\psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2)^2 + \lambda_3(\psi_1^\dagger\psi_2)(\psi_2^\dagger\psi_1) + \lambda_4(\phi^\dagger\phi)^2 \\ &+\sigma_1(\phi^\dagger\phi)(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2) + [\sigma_2(\phi^\dagger\psi_1)(\phi^\dagger\psi_2) + h.c.] \\ &+\sigma_3[(\phi^\dagger\psi_1)(\psi_1^\dagger\phi) + (\phi^\dagger\psi_2)(\psi_2^\dagger\phi)]\ , \end{aligned} \tag{5.3}$$

where the soft-breaking terms are colored blue (gray) and μ_3, μ_6, μ_7 as well as σ_2 are complex[1] and not real as the other coefficients.

Now let us make a separation of favored and disfavored cases. By parameter counting we cannot disfavor any of our mass matrices for 3 Higgs because if we have a look at them (table C.6), we can see that the number of Yukawa couplings varies from 3 to 5 and because there is no need that the up and down quarks have the same transformation behavior under D_5, we can choose one mass matrix having 5 and the the other one 3 Yukawa couplings which makes together with the two independent[2] VEVs 10 parameters like the number of parameters describing the quark sector which is shown in table 5.1. For example, if we had altogether 6 parameters it could be hard to describe the quark sector with its 10 parameters but this does not mean that it is impossible through a "piece of luck" or the correct theory.

Quark Masses	6
Mixing Angles	3
CP-Phase	1
Number of Parameters	10

Table 5.1: Parameter Counting in the Quark Sector

Now we go into more details for the example of the singlet $\phi_1 \sim \mathbf{1}_1$ and doublet $\psi_1 \sim \mathbf{2}_1$, where we will use the notation of table C.6. First of all, we eliminate all mass matrices containing a zero eigenvalue because our model should be valid at tree-level. Then only the matrices M_1, M_6, M_9, M_{11}, M_{12}, M_{14}, M_{15} and M_{18} are left over. But if we also call for some unification we can discard M_1 and M_6 because in these cases only two fermions of one chirality transform as a D_5 doublet and the rest as singlets. Furthermore the texture of M_{15}

[1] By a phase transformation of the Higgs fields it can be achieved that one of the complex coefficients will be real-valued without making real coefficients complex.

[2] We have to reduce the number of VEV by one because we have the constraint that the sum of the VEV squared must be the electroweak scale in order not to violate experimental facts, i.e. $\sum_i v_i^2 = (246\text{ GeV})^2$.

and M_{18} is phenomenologically unfavorable. E.g. to achieve one of the textures in [29], we have to set one of the Yukawa couplings or the VEVs to zero which then leads to a zero eigenvalue. In addition we only have 3 Yukawa couplings which can make it difficult to achieve the experimental bounds just because of the number of parameters. But this we will see in more detail in the next paragraph. Therefore only M_9, M_{11}, M_{12} and M_{14} are left as possible mass matrices, where all have the elegance of equal transformation of L and L^c under D_5 what e.g. can argue for LR-models. Apart from this M_{12} and M_{14} have a block structure which would e.g. separate the 3^{rd} generation from the first two and is in this way able to explain why the top quark is so much heavier than the other ones. For the other combination of a D_5 singlet and doublet we can see that ϕ_1 and ψ_2 lead to the same result as before, whereas this time M_9 and M_{11} are block diagonal. For ϕ_2, ψ_1 and ϕ_2, ψ_2 M_{10} and M_{13} are interesting candidates, where M_{13} has block structure for the first case and M_{10} for the second one. Again we want to remind the reader that the structure of the Higgs potential does not change its form by the choice of the composition of the three Higgs.

But our considerations are just qualitative. We cannot exclude something definitively before it is calculated in detail.

Chapter 6

The Four Higgs Model

6.1 Possibilities with Four Higgs

Another way to avoid unobserved GB can be adding of a further Higgs boson. This may also help us to get rid of the additional accidental symmetries, plus has the bonus that the model is self-consistent in the meaning that there are no additional terms “falling from the sky” like soft-breaking terms in the chapter before.

But in return, the potentials will have numerous parameters and are therefore not anymore analytic solvable. Only cases of special interest can be calculated and not anymore the whole potential in general, like we did before for three Higgs.

For four Higgs, in principle, three different possibilities exist to transform under D_5. But we can a priori exclude the case of four singlets since, as we already pointed out, the mass matrices containing only $\langle\phi_1\rangle$ and $\langle\phi_2\rangle$ have a zero eigenvalue and/or do not allow a description of the mass hierarchy. Then, the two possibilities left are:

1. One doublet and two singlets or
2. Two doublets

In the first case we have 24 real parameters and in the second one 20. This should be compared to the case before with soft-breaking terms where we had 18 real parameters.

6.2 One D_5 Doublet and Two Singlets

For this case we add to our original three Higgs potential (4.1) ($\phi_1 \sim 1_1$ and $\psi_{1[2]} \sim 2_{1[2]}$) the other D_5 Higgs singlet ($\phi_2 \sim 1_2$) and get the following Higgs potential, where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and can transform[1] as 2_1 or 2_2:

[1] The potential has the same form by taking the doublet 2_1 or 2_2.

$$
\begin{aligned}
V = \; & -\tfrac{1}{2}\mu_1^2(\psi_1^\dagger\psi_1+\psi_2^\dagger\psi_2)-\mu_2^2(\phi_1^\dagger\phi_1) \\
& +\tfrac{1}{4}\lambda_1(\psi_1^\dagger\psi_1+\psi_2^\dagger\psi_2)^2-\tfrac{1}{4}\lambda_2(\psi_1^\dagger\psi_1-\psi_2^\dagger\psi_2)^2 \\
& +\lambda_3(\psi_1^\dagger\psi_2)(\psi_2^\dagger\psi_1)+\lambda_4(\phi_1^\dagger\phi_1)^2 \\
& +\tfrac{1}{2}\sigma_1(\phi_1^\dagger\phi_1)(\psi_1^\dagger\psi_1+\psi_2^\dagger\psi_2)+\sigma_2[(\phi_1^\dagger\psi_1)(\phi_1^\dagger\psi_2)+h.c.] \\
& +\tfrac{1}{2}\sigma_3[(\phi_1^\dagger\psi_1)(\psi_1^\dagger\phi_1)+(\phi_1^\dagger\psi_2)(\psi_2^\dagger\phi_1)]+ \\
& \\
& +\tau_1\phi_2^\dagger\phi_2+\tau_2(\phi_2^\dagger\phi_2)^2 \\
& +\tau_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)+\tau_4(\phi_2^\dagger\phi_2)(\psi_1^\dagger\psi_1+\psi_2^\dagger\psi_2) \\
& +[\tau_5(\phi_2^\dagger\phi_1)^2+h.c.]+[\tau_6(\phi_2^\dagger\phi_1)(\psi_1^\dagger\psi_1-\psi_2^\dagger\psi_2)+h.c.] \\
& +\tau_7(\phi_2^\dagger\phi_1)(\phi_1^\dagger\phi_2)+[\tau_8(\psi_2^\dagger\phi_2)(\psi_1^\dagger\phi_2)+h.c.] \\
& +\tau_9[(\psi_2^\dagger\phi_2)(\phi_2^\dagger\psi_2)+(\psi_1^\dagger\phi_2)(\phi_2^\dagger\psi_1)] \\
& +[\tau_{10}[(\psi_2^\dagger\phi_2)(\phi_1^\dagger\psi_2)-(\psi_1^\dagger\phi_2)(\phi_1^\dagger\psi_1)]+h.c.] \\
& +[\tau_{11}[(\psi_2^\dagger\phi_2)(\psi_1^\dagger\phi_1)-(\psi_1^\dagger\phi_2)(\psi_2^\dagger\phi_1)]+h.c.]\;,
\end{aligned}
\tag{6.1}
$$

where the second half are the added terms. As mentioned before, this potential has 24 real parameters. If we have a closer look at it, we can see that there is still a further $U(1)$ remaining. With the charge assignment

$$
\begin{array}{l}
\phi_1 \to \phi_1\, e^{i\rho_1}\;, \\
\phi_2 \to \phi_2\, e^{i\rho_2}\;, \\
\psi_1 \to \psi_1\, e^{i\rho_3}\;, \\
\psi_2 \to \psi_2\, e^{i\rho_4}\;,
\end{array}
\tag{6.2}
$$

we obtain from the Higgs potential

$$
\begin{array}{rr}
\sigma_2: & -2\rho_1+\rho_3+\rho_4=0 \\
\tau_5,\,\tau_6,\,\tau_{10}: & \rho_1-\rho_2=0 \\
\tau_8: & 2\rho_2-\rho_3-\rho_4=0 \\
\tau_{11}: & \rho_1+\rho_2-\rho_3-\rho_4=0
\end{array}
\tag{6.3}
$$

and so can illustrate this accidental $U(1)$ symmetry by:

$$
\begin{array}{rclrcl}
Q(\phi_1)^{\rho_2} & = & 1\,, & Q(\phi_1)^{\rho_3} & = & 0\,, \\
Q(\phi_2)^{\rho_2} & = & 1\,, & Q(\phi_2)^{\rho_3} & = & 0\,, \\
Q(\psi_1)^{\rho_2} & = & 0\,, & Q(\psi_1)^{\rho_3} & = & 1\,, \\
Q(\psi_2)^{\rho_2} & = & 2\,, & Q(\psi_2)^{\rho_3} & = & -1\,.
\end{array}
\xrightarrow{\widehat{\rho_3}=\rho_2+\rho_3}
\begin{array}{rclrcl}
Q(\phi_1)^{\rho_2} & = & 1\,, & Q(\phi_1)^{\widehat{\rho_3}} & = & 0\,, \\
Q(\phi_2)^{\rho_2} & = & 1\,, & Q(\phi_2)^{\widehat{\rho_3}} & = & 0\,, \\
Q(\psi_1)^{\rho_2} & = & 1\,, & Q(\psi_1)^{\widehat{\rho_3}} & = & 1\,, \\
Q(\psi_2)^{\rho_2} & = & 1\,, & Q(\psi_2)^{\widehat{\rho_3}} & = & -1\,.
\end{array}
\tag{6.4}
$$

where $\rho_1 = \rho_2$ and $\rho_4 = 2\rho_2 - \rho_3$. Through the definition of $\widehat{\rho_3}$ we now can identify ρ_2 as the charge of $U(1)_Y$ and see that the additional symmetry will be broken if ψ_1 or ψ_2 acquires a VEV and is therefore unwanted. The reason is that we maximally can[2] give ϕ_1, ϕ_2 and ψ_i, with i=1 or 2, a VEV if we do not want to break the additional symmetry. But then we have problems with explaining the mass hierarchy and do not to get a mass eigenvalue equal to zero at tree-level from the mass matrices in table C.6.

Therefore this possibility will not help us to avoid additional Goldstone-bosons.

6.3 Two D_5 Doublets

The other possibility for four Higgs is to take the two doublets $\psi^1 = \begin{pmatrix} \psi_1^1 \\ \psi_2^1 \end{pmatrix} \sim \mathbf{2}_1$ and $\psi^2 = \begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix} \sim \mathbf{2}_2$ under D_5. The Higgs potential then has 20 real parameters and the following form:

$$
\begin{aligned}
V = \; & -\mu_1^2(\psi_1^{1\dagger}\psi_1^1 + \psi_2^{1\dagger}\psi_2^1) - \mu_2^2(\psi_1^{2\dagger}\psi_1^2 + \psi_2^{2\dagger}\psi_2^2) \\
& +\lambda_1(\psi_1^{1\dagger}\psi_1^1 + \psi_2^{1\dagger}\psi_2^1)^2 + \lambda_2(\psi_1^{1\dagger}\psi_1^1 - \psi_2^{1\dagger}\psi_2^1)^2 \\
& +\lambda_3(\psi_1^{1\dagger}\psi_2^1)(\psi_2^{1\dagger}\psi_1^1) + \lambda_4(\psi_1^{2\dagger}\psi_1^2 + \psi_2^{2\dagger}\psi_2^2)^2 \\
& +\lambda_5(\psi_1^{2\dagger}\psi_1^2 - \psi_2^{2\dagger}\psi_2^2)^2 + \lambda_6(\psi_2^{2\dagger}\psi_1^2)(\psi_1^{2\dagger}\psi_2^2) \\
& +\lambda_7(\psi_1^{1\dagger}\psi_1^1 + \psi_2^{1\dagger}\psi_2^1)(\psi_1^{2\dagger}\psi_1^2 + \psi_2^{2\dagger}\psi_2^2) + \lambda_8(\psi_1^{1\dagger}\psi_1^1 - \psi_2^{1\dagger}\psi_2^1)(\psi_1^{2\dagger}\psi_1^2 - \psi_2^{2\dagger}\psi_2^2) \\
& + \left[\lambda_9(\psi_1^{1\dagger}\psi_1^2)(\psi_2^{1\dagger}\psi_2^2) + h.c.\right] + \left[\lambda_{10}(\psi_1^{1\dagger}\psi_2^2)(\psi_2^{1\dagger}\psi_1^2) + h.c.\right] \\
& + \left[\lambda_{11}((\psi_1^{1\dagger}\psi_2^2)(\psi_1^{1\dagger}\psi_2^1) + (\psi_2^{1\dagger}\psi_1^2)(\psi_2^{1\dagger}\psi_1^1)) + h.c.\right] \\
& + \left[\lambda_{12}((\psi_1^{1\dagger}\psi_1^2)(\psi_2^{2\dagger}\psi_1^2) + (\psi_2^{1\dagger}\psi_2^2)(\psi_1^{2\dagger}\psi_2^2)) + h.c.\right] \\
& +\lambda_{13}[(\psi_1^{2\dagger}\psi_1^1)(\psi_1^{1\dagger}\psi_1^2) + (\psi_2^{2\dagger}\psi_2^1)(\psi_2^{1\dagger}\psi_2^2)] \\
& +\lambda_{14}[(\psi_2^{2\dagger}\psi_1^1)(\psi_1^{1\dagger}\psi_2^2) + (\psi_1^{2\dagger}\psi_2^1)(\psi_2^{1\dagger}\psi_1^2)]
\end{aligned}
\tag{6.5}
$$

6.3.1 Symmetries of the Higgs Potential

In the following we see that the potential (6.5) has no accidental symmetry anymore. This changes of course if we choose some parameters in an unfavorable way. E.g. if we choose $\lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = 0$ the Higgs potential has a further $U(1)^4$ symmetry. For this we

[2] If $\langle\psi_1\rangle = \langle\psi_2\rangle = 0$ we could also drop the doublet in order not to confuse the issue.

introduce the charges

$$\begin{aligned} \psi_1^1 &\to \psi_1^1\, e^{i\rho_1} \ , \\ \psi_2^1 &\to \psi_2^1\, e^{i\rho_2} \ , \\ \psi_1^2 &\to \psi_1^2\, e^{i\rho_3} \ , \\ \psi_2^2 &\to \psi_2^2\, e^{i\rho_4} \ , \end{aligned} \tag{6.6}$$

and obtain from the Higgs potential

$$\begin{aligned} \lambda_9: &\quad -\rho_1-\rho_2+\rho_3+\rho_4=0 \\ \lambda_{10}: &\quad -\rho_1-\rho_2+\rho_3+\rho_4=0 \\ \lambda_{11}: &\quad -2\rho_1+\rho_2+\rho_4=0 \\ &\quad \rho_1-2\rho_2+\rho_3=0 \\ \lambda_{12}: &\quad -\rho_1+2\rho_3-\rho_4=0 \\ &\quad -\rho_2-\rho_3+2\rho_4=0 \end{aligned} \tag{6.7}$$

This means that these terms break our accidental symmetry. All terms have a $U(1)$ symmetry, whereas the λ_9 and the λ_{10} term possess the same one. Therefore we have to discard cases which force this parameters to be zero (in the case of λ_9 and λ_{10} at least one of them). But we will revert to this in more detail in section 6.3.3. Theoretically it could be possible that a VEV configuration exists which does not break this accidental symmetry. Therefore we discuss in the following how they will be broken.

On the example of the λ_{11} term we can see that every VEV except the one of ψ_1^1 breaks the additional $U(1)$ symmetry:

$$\begin{array}{llll} Q(\psi_1^1)^{\rho_1} = 1\ , & Q(\psi_1^1)^{\rho_2} = 0\ , & & Q(\psi_1^1)^{\rho_1} = 1\ , \quad Q(\psi_1^1)^{\widehat{\rho_2}} = 0\ , \\ Q(\psi_2^1)^{\rho_1} = 0\ , & Q(\psi_2^1)^{\rho_2} = 1\ , & \xrightarrow{\widehat{\rho_2}=\rho_1+\rho_2} & Q(\psi_2^1)^{\rho_1} = 1\ , \quad Q(\psi_2^1)^{\widehat{\rho_2}} = 1\ , \\ Q(\psi_1^2)^{\rho_1} = -1\ , & Q(\psi_1^2)^{\rho_2} = 2\ , & & Q(\psi_1^2)^{\rho_1} = 1\ , \quad Q(\psi_1^2)^{\widehat{\rho_2}} = 2\ , \\ Q(\psi_2^2)^{\rho_1} = 2\ , & Q(\psi_2^2)^{\rho_2} = -1\ . & & Q(\psi_2^2)^{\rho_1} = 1\ , \quad Q(\psi_2^2)^{\widehat{\rho_2}} = -1\ . \end{array} \tag{6.8}$$

This we want to summarize below:

- $\lambda_{9/10}$ term:

$$\begin{aligned} U(1)^3 &\xrightarrow{\langle\psi_2^2\rangle} \emptyset \\ U(1)^3 &\xrightarrow{\langle\psi_2^1\rangle} U'(1) \\ U(1)^3 &\xrightarrow{\langle\psi_1^2\rangle} U''(1) \\ U(1)^3 &\xrightarrow{\langle\psi_2^1\rangle,\langle\psi_1^2\rangle} \emptyset \end{aligned} \tag{6.9}$$

- λ_{11} term:

$$U(1)^2 \xrightarrow{\langle\psi_2^1\rangle,\langle\psi_1^2\rangle \text{ or } \langle\psi_2^2\rangle} \emptyset \tag{6.10}$$

- λ_{12} term:

$$U(1)^2 \xrightarrow{\langle\psi_2^1\rangle,\langle\psi_1^2\rangle \text{ or } \langle\psi_2^2\rangle} \emptyset \tag{6.11}$$

As we can see we always break these symmetries if we use 2 or more VEVs. But this we have to do because of the fermion mass matrices. Therefore we have no other choice as to protect the $\lambda_{9[10]}$, λ_{11} and λ_{12} term in order not to get an accidental symmetry.

6.3.2 Minimization Conditions

Now parametrize the fields in the following way

$$\psi_k^l = \begin{pmatrix} \psi_k^{l\,+} \\ \frac{1}{\sqrt{2}}\left(\psi_k^{l\,r} + i\,\psi_k^{l\,i}\right) \end{pmatrix} , \tag{6.12}$$

where k, $l \in \{1, 2\}$ and the VEV are denoted as

$$\begin{aligned}
\langle\psi_1^1\rangle &= v_1^1 = v_1^1 + i\cdot 0 = \langle\psi_1^{1\,r}\rangle + i\,\langle\psi_1^{1\,i}\rangle \ , \\
\langle\psi_2^1\rangle &= v_2^1 e^{i\alpha} = v_2^1\cos\alpha + i\,v_2^1\sin\alpha = \langle\psi_2^{1\,r}\rangle + i\,\langle\psi_2^{1\,i}\rangle \ , \\
\langle\psi_1^2\rangle &= v_1^2 e^{i\beta} = v_1^2\cos\beta + i\,v_1^2\sin\beta = \langle\psi_1^{2\,r}\rangle + i\,\langle\psi_1^{2\,i}\rangle \ \text{ and} \\
\langle\psi_2^2\rangle &= v_2^2 e^{i\gamma} = v_2^2\cos\gamma + i\,v_2^2\sin\gamma = \langle\psi_2^{2\,r}\rangle + i\,\langle\psi_2^{2\,i}\rangle \ .
\end{aligned} \tag{6.13}$$

Then the first derivatives of V at the minimum are given as

$$\begin{aligned}
\left.\frac{\partial V}{\partial\psi_1^{1r}}\right|_{min} = \ & \tfrac{1}{2}\left(\lambda_7 - \lambda_8 + \lambda_{14}\right) v_1^1 (v_2^2)^2 \\
& + \Big[\Big(\tfrac{1}{2}\left((\lambda_9^r + \lambda_{10}^r)\cos(\alpha-\beta-\gamma) + (\lambda_9^i + \lambda_{10}^i)\sin(\alpha-\beta-\gamma)\right) v_1^2 \\
& + \left(-\lambda_{11}^i\sin(\alpha+\gamma) + \lambda_{11}^r\cos(\alpha+\gamma)\right) v_1^1\Big) v_2^1 + \\
& -\tfrac{1}{2}\left(\lambda_{12}^i\sin(2\beta-\gamma) + \lambda_{12}^r\cos(2\beta-\gamma)\right)(v_1^2)^2\Big] v_2^2 \\
& + \left[\tfrac{1}{2}\left(\lambda_{11}^i\sin(2\alpha-\beta) + \lambda_{11}^r\cos(2\alpha-\beta)\right) v_1^2 + \left(\lambda_1 - \lambda_2 + \tfrac{1}{2}\lambda_3\right) v_1^1\right](v_2^1)^2 \\
& + \tfrac{1}{2}\left(\lambda_7 + \lambda_8 + \lambda_{13}\right) v_1^1 (v_1^2)^2 + (\lambda_1 + \lambda_2)(v_1^1)^3 - {\mu_1}^2 v_1^1
\end{aligned} \tag{6.14}$$

$$\begin{aligned}
\left.\frac{\partial V}{\partial\psi_1^{1i}}\right|_{min} = \ & \Big[\Big(\tfrac{1}{2}\left((\lambda_9^i + \lambda_{10}^i)\cos(\alpha-\beta-\gamma) - (\lambda_9^r + \lambda_{10}^r)\sin(\alpha-\beta-\gamma)\right) v_1^2 \\
& + \left(\lambda_{11}^i\cos(\alpha+\gamma) + \lambda_{11}^r\sin(\alpha+\gamma)\right) v_1^1\Big) v_2^1 + \\
& \tfrac{1}{2}\left(\lambda_{12}^r\sin(2\beta-\gamma) + \lambda_{12}^i\cos(2\beta-\gamma)\right)(v_1^2)^2\Big] v_2^2 \\
& + \tfrac{1}{2}\left[-\lambda_{11}^i\cos(2\alpha-\beta) + \lambda_{11}^r\sin(2\alpha-\beta)\right] v_1^2 (v_2^1)^2
\end{aligned} \tag{6.15}$$

$$\left.\frac{\partial V}{\partial \psi_2^{1r}}\right|_{min} = \left[\tfrac{1}{2}\left(\lambda_7+\lambda_8+\lambda_{13}\right)v_2^1\cos\left(\alpha\right)+\tfrac{1}{2}\left(\lambda_{12}^r\cos\left(\beta-2\gamma\right)+\lambda_{12}^i\sin\left(\beta-2\gamma\right)\right)v_1^2\right]\left(v_2^2\right)^2$$
$$+\tfrac{1}{2}\left[-\lambda_{11}^i\sin\left(\gamma\right)\left(v_1^1\right)^2+\lambda_{11}^r\cos\left(\gamma\right)\left(v_1^1\right)^2+\right.$$
$$\left.\left(-(\lambda_9^i+\lambda_{10}^i)\sin\left(\beta+\gamma\right)+(\lambda_9^r+\lambda_{10}^r)\cos\left(\beta+\gamma\right)\right)v_1^1v_1^2\right]v_2^2$$
$$+\left[\left(\lambda_1+\lambda_2\right)\left(v_2^1\right)^3+\left(\tfrac{1}{2}\left(\lambda_7-\lambda_8+\lambda_{14}\right)\left(v_1^2\right)^2+\left(\lambda_1-\lambda_2+\tfrac{1}{2}\lambda_3\right)\left(v_1^1\right)^2-{\mu_1}^2\right)v_2^1\right]\cos\left(\alpha\right)$$
$$+\left(\lambda_{11}^i\sin\left(\alpha-\beta\right)+\lambda_{11}^r\cos\left(\alpha-\beta\right)\right)v_1^1v_1^2v_2^1 \tag{6.16}$$

$$\left.\frac{\partial V}{\partial \psi_2^{1i}}\right|_{min} = \tfrac{1}{2}\left[\left(\lambda_7+\lambda_8+\lambda_{13}\right)v_2^1\sin\left(\alpha\right)+\left(-\lambda_{12}^r\sin\left(\beta-2\gamma\right)+\lambda_{12}^i\cos\left(\beta-2\gamma\right)\right)v_1^2\right]\left(v_2^2\right)^2$$
$$+\tfrac{1}{2}\left[-\lambda_{11}^r\sin\left(\gamma\right)\left(v_1^1\right)^2-\lambda_{11}^i\cos\left(\gamma\right)\left(v_1^1\right)^2+\right.$$
$$\left.\left((\lambda_9^i+\lambda_{10}^i)\cos\left(\beta+\gamma\right)+(\lambda_9^r+\lambda_{10}^r)\sin\left(\beta+\gamma\right)\right)v_1^1v_1^2\right]v_2^2$$
$$+\left[\left(\lambda_1+\lambda_2\right)\left(v_2^1\right)^3+\left(\tfrac{1}{2}\left(\lambda_7-\lambda_8+\lambda_{14}\right)\left(v_1^2\right)^2+\right.\right.$$
$$\left.\left.\left(\lambda_1-\lambda_2+\tfrac{1}{2}\lambda_3\right)\left(v_1^1\right)^2-{\mu_1}^2\right)v_2^1\right]\sin\left(\alpha\right)$$
$$+\left[\lambda_{11}^i\cos\left(\alpha-\beta\right)-\lambda_{11}^r\sin\left(\alpha-\beta\right)\right]v_1^1v_1^2v_2^1 \tag{6.17}$$

$$\left.\frac{\partial V}{\partial \psi_1^{2r}}\right|_{min} = \left[\left(\lambda_4-\lambda_5+\tfrac{1}{2}\lambda_6\right)v_1^2\cos\left(\beta\right)+\tfrac{1}{2}\left(\lambda_{12}^i\sin\left(\alpha-2\gamma\right)+\lambda_{12}^r\cos\left(\alpha-2\gamma\right)\right)v_2^1\right]\left(v_2^2\right)^2$$
$$+\left[\tfrac{1}{2}\left((\lambda_9^r+\lambda_{10}^r)\cos\left(\alpha-\gamma\right)+(\lambda_9^i+\lambda_{10}^i)\sin\left(\alpha-\gamma\right)\right)v_1^1v_2^1\right.$$
$$\left.+\left(-\lambda_{12}^i\sin\left(\beta-\gamma\right)+\lambda_{12}^r\cos\left(\beta-\gamma\right)\right)v_1^1v_1^2\right]v_2^2$$
$$+\left[\tfrac{1}{2}\left(\lambda_7-\lambda_8+\lambda_{14}\right)v_1^2(v_2^1)^2+\left(\lambda_4+\lambda_5\right)\left(v_1^2\right)^3+\right.$$
$$\left.\left(\tfrac{1}{2}\left(\lambda_7+\lambda_8+\lambda_{13}\right)\left(v_1^1\right)^2-{\mu_2}^2\right)v_1^2\right]\cos\left(\beta\right)$$
$$+\tfrac{1}{2}\left(\lambda_{11}^i\sin\left(2\alpha\right)+\lambda_{11}^r\cos\left(2\alpha\right)\right)v_1^1(v_2^1)^2 \tag{6.18}$$

$$
\begin{aligned}
\left.\frac{\partial V}{\partial \psi_1^{2i}}\right|_{min} = \; & \left[\left(\lambda_4 - \lambda_5 + \tfrac{1}{2}\lambda_6\right) v_1^2 \sin(\beta) + \tfrac{1}{2}\left(\lambda_{12}^i \cos(\alpha - 2\gamma) - \lambda_{12}^r \sin(\alpha - 2\gamma)\right) v_2^1\right] (v_2^2)^2 \\
& + \left[\tfrac{1}{2}\left(-(\lambda_9^i + \lambda_{10}^i)\cos(\alpha - \gamma) + (\lambda_9^r + \lambda_{10}^r)\sin(\alpha - \gamma)\right) v_1^1 v_2^1\right. \\
& \left. + \left(-\lambda_{12}^r \sin(\beta - \gamma) - \lambda_{12}^i \cos(\beta - \gamma)\right) v_1^1 v_1^2\right] v_2^2 \\
& + \left[\tfrac{1}{2}(\lambda_7 - \lambda_8 + \lambda_{14}) v_1^2 (v_2^1)^2 + (\lambda_4 + \lambda_5)(v_1^2)^3 + \left(\tfrac{1}{2}(\lambda_7 + \lambda_8 + \lambda_{13})(v_1^1)^2 - {\mu_2}^2\right) v_1^2\right] \sin(\beta) \\
& + \tfrac{1}{2}\left[\lambda_{11}^r \sin(2\alpha) - \lambda_{11}^i \cos(2\alpha)\right] v_1^1 (v_2^1)^2
\end{aligned} \tag{6.19}
$$

$$
\begin{aligned}
\left.\frac{\partial V}{\partial \psi_2^{2r}}\right|_{min} = \; & (\lambda_4 + \lambda_5)\cos(\gamma)(v_2^2)^3 + \\
& \left[\left(\tfrac{1}{2}(\lambda_7 + \lambda_8 + \lambda_{13}) {v_2^1}^2 + \left(\lambda_4 - \lambda_5 + \tfrac{1}{2}\lambda_6\right)(v_1^2)^2 + \tfrac{1}{2}(\lambda_7 - \lambda_8 + \lambda_{14})(v_1^1)^2 - {\mu_2}^2\right) \cos(\gamma)\right. \\
& \left. + \left(\lambda_{12}^i \sin(-\gamma + \beta + \alpha) + \lambda_{12}^r \cos(-\gamma + \beta + \alpha)\right) v_1^2 v_2^1\right] v_2^2 \\
& \tfrac{1}{2}\left[-\lambda_{11}^i \sin(\alpha) v_2^1 (v_1^1)^2 + \lambda_{11}^r \cos(\alpha) v_2^1 (v_1^1)^2 + \left(-\lambda_{12}^i \sin(2\beta) + \lambda_{12}^r \cos(2\beta)\right) v_1^1 (v_1^2)^2\right] \\
& + \tfrac{1}{2}\left[(\lambda_9^r + \lambda_{10}^r)\cos(\alpha - \beta) + (\lambda_9^i + \lambda_{10}^i)\sin(\alpha - \beta)\right] v_1^1 v_1^2 v_2^1
\end{aligned} \tag{6.20}
$$

$$
\begin{aligned}
\left.\frac{\partial V}{\partial \psi_2^{2i}}\right|_{min} = \; & (\lambda_4 + \lambda_5)\sin(\gamma)(v_2^2)^3 \\
& + \left[\left(\tfrac{1}{2}(\lambda_7 + \lambda_8 + \lambda_{13})(v_2^1)^2 + \left(\lambda_4 - \lambda_5 + \tfrac{1}{2}\lambda_6\right)(v_1^2)^2 + \tfrac{1}{2}(\lambda_7 - \lambda_8 + \lambda_{14})(v_1^1)^2 - {\mu_2}^2\right) \sin(\gamma)\right. \\
& \left. + \left(\lambda_{12}^r \sin(\alpha + \beta - \gamma) - \lambda_{12}^i \cos(\alpha + \beta - \gamma)\right) v_1^2 v_2^1\right] v_2^2 \\
& \tfrac{1}{2}\left[-\lambda_{11}^r \sin(\alpha) v_2^1 (v_1^1)^2 - \lambda_{11}^i \cos(\alpha) v_2^1 (v_1^1)^2 + \left(\lambda_{12}^i \cos(2\beta) + \lambda_{12}^r \sin(2\beta)\right) v_1^1 (v_1^2)^2\right] \\
& + \tfrac{1}{2}\left[-(\lambda_9^i + \lambda_{10}^i)\cos(\alpha - \beta) + (\lambda_9^r + \lambda_{10}^r)\sin(\alpha - \beta)\right] v_1^1 v_1^2 v_2^1
\end{aligned} \tag{6.21}
$$

Here we used the notation $\lambda_j = \lambda_j^r + i\,\lambda_j^i$ to split the complex Yukawa couplings into their real and imaginary parts.

Since the potential has a very complicated structure we cannot make a full discussion of it. But e.g. if we assume that all VEVs are real we can make some predictions.

6.3.3 Restrictions on the Potential

A systematic investigation of meaningful VEV structures leads to the result that certain ones are phenomenologically forbidden because they force some parameters to be zero[3] which leads to accidental symmetries. This VEV structures are summarized for real VEVs, i.e. $\alpha = \beta = \gamma = 0$ in table 6.1 and the origin is shown below:

- $v_1^1 = v_2^2 = 0$:

$$\begin{array}{lcl} (6.14) & \Rightarrow & \frac{1}{2}v_1^2(v_2^1)^2\lambda_{11}^r = 0 \\ (6.15) & \Rightarrow & -\frac{1}{2}v_1^2(v_2^1)^2\lambda_{11}^i = 0 \end{array}$$

- $v_1^1 = v_1^2 = 0$:

$$\begin{array}{lcl} (6.18) & \Rightarrow & \frac{1}{2}v_2^1(v_2^2)^2\lambda_{12}^r = 0 \\ (6.19) & \Rightarrow & \frac{1}{2}v_2^1(v_2^2)^2\lambda_{12}^i = 0 \end{array}$$

- $v_2^1 = v_1^2 = 0$:

$$\begin{array}{lcl} (6.16) & \Rightarrow & \frac{1}{2}v_2^2(v_1^1)^2\lambda_{11}^r = 0 \\ (6.17) & \Rightarrow & -\frac{1}{2}v_2^2(v_1^1)^2\lambda_{11}^i = 0 \end{array}$$

- $v_2^1 = v_2^2 = 0$:

$$\begin{array}{lcl} (6.20) & \Rightarrow & \frac{1}{2}v_1^1(v_1^2)^2\lambda_{12}^r = 0 \\ (6.21) & \Rightarrow & \frac{1}{2}v_1^1(v_1^2)^2\lambda_{12}^i = 0 \end{array}$$

All of them lead to a further $U(1)$ symmetry which would be broken if the remaining VEVs were unequal to zero and therefore to unwanted GB.

VEV Structure	Causes	Accidental Symmetry
$v_1^1 = v_2^2 = 0$	$\lambda_{11}^r = \lambda_{11}^i = 0$	$U'(1)$
$v_1^1 = v_1^2 = 0$	$\lambda_{12}^r = \lambda_{12}^i = 0$	$U''(1)$
$v_2^1 = v_1^2 = 0$	$\lambda_{11}^r = \lambda_{11}^i = 0$	$U'(1)$
$v_2^1 = v_2^2 = 0$	$\lambda_{12}^r = \lambda_{12}^i = 0$	$U''(1)$

Table 6.1: Unfavorable VEV-Structures

6.4 Four Higgs Model with Further Discrete Symmetry

Another possibility to restrict the number of parameters in the Higgs potential is to assume a auxiliary symmetry. Often a $\mathbb{Z}_2$ symmetry is used, see e.g. [85,86,102] viz invariance by the exchange of the doublets ψ^1 and ψ^2. But in our case this would force $\lambda_{11} = \lambda_{12} = 0$ which we already forbid.

But a possibility which work and which we will apply is the exchange of ψ^1 and ψ^2 together with the components of ψ^1 shown in table 6.2 a) or ψ^2 in table 6.2 b).

[3] The origin of that lies in the minimization conditions.

$$
\begin{array}{ccc}
\psi_1^1 & \longrightarrow & \psi_1^2 \\
\psi_2^1 & \longrightarrow & \psi_2^2 \\
\psi_1^2 & \longrightarrow & \psi_2^1 \\
\psi_2^2 & \longrightarrow & \psi_1^1 \\
 & \text{a)} &
\end{array}
\quad \text{or} \quad
\begin{array}{ccc}
\psi_1^1 & \longrightarrow & \psi_2^2 \\
\psi_2^1 & \longrightarrow & \psi_1^2 \\
\psi_1^2 & \longrightarrow & \psi_1^1 \\
\psi_2^2 & \longrightarrow & \psi_2^1 \\
 & \text{b)} &
\end{array}
$$

Table 6.2: Possible Additional Discrete Symmetry of the Two Doublet Potential

Both symmetries lead to the same restrictions on our Higgs potential, e.g. for the λ_1 and λ_4 term it holds:

$$
\begin{aligned}
&\lambda_1(\psi_1^{1\dagger}\psi_1^1 + \psi_2^{1\dagger}\psi_2^1)^2 \xrightarrow{\text{6.4 a) or b)}} \lambda_1(\psi_1^{2\dagger}\psi_1^2 + \psi_2^{2\dagger}\psi_2^2)^2 \ , \\
&\lambda_4(\psi_1^{2\dagger}\psi_1^2 + \psi_2^{2\dagger}\psi_2^2)^2 \xrightarrow{\text{6.4 a) or b)}} \lambda_4(\psi_1^{1\dagger}\psi_1^1 + \psi_2^{1\dagger}\psi_2^1)^2 \ .
\end{aligned}
\tag{6.22}
$$

So if the potential has to be invariant under this auxiliary symmetry then it will hold that $\lambda_1 = \lambda_4$. For the other terms we get:

$$
\begin{array}{ll}
\mu_1^2 = \mu_2^2 \ , & \lambda_8 = 0 \ , \\
\lambda_1 = \lambda_4 \ , & \lambda_9 = \lambda_{10}^* \ , \\
\lambda_2 = \lambda_5 \ , & \lambda_{11} = \lambda_{12}^* \ , \\
\lambda_3 = \lambda_6 \ , & \lambda_{13} = \lambda_{14} \ .
\end{array}
\tag{6.23}
$$

6.4.1 Higgs Potential

Accordingly we get a "new", restricted Higgs potential, namely

$$
\begin{aligned}
V_{dis} = \ & -\mu_1^2[(\psi_1^{1\dagger}\psi_1^1 + \psi_2^{1\dagger}\psi_2^1) + (\psi_1^{2\dagger}\psi_1^2 + \psi_2^{2\dagger}\psi_2^2)] \\
& +\lambda_1[(\psi_1^{1\dagger}\psi_1^1 + \psi_2^{1\dagger}\psi_2^1)^2 + (\psi_1^{2\dagger}\psi_1^2 + \psi_2^{2\dagger}\psi_2^2)^2] \\
& +\lambda_2[(\psi_1^{1\dagger}\psi_1^1 - \psi_2^{1\dagger}\psi_2^1)^2 + (\psi_1^{2\dagger}\psi_1^2 - \psi_2^{2\dagger}\psi_2^2)^2] \\
& +\lambda_3[(\psi_1^{1\dagger}\psi_2^1)(\psi_2^{1\dagger}\psi_1^1) + (\psi_2^{2\dagger}\psi_1^2)(\psi_1^{2\dagger}\psi_2^2)] \\
& +\lambda_7(\psi_1^{1\dagger}\psi_1^1 + \psi_2^{1\dagger}\psi_2^1)(\psi_1^{2\dagger}\psi_1^2 + \psi_2^{2\dagger}\psi_2^2) \\
& + \Big[\lambda_9[(\psi_1^{1\dagger}\psi_1^2)(\psi_2^{1\dagger}\psi_2^2) + (\psi_2^{2\dagger}\psi_1^1)(\psi_1^{2\dagger}\psi_2^1)] + h.c.\Big] \\
& + \Big[\lambda_{11}[(\psi_1^{1\dagger}\psi_2^2)(\psi_1^{1\dagger}\psi_2^1) + (\psi_2^{1\dagger}\psi_1^2)(\psi_2^{1\dagger}\psi_1^1) + (\psi_1^{2\dagger}\psi_1^1)(\psi_1^{2\dagger}\psi_2^2) + (\psi_2^{2\dagger}\psi_2^1)(\psi_2^{2\dagger}\psi_1^2)] + h.c.\Big] \\
& +\lambda_{13}[(\psi_1^{2\dagger}\psi_1^1)(\psi_1^{1\dagger}\psi_1^2) + (\psi_2^{2\dagger}\psi_2^1)(\psi_2^{1\dagger}\psi_2^2) + (\psi_2^{2\dagger}\psi_1^1)(\psi_1^{1\dagger}\psi_2^2) + (\psi_1^{2\dagger}\psi_2^1)(\psi_2^{1\dagger}\psi_1^2)]
\end{aligned}
\tag{6.24}
$$

With this restricted Higgs potential V_{dis} we will now continue our investigations for special cases and assumptions on our Higgs potential and/or the VEV structure because the number of parameter is still too much to discuss it in general.

6.4.2 Minimization Conditions

For this and the following chapters we are not using anymore the notation of (6.13) for the VEVs. The reason is simply because some of the VEV configurations we are considering, assuming a phase for ψ_1^1 what we excluded by our choice in (6.13). But w.l.o.g. we can also rotate our Higgs fields in a way that the phase of another VEV disappears and as a consequence in general, therefore ψ_1^1 obtain a phase.

Therefore we give every VEV a phase but keep in mind that we can choose one of them to be zero. By using this convention and using notation (6.25),

$$\begin{aligned} v_1^1 &= v\, e^{i\alpha} \\ v_2^1 &= v\, e^{i\beta} \\ v_1^2 &= v\, e^{i\gamma} \\ v_2^2 &= v\, e^{i\delta} \end{aligned} \tag{6.25}$$

it follows for the first derivatives of V at the minimum:

$$\begin{aligned} \left.\frac{\partial V}{\partial \psi_1^{1r}}\right|_{min} = \; & v_2^2 v_2^1 v_1^2 \cos\left(-\beta+\gamma+\delta\right) \lambda_9^r + \\ & \left(\tfrac{1}{2}\cos\left(2\beta-\gamma\right) v_1^2 (v_1^1)^2 + \cos\left(-\alpha+\beta+\delta\right) v_2^2 v_2^1 v_1^1 + \tfrac{1}{2}\cos\left(-2\gamma+\delta\right) v_2^2 (v_1^2)^2\right) \lambda_{11}^r + \\ & \left(\tfrac{1}{2}\sin\left(2\beta-\gamma\right) v_1^2 (v_1^1)^2 - \sin\left(-\alpha+\beta+\delta\right) v_2^2 v_2^1 v_1^1 - \tfrac{1}{2}\sin\left(-2\gamma+\delta\right) v_2^2 (v_1^2)^2\right) \lambda_{11}^i + \\ & \left((\lambda_1+\lambda_2)(v_1^1)^2 + \left(\lambda_1-\lambda_2+\tfrac{1}{2}\lambda_3\right)(v_1^1)^2 + \tfrac{1}{2}\left(\lambda_7+\lambda_{13}\right)[(v_1^2)^2+(v_2^2)^2] - {\mu_1}^2\right) v_1^1 \cos\alpha \end{aligned} \tag{6.26}$$

$$\begin{aligned} \left.\frac{\partial V}{\partial \psi_1^{1i}}\right|_{min} = \; & v_2^2 v_2^1 v_1^2 \sin\left(-\beta+\gamma+\delta\right) \lambda_9^r + \\ & \left(\sin\left(-\alpha+\beta+\delta\right) v_2^2 v_2^1 v_1^1 + \tfrac{1}{2}\sin\left(2\beta-\gamma\right) v_1^2 (v_1^1)^2 - \tfrac{1}{2}\sin\left(-2\gamma+\delta\right) v_2^2 (v_1^2)^2\right) \lambda_{11}^r + \\ & \left(\cos\left(-\alpha+\beta+\delta\right) v_2^2 v_2^1 v_1^1 - \tfrac{1}{2}\cos\left(2\beta-\gamma\right) v_1^2 (v_1^1)^2 - \tfrac{1}{2}\cos\left(-2\gamma+\delta\right) v_2^2 (v_1^2)^2\right) \lambda_{11}^i + \\ & \left((\lambda_1+\lambda_2)(v_1^1)^2 + \left(\lambda_1-\lambda_2+\tfrac{1}{2}\lambda_3\right)(v_1^1)^2 + \tfrac{1}{2}\left(\lambda_7+\lambda_{13}\right)[(v_1^2)^2+(v_2^2)^2] - {\mu_1}^2\right) v_1^1 \sin\alpha \end{aligned} \tag{6.27}$$

$$\begin{aligned} \left.\frac{\partial V}{\partial \psi_2^{1r}}\right|_{min} = \; & v_2^2 v_1^2 \cos\left(-\alpha+\gamma+\delta\right) v_1^1 \lambda_9^r + \\ & \left(\cos\left(\alpha-\beta+\gamma\right) v_1^2 v_2^1 v_1^1 + \tfrac{1}{2}\cos\left(-2\alpha+\delta\right) v_2^2 (v_1^1)^2 + \tfrac{1}{2}\cos\left(-\gamma+2\delta\right) (v_2^2)^2 v_1^2\right) \lambda_{11}^r + \\ & \left(-\sin\left(\alpha-\beta+\gamma\right) v_1^2 v_2^1 v_1^1 - \tfrac{1}{2}\sin\left(-2\alpha+\delta\right) v_2^2 (v_1^1)^2 + \tfrac{1}{2}\sin\left(-\gamma+2\delta\right) (v_2^2)^2 v_1^2\right) \lambda_{11}^i + \\ & \left(\left(\lambda_1-\lambda_2+\tfrac{1}{2}\lambda_3\right) {v_1^1}^2 + \left(\tfrac{1}{2}\lambda_7+\tfrac{1}{2}\lambda_{13}\right)[(v_1^2)^2+(v_2^2)^2] - {\mu_1}^2 + (\lambda_1+\lambda_2)(v_2^1)^2\right) v_2^1 \cos\beta \end{aligned} \tag{6.28}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial \psi_2^{1i}}\right|_{min} = \; & v_2^2 v_1^2 \sin\left(-\alpha+\gamma+\delta\right) v_1^1 \lambda_9^r + \\ & \left(\sin\left(\alpha-\beta+\gamma\right) v_1^2 v_2^1 v_1^1 - \tfrac{1}{2}\sin\left(-2\alpha+\delta\right) v_2^2 (v_1^1)^2 + \tfrac{1}{2}\sin\left(-\gamma+2\delta\right) (v_2^2)^2 v_1^2\right) \lambda_{11}^r + \\ & \left(\cos\left(\alpha-\beta+\gamma\right) v_1^2 v_2^1 v_1^1 - \tfrac{1}{2}\cos\left(-2\alpha+\delta\right) v_2^2 (v_1^1)^2 - \tfrac{1}{2}\cos\left(-\gamma+2\delta\right) (v_2^2)^2 v_1^2\right) \lambda_{11}^i + \\ & \left(\left(\lambda_1-\lambda_2+\tfrac{1}{2}\lambda_3\right) {v_1^1}^2 + \left(\tfrac{1}{2}\lambda_7+\tfrac{1}{2}\lambda_{13}\right)[(v_1^2)^2+(v_2^2)^2] - {\mu_1}^2 + (\lambda_1+\lambda_2)(v_2^1)^2\right) v_2^1 \sin\beta\end{aligned} \tag{6.29}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial \psi_1^{2r}}\right|_{min} = \; & v_2^2 v_2^1 \cos\left(-\alpha-\beta+\delta\right) v_1^1 \lambda_9^r + \\ & \left(\left(\tfrac{1}{2}\cos\left(\alpha-2\beta\right)(v_1^1)^2 + \cos\left(\alpha-\gamma+\delta\right) v_2^2 v_1^2\right) v_1^1 + \tfrac{1}{2}\cos\left(-\beta+2\delta\right)(v_2^2)^2 v_2^1\right) \lambda_{11}^r + \\ & \left(\left(-\tfrac{1}{2}\sin\left(\alpha-2\beta\right)(v_1^1)^2 - \sin\left(\alpha-\gamma+\delta\right) v_2^2 v_1^2\right) v_1^1 + \tfrac{1}{2}\sin\left(-\beta+2\delta\right)(v_2^2)^2 v_2^1\right) \lambda_{11}^i + \\ & \left((\lambda_1+\lambda_2)(v_1^2)^2 + \left(\lambda_1-\lambda_2+\tfrac{1}{2}\lambda_3\right)(v_2^2)^2 - {\mu_1}^2 + \tfrac{1}{2}(\lambda_7+\lambda_{13})[(v_1^1)^2+(v_1^1)^2]\right) v_1^2 \cos\gamma\end{aligned} \tag{6.30}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial \psi_1^{2i}}\right|_{min} = \; & -v_2^2 v_2^1 \sin\left(\delta-\alpha-\beta\right) v_1^1 \lambda_9^r + \\ & \left(\left(-\tfrac{1}{2}\sin\left(\alpha-2\beta\right)(v_1^1)^2 + \sin\left(\alpha-\gamma+\delta\right) v_2^2 v_1^2\right) v_1^1 + \tfrac{1}{2}\sin\left(-\beta+2\delta\right)(v_2^2)^2 v_2^1\right) \lambda_{11}^r + \\ & \left(\left(-\tfrac{1}{2}\cos\left(\alpha-2\beta\right)(v_1^1)^2 + \cos\left(\alpha-\gamma+\delta\right) v_2^2 v_1^2\right) v_1^1 - \tfrac{1}{2}\cos\left(-\beta+2\delta\right)(v_2^2)^2 v_2^1\right) \lambda_{11}^i + \\ & \left((\lambda_1+\lambda_2)(v_1^2)^2 + \left(\lambda_1-\lambda_2+\tfrac{1}{2}\lambda_3\right)(v_2^2)^2 - {\mu_1}^2 + \tfrac{1}{2}(\lambda_7+\lambda_{13})[(v_1^1)^2+(v_1^1)^2]\right) v_1^2 \sin\gamma\end{aligned} \tag{6.31}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial \psi_2^{2r}}\right|_{min} = \; & v_1^2 v_2^1 \cos\left(\alpha+\beta-\gamma\right) v_1^1 \lambda_9^r + \\ & \left(\tfrac{1}{2}\cos\left(\alpha-2\gamma\right)(v_1^2)^2 v_1^1 + \cos\left(-\beta-\gamma+\delta\right) v_2^2 v_1^2 v_2^1 + \tfrac{1}{2}\cos\left(2\alpha-\beta\right) v_2^1 (v_1^1)^2\right) \lambda_{11}^r + \\ & \left(-\tfrac{1}{2}\sin\left(\alpha-2\gamma\right)(v_1^2)^2 v_1^1 + \sin\left(-\beta-\gamma+\delta\right) v_2^2 v_1^2 v_2^1 + \tfrac{1}{2}\sin\left(2\alpha-\beta\right) v_2^1 (v_1^1)^2\right) \lambda_{11}^i + \\ & \left(\tfrac{1}{2}(\lambda_7+\lambda_{13})[(v_1^1)^2+(v_1^1)^2] + \left(\lambda_1-\lambda_2+\tfrac{1}{2}\lambda_3\right)(v_1^2)^2 - {\mu_1}^2 + (\lambda_1+\lambda_2)(v_2^2)^2\right) v_2^2 \cos\delta\end{aligned} \tag{6.32}$$

$$\begin{aligned}
\left.\frac{\partial V}{\partial \psi_2^{2i}}\right|_{min} = \; & v_1^2 v_2^1 \sin\left(\alpha+\beta-\gamma\right) v_1^1 \lambda_9^r + \\
& \left(-\tfrac{1}{2}\sin\left(\alpha-2\gamma\right)\left(v_1^2\right)^2 v_1^1 - \sin\left(-\beta-\gamma+\delta\right) v_2^2 v_1^2 v_2^1 + \tfrac{1}{2}\sin\left(2\alpha-\beta\right) v_2^1 (v_1^1)^2\right)\lambda_{11}^r + \\
& \left(-\tfrac{1}{2}\cos\left(\alpha-2\gamma\right)\left(v_1^2\right)^2 v_1^1 + \cos\left(-\beta-\gamma+\delta\right) v_2^2 v_1^2 v_2^1 - \tfrac{1}{2}\cos\left(2\alpha-\beta\right) v_2^1 (v_1^1)^2\right)\lambda_{11}^i + \\
& \left(\tfrac{1}{2}\left(\lambda_7+\lambda_{13}\right)\left[(v_1^1)^2+(v_1^1)^2\right] + \left(\lambda_1-\lambda_2+\tfrac{1}{2}\lambda_3\right)\left(v_1^2\right)^2 - {\mu_1}^2 + \left(\lambda_1+\lambda_2\right)\left(v_2^2\right)^2\right) v_2^2 \sin\delta
\end{aligned} \tag{6.33}$$

6.4.3 Various VEV and Parameter Configurations

Real VEVs but Complex Parameters

In this and the following sections we want to discuss some promising VEV configurations. To do this we will make assumptions on the VEV and/or the parameters to enable a calculation.

Annotation:
Under parameters we understand here the ones of the Higgs potential, so μ_1^2 and λ_i.

If we now assume the VEVs to be real in order to obtain as simple as possible minimization conditions and non-vanishing, we will only get two VEV configurations[4] which solve the minimization conditions, namely

$$v_1^1 = v_2^1 = v_1^2 = v_2^2 \equiv v \tag{6.34}$$

and

$$v_1^1 = v_2^1 = -v_1^2 = -v_2^2 \equiv v \; . \tag{6.35}$$

The full solution as well as the Higgs masses including their eigenvectors for (6.34) and (6.35) is given in appendix E.

But both VEV configurations force a $\mu\tau$–symmetry as described in section 2.4. This means that $\theta_{13} = 0$ and $\theta_{23} = \frac{\pi}{4}$ in the diagonalizing matrices, where we have used the usual parametrization shown in (A.3).

But actually it is not a surprise that both solutions lead to the same mixings because since the minus signs of (6.35) can be absorbed in the Yukawa couplings (see table C.6) they lead to the same lepton and quark mass matrices as solution (6.34).

Now we want to present a deeper insight on this mixing pattern. The reason why the diagonalizing matrices have such a form is as already mentioned a $\mu\tau$–symmetry (2-3 symmetry) in the mass matrices, so invariance under a S_2 permutation symmetry [15, 18]. A model based on S_3 that also explains this pattern can be found in [116].

Now, we want to consider possible mass matrices for the down-quarks, i.e. we have to select those having no zero eigenvalue. Then the phenomenologically possible ones have the following form:

[4] We show the solutions of the minimization conditions without the restriction on other parameters than the VEVs.

$$M = \begin{pmatrix} 0 & \pm A & A \\ \pm B & C & D \\ B & D & C \end{pmatrix} , \tag{6.36}$$

where we changed the original generation assignment like described in 3.4.1, strictly speaking we interchanged the first and the third generation assignment, i.e.

$$M = Q\, M_{15-18}\, Q^T \quad \text{,with } Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} . \tag{6.37}$$

But as shown in [18] it is not possible that the left- and right-handed quark mass matrix obtain a 2-3 symmetry since this would mean that there is no mixing between the third and the other two generations, so $\theta_{13} = \theta_{23} = 0$ what is manifestly false.

The same is true for the lepton mass matrix. Even if we consider the possibility that one neutrino is massless this will not help us since the mass matrices having one zero eigenvalue also possess a $\mu\tau$–symmetry. And if we pass to Majorana neutrinos and assume the existence of three right-handed neutrinos in a type–I seesaw scenario we will not avoid this mixing pattern.

Some possibilities to maintain a viable model are the introduction of a $SU(2)_L$ triplet and pass so to the type–II seesaw mechanism (see section 2.7.2 for references) or to introduce scalar singlets at high scales which have to respect the D_5 symmetry. But if we introduce them we have to break the flavor symmetry already at these high energies and not anymore as originally intended at the electroweak scale. Both possibilities then give contributions to the "original" mass matrices and can eliminate in this way their 2-3 symmetry.

But now we renounce on this and hope to find another VEV configuration which will allow us a better phenomenological description. For this we have to modify and/or broaden our assumption that all VEVs are real and the parameters are complex to get new VEV configurations.

Real VEV and Real Parameters

Maybe the more convenient assumption that all VEVs and parameters are real can help us to escape from these dilemma. For the scenario of real VEVs and parameters we obtain the reasonable solutions

$$\begin{array}{llll} v_1^1 = v_2^1 \equiv v \ , & v_1^1 = -v_2^1 \equiv v \ , & v_1^2 = v_2^2 \equiv v \ , & v_1^2 = -v_2^2 \equiv v \ , \\ v_1^2 \equiv v\,\omega \ , & v_1^2 \equiv v\,\omega \ , & v_1^1 \equiv v\,\omega \ , & v_1^1 \equiv v\,\omega \ , \\ v_2^2 \equiv \frac{v}{\omega} \ . & v_2^2 \equiv \frac{v}{\omega} \ . & v_2^1 \equiv \frac{v}{\omega} \ . & v_2^1 \equiv \frac{v}{\omega} \ . \end{array} \tag{6.38}$$

That we get such a ratio as solution can be understood by taking the ratio term by term of 6.26 and 6.28 as well as of 6.30 and 6.32 (since all VEVs and parameters are real the other minimization conditions are anyway zero). Then the resulting terms are only dependend of the ratios

$$v_1^1/v_2^1 \quad \text{and} \quad v_1^2/v_2^2 \ . \tag{6.39}$$

For the limit $\omega \to \pm 1$ we get 1 (expected) + 2 (unwanted) GB from the uncharged Higgs matrix and see that this limit does not lead to case (6.34) or (6.35) as one might assume. If we would already set $\omega \to \pm 1$ in the minimization condition, this would lead to the same equations as with (6.34) and (6.35). This also shows us that these solutions are independent ones and therefore not equivalent. Thus even if we want ω to be near ± 1 in order not to loose the results we got in section 6.4.3 we have to demand $\omega \neq \pm 1$. However, for this cases a problem with the Higgs mass arise. More precisely, two of the uncharged Higgs bosons do not acquire a mass above about 60 GeV and are therefore too small because phenomenology demands a mass above 115 GeV. To calculate the range of the Higgs masses, we assumed the following numerical ranges:

$$\begin{array}{rclrcl} |\omega| & = & 0.85..1.15\ , & \lambda_7 & = & -3..3\ , \\ v & = & 123\ \text{GeV}\ , & \lambda_9^r & = & -3..3\ , \\ \mu_1 & = & 200\ \text{GeV}\ , & \lambda_{13} & = & -3..3\ , \end{array} \tag{6.40}$$

Thereby, the two problematical Higgs masses do not depend much on the mass parameter μ_1, so a variation of ± 100 GeV does not affect them appreciable and we do not have to vary it. But since we get in the limit $\omega \to \pm 1$ two more GBs in the uncharged Higgs boson sector we anyway have to reject (6.38) as a possible ansatz for our model.

A further possibility would be that the VEVs of one doublet are pairwise equal except for a possible sign, whereas the VEV of the first doublet has to be unequal to the one of the second Higgs doublet, i.e.

$$\begin{array}{cc} v_1^1 = v_2^1 \equiv v\ , & v_1^1 = -v_2^1 \equiv v\ , \\ v_1^2 = v_2^2 \equiv u\ , & v_1^2 = -v_2^2 \equiv u\ , \\ u \neq v\ . & u \neq v\ . \end{array} \tag{6.41}$$

But the bad news is that all of these six VEV configuration (6.38) and (6.41) lead again to a $\mu\tau$-symmetry in the mass matrices ($\theta_{13} = 0$ and $\theta_{23} = \frac{\pi}{4}$) and are therefore phenomenologically disfavored.

Complex VEV and Real/Complex Parameters

Now we assume some of the VEVs to be complex and the parameters to be real what allows us to obtain spontaneous CP violation and filter out VEV configurations which do not fulfill the minimization conditions and/or their phases can be absorbed in the Yukawa couplings and hence yield nothing new. Then the plainest VEV configurations are (6.42) and (6.47) (by taking notice of the ratios (6.39) as solution and knowing that only one phase do not work).

$$\begin{array}{rcl} \langle\psi_1^1\rangle & = & v\, e^{i\alpha}\ , \\ \langle\psi_2^1\rangle & = & v\ , \\ \langle\psi_1^2\rangle & = & v\, e^{i\beta}\ , \\ \langle\psi_2^2\rangle & = & v\ . \end{array} \tag{6.42}$$

This leads to clearly determined values for α and β, thereby they fulfill the relations

$$\begin{array}{c} \alpha + 2\beta = 0\ , \\ e^{5i\beta} = 1\ . \end{array} \tag{6.43}$$

These relations reflecting the structure of our flavor symmetry D_5. One solution is for example

$$\begin{array}{rl} \alpha = & \frac{4\pi}{5}\ , \\ \beta = & -\frac{2\pi}{5}\ . \end{array} \tag{6.44}$$

In addition we get a constraint on λ_9^r from the minimization conditions, i.e.

$$\lambda_9^r = -\frac{1}{2}\lambda_{11}^r\left(1 \pm \sqrt{5}\right)\ . \tag{6.45}$$

Annotation:

If we insert (6.45) together with VEV configuration (6.42) in the Higgs potential (6.24) then the solutions (6.43) for α and β (we originally got from the minimization conditions) are the only solutions which let the Higgs potential invariant, i.e. with $\lambda_9^r = -\frac{1}{2}\lambda_{11}^r\left(1 \pm \sqrt{5}\right)$ it holds (shown for one solution of α and β) that

$$V_{dis}\left(\alpha = 0,\, \beta = 0\right) \Leftrightarrow V_{dis}\left(\alpha = 4\pi/5,\, \beta = -2\pi/5\right) \ . \tag{6.46}$$

So this are degenerate minima.

If we assume complex parameters instead of real ones in VEV configuration (6.42) and make a multiple Taylor expansion to the first order around the numerical values of α and β we calculated before, we observe that this does not help us to solve the problem.

The other mentioned VEV configuration is

$$\begin{aligned} \left\langle\psi_1^1\right\rangle &= v \ , \\ \left\langle\psi_2^1\right\rangle &= v \ , \\ \left\langle\psi_1^2\right\rangle &= v\,\omega\,e^{i\alpha} \ , \\ \left\langle\psi_2^2\right\rangle &= \tfrac{v}{\omega}\,e^{-i\alpha} \ , \end{aligned} \tag{6.47}$$

where the parameters are assumed to be complex[5]. This leads to clearly determined values of α, i.e. $\alpha = \pm\frac{\pi}{2}$. But one of the uncharged Higgs does not acquire a mass above roughly 85 GeV by using the parameterspace of (6.40). Therefore we also have to reject this VEV configuration.

6.5 A Viable Model

So far we have not found a suitable VEV configuration solving the minimization conditions and is able to explain the particle masses and mixings. Therefore, to show that it is nevertheless possible we try to fit the mass matrix numerically in order to obtain the masses and mixings within experimental bounds.

6.5.1 Particle Assignment and Notation

The first step to for our model is to choose the transformation properties of quarks and leptons under D_5. This is shown in the following:

$$Q_1 \sim \mathbf{1}_1 \ , \quad \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} \sim \mathbf{2}_2 \ , \quad \begin{matrix} u_1^c \sim \mathbf{1}_1 \ , & \begin{pmatrix} u_2^c \\ u_3^c \end{pmatrix} \sim \mathbf{2}_1 \ , \\ d_1^c \sim \mathbf{1}_1 \ , & \begin{pmatrix} d_2^c \\ d_3^c \end{pmatrix} \sim \mathbf{2}_1 \ , \end{matrix} \tag{6.48}$$

$$L_1 \sim \mathbf{1}_1 \ , \quad \begin{pmatrix} L_2 \\ L_3 \end{pmatrix} \sim \mathbf{2}_2 \ , \quad \begin{matrix} e_1^c \sim \mathbf{1}_1 \ , & \begin{pmatrix} e_2^c \\ e_3^c \end{pmatrix} \sim \mathbf{2}_1 \ , \\ \nu_1^c \sim \mathbf{1}_1 \ , & \begin{pmatrix} \nu_2^c \\ \nu_3^c \end{pmatrix} \sim \mathbf{2}_1 \ , \end{matrix} \tag{6.49}$$

With this assignment we obtain the following mass matrices, where we assume M_R to be real:

[5] This VEV configuration forces $\lambda_{11}^r = 0$ and therefore $\lambda_{11}^i \neq 0$ in order not to obtain an accidental symmetry in the Higgs potential.

$$M_{u,\nu} = \begin{pmatrix} 0 & \kappa_3^{u,\nu}\,\overline{\langle\psi_1^1\rangle} & \kappa_3^{u,\nu}\,\overline{\langle\psi_2^1\rangle} \\ \kappa_4^{u,\nu}\,\overline{\langle\psi_1^2\rangle} & \kappa_1^{u,\nu}\,\overline{\langle\psi_2^2\rangle} & \kappa_2^{u,\nu}\,\overline{\langle\psi_1^1\rangle} \\ \kappa_4^{u,\nu}\,\overline{\langle\psi_2^2\rangle} & \kappa_2^{u,\nu}\,\overline{\langle\psi_2^1\rangle} & \kappa_1^{u,\nu}\,\overline{\langle\psi_1^2\rangle} \end{pmatrix} ,$$

$$M_{d,e} = \begin{pmatrix} 0 & \xi_3^{d,e}\,\langle\psi_2^1\rangle & \xi_3^{d,e}\,\langle\psi_1^1\rangle \\ \xi_4^{d,e}\,\langle\psi_2^2\rangle & \xi_1^{d,e}\,\langle\psi_1^2\rangle & \xi_2^{d,e}\,\langle\psi_2^1\rangle \\ \xi_4^{d,e}\,\langle\psi_1^2\rangle & \xi_2^{d,e}\,\langle\psi_1^1\rangle & \xi_1^{d,e}\,\langle\psi_2^2\rangle \end{pmatrix} , \tag{6.50}$$

$$M_R = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & B & 0 \end{pmatrix} .$$

6.5.2 Model with Dirac Neutrinos

Now that we have obtained the mass matrices we try to find by a numerical analysis values for the Yukawa couplings and VEVs allowing us to describe the masses and mixings within their experimental bounds [117]. This we will do for the assumption of Dirac neutrinos. As solution for our investigation we have received the values below, whereas we round after the fourth significant decimal place and give the masses at the energy scale of m_W.

The VEVs are therefore:

$$\begin{aligned} \langle\psi_1^1\rangle &= 133.8841\,e^{-i\,0.04008}\ \text{GeV} , & \langle\psi_1^2\rangle &= 128.8769\,e^{i\,0.05583}\ \text{GeV} , \\ \langle\psi_2^1\rangle &= 112.8433\,e^{i\,0.05335}\ \text{GeV} , & \langle\psi_2^2\rangle &= 117.1846\,e^{-i\,0.03769}\ \text{GeV} . \end{aligned} \tag{6.51}$$

The up-type quark Yukawa couplings are:

$$\begin{aligned} \kappa_1^u &= 0.6878 - i\,0.001249 , \\ \kappa_2^u &= -0.6942 + i\,0.003655 , \\ \kappa_3^u &= (0.2670 - i\,3.7909)\cdot 10^{-4} , \\ \kappa_4^u &= 0.15497\cdot 10^{-3} . \end{aligned} \tag{6.52}$$

The down-type quark Yukawa couplings are:

$$\begin{aligned} \xi_1^d &= 0.01228 + i\,0.0002213 , \\ \xi_2^d &= -0.01285 - i\,0.00003 , \\ \xi_3^d &= (9.4676 - i\,0.3100)\cdot 10^{-5} , \\ \xi_4^d &= 0.0001167 + i\,0.00003840 . \end{aligned} \tag{6.53}$$

The neutrino Yukawa couplings are:

$$\begin{aligned} \kappa_1^\nu &= (-0.1206 - i\,0.07110)\cdot 10^{-11}/3.45 , \\ \kappa_2^\nu &= (0.1098 + i\,0.06753)\cdot 10^{-11}/3.45 , \\ \kappa_3^\nu &= (0.1630 - i\,0.07656)\cdot 10^{-11}/3.45 , \\ \kappa_4^\nu &= (0.06504 + i\,0.1794)\cdot 10^{-11}/3.45 . \end{aligned} \tag{6.54}$$

The charged lepton Yukawa couplings are:

$$\begin{aligned} \xi_1^e &= 0.006914 + i\,0.00009900 , \\ \xi_2^e &= -0.007469 - i\,0.0007630 , \\ \xi_3^e &= 0.00008625 + i\,0.00005939 , \\ \xi_4^e &= 0.00001707 + i\,8.0000\cdot 10^{-7} . \end{aligned} \tag{6.55}$$

Quarks

For this values we can now calculate the quark masses, mixings and $\mathcal{J}_{CP}$ and one can see that they are in agreement with experiments:

$$\begin{aligned} m_u &= 0.002201 \text{ GeV} , & m_d &= 0.004399 \text{ GeV} , \\ m_c &= 0.8098 \text{ GeV} , & m_s &= 0.08002 \text{ GeV} , \\ m_t &= 170.6638 \text{ GeV} , & m_b &= 3.1039 \text{ GeV} . \end{aligned} \tag{6.56}$$

$$\begin{aligned} s_{12} &= 0.2249 , \\ s_{23} &= 0.04144 , \\ s_{13} &= 0.003683 , \\ \\ \mathcal{J}_{CP} &= 2.98 \cdot 10^{-5} . \end{aligned} \tag{6.57}$$

Leptons

The values for charged leptons are shown below:

$$\begin{aligned} m_e &= 0.0005111 \text{ GeV} , \\ m_\mu &= 0.1060 \text{ GeV} , \\ m_\tau &= 1.7799 \text{ GeV} . \end{aligned} \tag{6.58}$$

And for neutrinos one gets:

$$\begin{aligned} \Delta m_{21}^2 &= 2.22 \cdot 10^{-3} \text{ eV}^2 , & \sum_i m_i &= 0.284 \text{ eV} , \\ \Delta m_{32}^2 &= 7.94 \cdot 10^{-5} \text{ eV}^2 , & R &= \tfrac{\Delta m_{21}^2}{\Delta m_{32}^2} = 27.97 . \end{aligned} \tag{6.59}$$

$$\begin{aligned} s_{12}^2 &= 0.2999 , \\ s_{23}^2 &= 0.4992 , \\ s_{13}^2 &= 0.0274 , \\ \\ \mathcal{J}_{CP} &= -0.01548 . \end{aligned} \tag{6.60}$$

Annotation:

In some publications the inverse of our definition of the ratio R is used, so

$$R = \frac{\Delta m_\odot^2}{\Delta m_A^2} = \frac{m_2^2 - m_1^2}{|m_3^2 - m_1^2|} . \tag{6.61}$$

6.5.3 Model with Majorana Neutrinos

The model we have shown before assumed the neutrinos to be Dirac particles. For the case of Majorana neutrinos in a type-I seesaw scenario we obtain the follwowing values [117]:

The VEVs are:

$$\begin{aligned} \langle \psi_1^1 \rangle &= 129.0887\, e^{-i\,0.13549} \text{ GeV} , & \langle \psi_1^2 \rangle &= 124.0342\, e^{i\,0.06774} \text{ GeV} , \\ \langle \psi_2^1 \rangle &= 119.3079\, e^{i\,0.10239} \text{ GeV} , & \langle \psi_2^2 \rangle &= 124.3133\, e^{-i\,0.10779} \text{ GeV} . \end{aligned} \tag{6.62}$$

The up-type quark Yukawa couplings are:

$$\begin{aligned}
\kappa_1^u &= 0.6816 + i\,0.00006140 \ , \\
\kappa_2^u &= -0.6884 - i\,0.001245 \ , \\
\kappa_3^u &= 0.00006971 - i\,0.0003234 \ , \\
\kappa_4^u &= 0.0001795 + i\,1.2705 \cdot 10^{-21} \ .
\end{aligned} \tag{6.63}$$

The down-type quark Yukawa couplings are:

$$\begin{aligned}
\xi_1^d &= 0.01221 + i\,0.0003440 \ , \\
\xi_2^d &= -0.01275 - i\,0.0000128 \ , \\
\xi_3^d &= 0.00009521 + i\,0.00001006 \ , \\
\xi_4^d &= 0.0001147 + i\,0.00004308 \ .
\end{aligned} \tag{6.64}$$

The parameters of the right-handed neutrino mass matrix are:

$$\begin{aligned}
A &= -92.3382 \cdot 1.55 \cdot 10^{11} \text{ GeV} \ , \\
B &= -105.6449 \cdot 1.55 \cdot 10^{11} \text{ GeV} \ .
\end{aligned} \tag{6.65}$$

The Dirac neutrino Yukawa couplings are:

$$\begin{aligned}
\kappa_1^\nu &= 0.01507 + i\,0.1697 \ , \\
\kappa_2^\nu &= 0.1272 + i\,0.004653 \ , \\
\kappa_3^\nu &= -0.08700 + i\,0.01851 \ , \\
\kappa_4^\nu &= -0.1386 + i\,0.1299 \ .
\end{aligned} \tag{6.66}$$

The charged lepton Yukawa couplings are:

$$\begin{aligned}
\xi_1^e &= 0.006774 + i\,0.001060 \ , \\
\xi_2^e &= -0.007463 - i\,0.0005375 \ , \\
\xi_3^e &= -0.00004332 + i\,0.00007990 \ , \\
\xi_4^e &= 0.00001989 \ .
\end{aligned} \tag{6.67}$$

Quarks

The resulting masses, mixings and the Jarlskog invariant are:

$$\begin{aligned}
m_u &= 0.002200 \text{ GeV} \ , & m_d &= 0.004400 \text{ GeV} \ , \\
m_c &= 0.8100 \text{ GeV} \ , & m_s &= 0.08001 \text{ GeV} \ , \\
m_t &= 170.2021 \text{ GeV} \ , & m_b &= 3.1009 \text{ GeV} \ .
\end{aligned} \tag{6.68}$$

$$\begin{aligned}
s_{12} &= 0.2245 \ , \\
s_{23} &= 0.04136 \ , \\
s_{13} &= 0.003693 \ , \\
\\
\mathcal{J}_{CP} &= 2.93 \cdot 10^{-5} \ .
\end{aligned} \tag{6.69}$$

Leptons

The data for the charged leptons are:

$$\begin{aligned}
m_e &= 0.0005111 \text{ GeV} \ , \\
m_\mu &= 0.1060 \text{ GeV} \ , \\
m_\tau &= 1.7800 \text{ GeV} \ .
\end{aligned} \tag{6.70}$$

And the one for neutrinos includes the masses of the right-handed neutrinos N_i and the Majorana phases α and β as shown in parametrization A.2:

$$\begin{aligned}
\Delta m^2_{21} &= 2.23 \cdot 10^{-3}\ \mathrm{eV}^2\ , \quad \sum_i m_i = 0.109\ \mathrm{eV}\ , \\
\Delta m^2_{32} &= 7.93 \cdot 10^{-5}\ \mathrm{eV}^2\ , \quad R = \tfrac{\Delta m^2_{21}}{\Delta m^2_{32}} = 29.04\ .
\end{aligned} \tag{6.71}$$

$$\begin{aligned}
s^2_{12} &= 0.3000\ , \\
s^2_{23} &= 0.4997\ , \\
s^2_{13} &= 0.03994\ , \\
\\
\mathcal{J}_{CP} &= -0.03528\ .
\end{aligned} \tag{6.72}$$

$$\begin{aligned}
m_{N_1} &= 1.4312 \cdot 10^{13}\ \mathrm{GeV}\ , \\
m_{N_2} = m_{N_3} &= 1.6375 \cdot 10^{13}\ \mathrm{GeV}\ .
\end{aligned} \tag{6.73}$$

$$\begin{aligned}
\alpha &= -0.3724 \\
\beta &= -2.9196
\end{aligned} \tag{6.74}$$

So, with the values of section 6.5.2 and 6.5.3 we are able to reproduce the known masses and mixings for Dirac neutrinos as well as for a type-I seesaw scenario. However, since we have fitted the parameters of our mass matrices to obtain these quantities, the possibility to make a prediction is rather low.

Chapter 7

Conclusions and Outlook

The SM does not explain the three particle generations as well as their strong mass hierarchy. Also the difference between the mixings in the quark and the lepton sector is still a mystery. Here we have tried to motivate an explanation by using in addition to the SM group a discrete symmetry acting on the flavor space. In our case we have introduced the dihedral group D_5 since it has the advantage of discrete non-Abelian groups and an auspicious product structure.

With this symmetry we have calculated the possible structures of the mass matrices and of the Higgs sector. But as we have had to find out, for the assumption of one or two Higgs bosons we are not able to explain the mass hierarchy at tree-level, where our model should be valid as demanded. If we add a third Higgs boson this will be possible but we obtain an accidental symmetry which will be broken by all meaningful VEV configurations we have got from the minimization conditions and this consequently leads to further GBs which are not observed.

To solve this we have suggested to include soft-breaking terms to the three Higgs potential or to add a fourth Higgs boson. We have decided to take a fourth Higgs, where one has the possibility to take two singlets and one doublet or two doublets. However, it has turned out that the potential with the two singlets and the one doublet still has an accidental symmetry and that all possible configurations of four non-vanishing VEVs break it. Consequently, we expect a further GB which has not been seen in experiments.

Therefore, the only solution is to take four Higgs bosons transforming under D_5 as two doublets. For the resulting Higgs potential we then have to introduce a further discrete symmetry in order to restrict the parameter space since the great number of parameters does not show promise to find a viable model and a general discussion is anyway not possible anymore.

The resulting parameter space however still does not allow an analytical solution and we have added further assumptions like if they are real or complex or if we have a certain VEV configuration. This has allowed us to calculate special cases but all of them lead to a 2-3 symmetry in the mass matrices and thus to $\theta_{13} = 0$ in the CKM and PMNS matrix what is not found by experiments for the CKM matrix and to one and two too small Higgs masses respectively.

Then we have changed our strategy and turned away from the search for a meaningful VEV configuration to the fitting of the mass matrices.

After we have seen that we can generate the masses as well as the mixings within experimental bounds the next step would be a closer look at the Higgs sector, i.e. the stability, the Higgs masses as well as FCNCs and LFV.

Another aspect which we have not mentioned until now is the embedding of D_5 in a continuous symmetry, i.e. $SO(3)$ or $SU(3)$, since our purpose has been a phenomenologically viable model which is as simple as possible. However, if we want to embed it we can see that this is not possible for our Higgs sector since the three dimensional representation of $SO(3)$ has resolve into $\mathbf{1}_2 + \mathbf{2}_1$ and the five dimensional one into $\mathbf{1}_1 + \mathbf{2}_1 + \mathbf{2}_2$. The reason is that a model with three Higgs is phenomenologically not possible and for the next larger representation we would have to add a singlet which is of course possible and, on the aspect of embedding, suggestive. Another alternative is to embed D_5 in A_5. Since this is just possible with small variations in our mass matrices, we prefer that way.

As last point we want to mention that the particle assignment of our model does not allow an embedding in a GUT like $SO(10)$, however for LR or Pati-Salam models this would be possible.

Appendix A

Status Quo

A.1 Experimental Data

Now that we have a theoretical understanding about the SM and know that this theory does not give us all parameters, we want to present the "missing" ones which are measured in various experiments and can be found in [118] (or in [119], which is not up-to-date but instead with more details and good explanations).

A description of the status and the phenomenology of the SM is shown in [120].

A.2 Lepton Sector

Let us first begin with the lepton-sector and distinguish between charged leptons and the neutral ones, the neutrinos. Both exist in three flavors: electron-, muon- and tau-flavor.

A.2.1 Charged Leptons

The masses of the electron, muon and tau are shown in table A.1: For charged leptons in

$$\begin{aligned} m_e &= 0.51099892 \pm 0.00000004 \text{ MeV} \\ m_\mu &= 105.658369 \pm 0.000009 \text{ MeV} \\ m_\tau &= 1776.99^{+0.29}_{-0.26} \text{ MeV} \end{aligned}$$

Table A.1: Charged Lepton Masses

contrast to quarks and neutrinos up to now no mixing is observed. But this possibility will still be considered in different models.

A.2.2 Neutral Leptons (Neutrinos)

The reason why we have separated neutrinos is their different behavior and properties compared to quarks and charged leptons. E.g. their mixing is, as we will see in table A.3, not small. Their masses, or more precise, their mass limits are summarized in table A.2.
But measured are actually the mass-squared differences, namely

$$\Delta m^2_{sol} = (8.1 \pm 1.0) \cdot 10^{-5} \text{ eV}^2 \ , \qquad \left|\Delta m^2_{atm}\right| = (2.2 \pm 1.1) \cdot 10^{-3} \text{ eV}^2 \ . \tag{A.1}$$

Compatible with these data are two scenarios: The so-called normal ordering (NO), the inverted ordering (IO). Neutrinos are quasi-degenerate when their mass is larger than the

$$\begin{array}{rcl} \langle m_{\nu_e} \rangle & < & 2.3 \text{ eV} \\ \langle m_{\nu_\mu} \rangle & < & 170 \text{ keV} \\ \langle m_{\nu_\tau} \rangle & < & 15.5 \text{ MeV } (CL = 95\%) \end{array}$$

Table A.2: Neutrino Mass Limits

Δm^2. We also have the possibility that the lightest neutrino is massless. In this limit, i.e. $m_1 = 0$ for NO and $m_3 = 0$ for IO these two scenarios will be called normal (NH) and inverted hierarchy (IH). This is illustrated in figure A.1 (without LSND data [121]).

The mixing of neutrino mass and flavor eigenstates is described by the Pontecorvo-Maki-

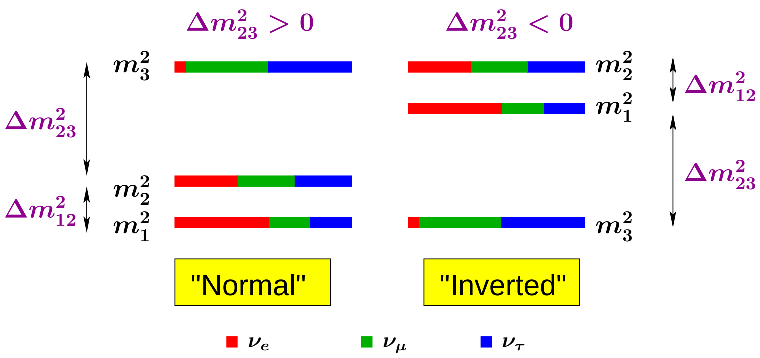

Figure A.1: Possible Neutrino Mass Orderings.

Nakagawa-Sakata (PMNS) matrix

$$U_{PMNS} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\text{atmospheric angle}} \underbrace{\begin{pmatrix} c_{13} & 0 & s_{13}\,e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}\,e^{i\delta} & 0 & c_{13} \end{pmatrix}}_{\text{reactor angle and Dirac CP phase}} \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{solar angle}} \underbrace{\mathrm{diag}(e^{i\alpha}, e^{i\beta}, 1)}_{\text{Majorana phases}}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \mathrm{diag}(e^{i\alpha}, e^{i\beta}, 1)\,, \tag{A.2}$$

where we have used the usual notations $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, δ is the Dirac CP violation phase, α and β are two possible Majorana CP violation phases. The values of the currently known mixing parameters are at 3σ:

$$\begin{array}{rcl} \sin^2\theta_{12} & = & 0.24 \ldots 0.41\,, \\ |U_{e3}|^2 = |s_{13}e^{-i\delta}|^2 & \leq & 0.044\,, \\ \sin^2\theta_{23} & = & 0.34 \ldots 0.68\,. \end{array}$$

Table A.3: Neutrino Mixing Angles

A.3 Quark Sector

The quark-sector consists of 6 different quarks separated by their flavors:
up-, down-, charm-, strange-, top- and bottom-quark.

Their masses are given in table A.4 at the scale of the Z-boson mass m_Z. In addition we

$m_u =$	1.7 ± 0.4 MeV	$m_c =$	0.62 ± 0.03 GeV	$m_t =$	171 ± 3 GeV
$m_d =$	3.0 ± 0.6 MeV	$m_s =$	54 ± 11 MeV	$m_b =$	2.87 ± 0.03 GeV

Table A.4: Quark Masses

have mixing in the quark-sector between the mass and flavor eigenstates, described by the Cabibbo-Kobayashi-Maskawa (CKM) matrix, where we use the same parametrization as for the PMNS matrix, i.e.

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} . \tag{A.3}$$

The experimentally determined limits are

$$(|V_{CKM}|) = \begin{pmatrix} 0.9739\ldots0.9751 & 0.221\ldots0.227 & 0.0029\ldots0.0045 \\ 0.221\ldots0.227 & 0.9730\ldots0.9744 & 0.039\ldots0.044 \\ 0.0048\ldots0.014 & 0.037\ldots0.043 & 0.9990\ldots0.9992 \end{pmatrix} . \tag{A.4}$$

This corresponds to the three angles and CP phase given in table A.5, by taking parametrization (A.3), where θ_{12} is the so-called Cabibbo angle.

$$\begin{aligned} s_{12} &= 0.2243 \pm 0.0016 \\ s_{23} &= 0.0413 \pm 0.0015 \\ s_{13} &= 0.0037 \pm 0.0005 \end{aligned}$$

$$\delta = 60^\circ \pm 14^\circ$$

Table A.5: Quark Mixing Angles

A.4 Gauge and Higgs Bosons

In the SM we have four gauge bosons with masses shown in table A.6. And last but not

$$\begin{aligned} m_\gamma &< 6 \cdot 10^{-17}\ \text{eV} \\ m_g &= 0\ ^1 \\ m_{W^\pm} &= 80.425 \pm 0.038\ \text{GeV} \\ m_{Z^0} &= 91.1876 \pm 0.0021\ \text{GeV} \end{aligned}$$

Table A.6: Gauge Boson Masses

least the Higgs boson mass or more precise its mass limit is

$$114.4\,\text{GeV} < m_H < 246\,\text{GeV}\,(CL = 95\%) , \tag{A.5}$$

[1] This is a theoretical value and not a measured one.

where the lower limit comes from the until now reached energies at CERN and the upper bound from electroweak data like the W and top quark masses.

Appendix B

Group Theory

B.1 Elements of Group Theory

General Definitions

In this section we want to provide a basis of group theory. More about group theory can be found e.g. in [122–124] or especially for Lie groups and grand unified theories in [4, 36, 125].

A *finite* (or *infinite*) *group* G is a finite (or infinite) set of elements together with a multiplication law that satisfies the four lower fundamental properties of closure, associativity, the identity property, and the inverse property. Elements A, B, C,... with binary operation between A and B denoted AB form a group if they fulfill

1. *Closure*:
   If A and B are two elements in G, then the product AB is also in G.

2. *Associativity*:
   The defined multiplication is associative, i.e., for all A, B, C in G, $(AB)C = A(BC)$.

3. *Identity*:
   There is an identity element E such that $EA = AE = A$ for every element A in G.

4. *Inverse*:
   There must be an inverse or reciprocal of each element. Therefore, the set must contain an element $B = A^{-1}$ such that $AA^{-1} = A^{-1}A = E$ for each element of G.

A group must contain at least one element, with the unique (up to isomorphism) single-element group known as the *trivial group*.

Hence we will restrict us to finite groups with more than one element except it is explicitly stated.

If two elements *commute* with each other they fulfill the relation

$$AB = BA \; . \tag{B.1}$$

The identity of the group commutes with all their elements. If all elements commute with each other the group is called *Abelian*.

Now we denote $AA = A^2$ and similar for A^n as well as $A^{-n} = (A^{-1})^n = (A^n)^{-1}$. An element is called to be of *order n* if there exists a smallest positive integer n for what the relation

$$A^n = E \tag{B.2}$$

holds. For an element A which is of order n, all elements

$$A, A^2, ..., A^{n-1}, A^n = E \tag{B.3}$$

are distinct. A group formed by such elements is called *cyclic group* and the element is the *generator of the group*.

Two groups are *isomorphic* ($G \approx G^{'}$) if there exists a one-to-one correspondence preserved under combinations.

To write all this in a convenient way, a *multiplication table* which gives all properties of a finite group is used. In this quadratic array the rows and columns are labeled according to elements of the group.

	E	A	B	$\ldots$
E	$E^2 = E$	$EA = A$	$EB = B$	$\ldots$
A	$AE = A$	A^2	AB	$\ldots$
B	$BE = B$	BA	B^2	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table B.1: Multiplication Table for a Cyclic Group

Like one can verify, every element of the group appears only once per row and column. This theorem is called *rearrangement theorem*. And if the multiplication table is symmetric with respect to their main diagonal it is an Abelian group.

The *order of a group* is the number of element in this group and denoted as h.

Subgroups and Cayley's Theorem

If we select a subset $\mathscr{H}$ of elements of a group G and this set of elements forms a group under the same law of transformations as G, we call $\mathscr{H}$ a *subgroup* of G. Every group has two trivial subgroups: the group itself and the group consisting only of the neutral element E. These groups are called *improper*. Every other subgroup except the two mentioned before are called *proper*.

Therefore, if $G \supset \mathscr{H}_1 \supset \mathscr{H}_2$ then $\mathscr{H}_2$ is also a subgroup of G.

If additional for any element A of the subgroup $\mathscr{H}$ holds that

$$A \mathscr{H} A^{-1} = \mathscr{H} \tag{B.4}$$

the subgroup is called to be a *normal subgroup* or equivalent an *invariant subgroup* or *self-conjugate subgroup*.

It should be annotated that all subgroups of Abelian groups are normal. In addition, the symmetric groups S_n (finite) are of particular importance because they exhaust the possible structure of finite groups. This is shown by

- *Cayley's theorem*:
  Every group of order n is isomorphic with a subgroup of S_n.

Conjugate Classes

If two elements A and B of G are *conjugate* then it's possible to find an element X in G such that

$$X\,A\,X^{-1} = B \tag{B.5}$$

is fulfilled. If $X = E$ we see that A is conjugate by itself. This leads to some equivalence relations ($\equiv$):

- $A \equiv A$
- If $A \equiv B$ then $B \equiv A$
- If $A \equiv B$ and $B \equiv C$ then $A \equiv C$

To separate a set into *classes* $\mathscr{C}_i$ such an equivalence relation is used. The number of distinct elements of a class is called *order of the class* $h_{\mathscr{C}_i}$ and if $h > 1$ then $h_{\mathscr{C}_i} \geq 2$. The identity E always forms a class by itself. And in the special case of Abelian groups, each element forms a class by itself because for any A,B of G $ABA^{-1} = B$ is true.

Moreover, all elements of a class have the same order and all classes of G are disjoint $\bigcap\limits_i \mathscr{C}_i = \emptyset$. This leads to

$$\sum_{\mathscr{C}_i} h_{\mathscr{C}_i} = h \tag{B.6}$$

Invariant Subgroups, Factor Groups, Homomorphism

For any element A and subgroup $\mathscr{H}$ of G it is possible to generate a set $A\mathscr{H}A^{-1}$ which is again a subgroup of G and is called to be a *conjugate subgroup* of $\mathscr{H}$ in G. If especially $A\mathscr{H}A^{-1} = \mathscr{H}$ holds, $\mathscr{H}$ is said to be an *invariant subgroup* in G. A group which has no invariant subgroup is called *simple*. If non of its invariant subgroups are Abelian its said to be *semisimple*. All subgroups of Abelian groups are obviously invariant. Also, we note that the product of two cosets of an invariant subgroup is again a coset.

When we treat the cosets of $\mathscr{H}$ as elements and define a product for their multiplication the cosets of the invariant subgroups form a group, the *factor group* (=*quotient group*) $G/\mathscr{H}$.

A group *homomorphism* is a map between two groups $G \to H$ such that

1. the identity of G is mapped to the one of H and
2. the group operation is preserved.

The *kernel* of G is the set of distinct elements mapped on the identity of H. For each element of H the number of elements in G mapping on it is constant. Therefore $\frac{h_G}{h_H} = m$ is called *multiplicity*.

It should be annotated that a homomorphism must also preserve the inverse map.

Direct Product

A group G is a *direct product* if its subgroups $\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n$ fulfill the following conditions:

1. The elements of different subgroups commute.
2. Every element g of G can only be written as $g = h_1...h_n$,

where h_i is an element of $\mathscr{H}_i$. A direct product is also called *Kronecker product* or *tensor product.* This direct product can be written as

$$G = \mathscr{H}_1 \times \mathscr{H}_2 \times ... \times \mathscr{H}_n \ , \tag{B.7}$$

whereas $\mathscr{H}_1, \mathscr{H}_2, ...$ is labeled as *direct factors* of G.

The Kronecker products for the group D_5 we will use are shown in table C.3.

Representations, Character Table

Group Representations

A set of a homomorphical map of a group onto a group of operators is denoted as $\mu = \{D(G)\}$ and called *representation of a group G.* Most of the groups have many different representations, possibly on different vector spaces. If their representations are similar they are considered to be *equivalent*.

Any representation of G can be restricted to a representation of any subgroup $\mathscr{H}$ of G. Furthermore, any representation on $\mathscr{H}$ can be extended to a representation of G on a larger vector space called *induced representation*.

Character Table

A possible description of a finite group is its *character table* which contains the characters of all irreducible representations (for definition see next section) for all conjugate classes. As mentioned before the character of a representation is

$$\chi^{(\mu)}(R) = tr\left[D^{(\mu)}(R)\right] \ . \tag{B.8}$$

The advantage of using traces is that they are invariant under transformation. That leads to the fact that all elements of one class have the same character just like equivalent representations.

The *trivial representation* which is a one-dimensional representation and whose characters are all one exist in all groups and is denoted as $\mathbf{1}_1$.

The character table of D_5 is given as table C.2.

Irreducible Representation

An *irreducible representation* is a group representation that has no nontrivial invariant subspaces, e.g. $O(n)$ has an irreducible representation on $\mathbb{R}^n$. Except finite or semisimple Lie groups it is not in general possible to break up their representation into a direct sum of irreducible representations.

Irreducible representations have some properties which can be derived by Schur's lemma. One version of *Schur's lemma* is that if D and D' are irreducible representations and A is a linear map such that $A\,D(R) = D'(R)A$ then $A = 0$ or A is invertible.

A number of these implications for irreducible representations are formalized below.

1. The *group orthogonality theorem*

$$\sum_R D^{(\mu)}(R)_{mn} D^{(\nu)}(R^{-1})_{m'n'} = \frac{h}{n_\mu} \delta_{\mu\nu}\delta_{mn'}\delta_{nm'} \ , \tag{B.9}$$

in which n_μ denotes the *dimension of the μth representation.*

2. The *completeness relation* for the columns of the character table:

$$\frac{h}{h_{\mathscr{C}_i}} \delta_{ji'} = \sum_\mu \chi^{(\mu)}_{\mathscr{C}_i} \chi^{(\mu)}_{\mathscr{C}_j} \ , \tag{B.10}$$

where i' is the index of the class containing the inverse elements of $\mathscr{C}_i$.

3. *Character orthogonality relation* for the rows of the character table:

$$\sum_R \chi^\mu(R)\chi^\nu(R^{-1}) = h\,\delta_{\mu\nu} \tag{B.11}$$

4. For unitary representations:

$$\frac{h}{h_{\mathscr{C}_i}} = \sum_\mu |\chi^{(\mu)}_{\mathscr{C}_i}|^2 \tag{B.12}$$

From now on we consider irreducible representation unless otherwise mentioned.

If there exist a similarity transformation between two representations $\mu = \{D^{(\mu)}(R)\}$ and $\nu = \{D^{(\nu)}(R)\}$ like $D^{(\nu)}(R) = X D^{(\mu)}(R) X^{-1}$ for all elements R of G then they are called *equivalent.* Every representation μ of each group has its *complex conjugate representation* $\bar{\mu}$ and for their matrix representations $D^{(\bar{\mu})}(R) = D^{(\mu)}(R)^*$ respectively for all elements R of G. Similar an *adjoint representation* μ^{ad} is defined and for the matrix representation $D^{(\mu^{ad})}(R) = \left[D^{(\mu)}(R)^T\right]^{-1}$, $\forall R \in G$. It is important to mention that for finite groups the complex conjugate and the adjoint representation are corresponding since all representations are unitary and that all have the same dimension as μ.

For the case that the number of all distinct representation matrices is equal to the group order the representation is called *faithful*.

B.2 Clebsch-Gordan Coefficients

The *Clebsch-Gordan Coefficients* (CG) denoted[1] by $\left(\begin{array}{cc|cc} \mu & \nu & \tau & \alpha \\ i & j & k & \end{array}\right)$. That is the coefficient in front of the k^{th} component of a vector transforming under $D^{(\tau)}(R)$ which is the product of the i^{th} component of a vector transforming under $D^{(\mu)}(R)$ and the j^{th} component of a vector transforming under $D^{(\nu)}(R)$, where α is the so-called *multiplicity* running from 1 to $(\mu\nu\tau)$. The inverse of the CG is defined as

[1] The notation and definition we are using is different compared to the most textbooks of mathematic. There is often the notation $(\mu i, \nu j|\tau\alpha k)$ used and the inverse of the CG we are using is called a Clebsch-Gordan coefficient.

$$\left(\begin{array}{cc|cc} \mu & \nu & \tau & \alpha \\ i & j & k & \end{array}\right)^{-1} := \left(\begin{array}{cc|cc} \tau & \alpha & \mu & \nu \\ k & & i & j \end{array}\right) \stackrel{2}{=} \overline{\left(\begin{array}{cc|cc} \mu & \nu & \tau & \alpha \\ i & j & k & \end{array}\right)} . \tag{B.13}$$

From the general equation

$$\sum_{i,j,k,l} \left(\begin{array}{cc|cc} \mu & \nu & \tau' & \alpha' \\ i & k & s' & \end{array}\right) \underbrace{D^{(\mu)}(R)_{ij} D^{(\nu)}(R)_{kl}}_{D^{(\mu\times\nu)}(R)_{ik,jl}} \left(\begin{array}{cc|cc} \tau & \alpha & \mu & \nu \\ s & & j & l \end{array}\right) = D^{(\tau)}(R)_{ss'}\delta_{\tau\tau'}\delta_{\alpha\alpha'} \tag{B.14}$$

we can see that the CG are the transformation matrices block-diagonalizing the representation matrices of the product $\mu \times \nu$.

Two important relations for CG are given through the orthonormalization and completeness relation:

$$\sum_{j=1}^{N_\mu}\sum_{l=1}^{N_\nu} \overline{\left(\begin{array}{cc|cc} \mu & \nu & \tau & \alpha \\ j & l & k & \end{array}\right)} \left(\begin{array}{cc|cc} \mu & \nu & \tau' & \alpha' \\ j & l & k' & \end{array}\right) = \delta_{\alpha\alpha'}\delta_{kk'}\delta_{\tau\tau'} \tag{B.15}$$

$$\sum_{\tau,\alpha,k} \left(\begin{array}{cc|cc} \mu & \nu & \tau & \alpha \\ i & l & k & \end{array}\right) \overline{\left(\begin{array}{cc|cc} \mu & \nu & \tau & \alpha \\ j' & l' & k & \end{array}\right)} = \delta_{jj'}\delta_{ll'} \tag{B.16}$$

For the practical calculation of the CG the following formula is used

$$\left(\begin{array}{cc|cc} \mu & \nu & \tau & \alpha \\ a & b & i & \end{array}\right) = \sqrt{\frac{N_\tau}{h}} \frac{\sum\limits_{R\in G} \overline{D^{(\mu)}(R)_{aj}}\, \overline{D^{(\nu)}(R)_{bk}}\, D^{(\tau)}(R)_{il}}{\sqrt{\sum\limits_{R\in G} D^{(\mu)}(R)_{jj}\, D^{(\nu)}(R)_{kk}\, \overline{D^{(\tau)}(R)_{ll}}}} , \tag{B.17}$$

where j, k and l are chosen in a way that the denominator is real and > 0.

The CG for our group D_5 are shown in section C.3.

[2]Valid for finite groups (all representations are unitary).

Appendix C

Flavor Symmetry D_5

C.1 Properties of D_5

C.1.1 Complex Generators of D_5

The generator relations for the group D_5 are $A^5 = 1$, $B^2 = 1$ and $ABA = B$ or in other words, a representation is given for $n = 5$ by

$$< A, B|\, A^n = 1,\ B^2 = 1,\ (AB)^n = AB > \quad . \tag{C.1}$$

$$2_1: \quad A = \begin{pmatrix} e^{i\frac{2\pi}{5}} & 0 \\ 0 & e^{-i\frac{2\pi}{5}} \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$2_2: \quad A = \begin{pmatrix} e^{i\frac{4\pi}{5}} & 0 \\ 0 & e^{-i\frac{4\pi}{5}} \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Table C.1: Complex Generators of D_5

C.1.2 Character Table of D_5

classes	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
G	$\mathbb{1}$	B	A	A^2
$h_{\mathcal{C}_i}$	1	5	2	2
$n_{\mathcal{C}_i}$	1	2	5	5
1_1	1	1	1	1
1_2	1	-1	1	1
2_1	2	0	$\frac{1}{2}(-1+\sqrt{5})$	$\frac{1}{2}(-1-\sqrt{5})$
2_2	2	0	$\frac{1}{2}(-1-\sqrt{5})$	$\frac{1}{2}(-1+\sqrt{5})$

Table C.2: Character Table

At which for every representation μ the character $\chi^{(\mu)}_{\mathcal{C}_i}$ is given for each class $\mathcal{C}_i$, $h_{\mathcal{C}_i}$ is the number of distinct elements,

$n_{\mathcal{C}_i}$ the order of the element and
G the representative of the class in terms of the generators A and B.

C.1.3 Kronecker Products of D_5

$$
\begin{aligned}
1_1 \times 1_1 &= 1_1 \\
1_2 \times 1_1 &= 1_2 \\
2_1 \times 1_1 &= 2_1 \\
2_2 \times 1_1 &= 2_2 \\
1_2 \times 1_2 &= 1_1 \\
2_1 \times 1_2 &= 2_1 \\
2_2 \times 1_2 &= 2_2 \\
2_1 \times 2_1 &= 1_1 + 1_2 + 2_2 \\
2_2 \times 2_1 &= 2_1 + 2_2 \\
2_2 \times 2_2 &= 1_1 + 1_2 + 2_1
\end{aligned}
$$

Table C.3: Kronecker Products of D_5

C.2 Similarity Transformation between Representations and their Complex Conjugates

The similarity transformation

$$\overline{R} = U\,R\,U^T \tag{C.2}$$

between representation matrix R and its conjugate $\overline{R}$ is done by the transformation matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{C.3}$$

and is for the representation 2_1 and 2_2 the same.

C.3 Clebsch-Gordan Coefficients For Complex Generators

In the following are the CG for complex generators given, where we use the notation $R^x O S$ (x stands for $\dagger$ or T) for $R^x \times S$, while O is the CG which "replaces" the $\times$ by performing the calculation.

$R^\dagger O S$

product		Coefficient
$1_1 \times 2_1$	2_1	$\begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \end{pmatrix}$
$1_2 \times 2_1$	2_1	$\begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \end{pmatrix}$
$2_1 \times 2_1$	1_1	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$
$2_1 \times 2_1$	1_2	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$
$2_1 \times 2_1$	2_2	$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$
$2_2 \times 2_1$	2_1	$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$
$2_2 \times 2_1$	2_2	$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$

product		Coefficient
$1_1 \times 2_2$	2_2	$\begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \end{pmatrix}$
$1_2 \times 2_2$	2_2	$\begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \end{pmatrix}$
$2_1 \times 2_2$	2_1	$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$
$2_1 \times 2_2$	2_2	$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$
$2_2 \times 2_2$	1_1	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$
$2_2 \times 2_2$	1_2	$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$
$2_2 \times 2_2$	2_1	$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$

product		Coefficient
$1_1 \times 1_1$	1_1	(1)
$1_2 \times 1_1$	1_2	(1)
$2_1 \times 1_1$	2_1	$\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$
$2_2 \times 1_1$	2_2	$\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$
$1_1 \times 1_2$	1_2	(1)
$1_2 \times 1_2$	1_1	(1)
$2_1 \times 1_2$	2_1	$\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{pmatrix}$
$2_2 \times 1_2$	2_2	$\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{pmatrix}$

Table C.4: CG for Complex Generators - $R^\dagger\, O\, S$

$R^T\,O\,S$

product		Coefficient
$1_1 \times 2_1$	2_1	$\begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \end{pmatrix}$
$1_2 \times 2_1$	2_1	$\begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \end{pmatrix}$
$2_1 \times 2_1$	1_1	$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$
$2_1 \times 2_1$	1_2	$\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$
$2_1 \times 2_1$	2_2	$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$
$2_2 \times 2_1$	2_1	$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$
$2_2 \times 2_1$	2_2	$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$

product		Coefficient
$1_1 \times 2_2$	2_2	$\begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \end{pmatrix}$
$1_2 \times 2_2$	2_2	$\begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \end{pmatrix}$
$2_1 \times 2_2$	2_1	$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$
$2_1 \times 2_2$	2_2	$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$
$2_2 \times 2_2$	1_1	$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$
$2_2 \times 2_2$	1_2	$\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$
$2_2 \times 2_2$	2_1	$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$

product		Coefficient
$1_1 \times 1_1$	1_1	(1)
$1_2 \times 1_1$	1_2	(1)
$2_1 \times 1_1$	2_1	$\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$
$2_2 \times 1_1$	2_2	$\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$
$1_1 \times 1_2$	1_2	(1)
$1_2 \times 1_2$	1_1	(1)
$2_1 \times 1_2$	2_1	$\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{pmatrix}$
$2_2 \times 1_2$	2_2	$\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{pmatrix}$

Table C.5: CG for Complex Generators - $R^T\,O\,S$

C.4 Mass Matrices

This are the mass matrices for down-type quarks and charged leptons obtained like described in section 3.4.1. There one can also find the proceeding to get the corresponding mass matrices for the up-type quarks and neutrinos.

L	L^C	Mass Matrix	Denomination
$(1_2, 1_1, 1_1)$	$(2_1, 1_{1[2]})$	$\begin{pmatrix} \kappa_1\psi_2^1 & -\kappa_1\psi_1^1 & \kappa_4\phi^{2[1]} \\ \kappa_2\psi_2^1 & \kappa_2\psi_1^1 & \kappa_5\phi^{1[2]} \\ \kappa_3\psi_2^1 & \kappa_3\psi_1^1 & \kappa_6\phi^{1[2]} \end{pmatrix}$	$M_{1[2]}$
$(1_2, 1_1, 1_1)$	$(2_2, 1_{1[2]})$	$\begin{pmatrix} \kappa_1\psi_2^2 & -\kappa_1\psi_1^2 & \kappa_4\phi^{2[1]} \\ \kappa_2\psi_2^2 & \kappa_2\psi_1^2 & \kappa_5\phi^{1[2]} \\ \kappa_3\psi_2^2 & \kappa_3\psi_1^2 & \kappa_6\phi^{1[2]} \end{pmatrix}$	$M_{3[4]}$
$(1_2, 1_2, 1_1)$	$(2_1, 1_{1[2]})$	$\begin{pmatrix} \kappa_1\psi_2^1 & -\kappa_1\psi_1^1 & \kappa_4\phi^{2[1]} \\ \kappa_2\psi_2^1 & -\kappa_2\psi_1^1 & \kappa_5\phi^{2[1]} \\ \kappa_3\psi_2^1 & \kappa_3\psi_1^1 & \kappa_6\phi^{1[2]} \end{pmatrix}$	$M_{5[6]}$
$(1_2, 1_2, 1_1)$	$(2_2, 1_{1[2]})$	$\begin{pmatrix} \kappa_1\psi_2^2 & -\kappa_1\psi_1^2 & \kappa_4\phi^{2[1]} \\ \kappa_2\psi_2^2 & -\kappa_2\psi_1^2 & \kappa_5\phi^{2[1]} \\ \kappa_3\psi_2^2 & \kappa_3\psi_1^2 & \kappa_6\phi^{1[2]} \end{pmatrix}$	$M_{7[8]}$
$(2_1, 1_1)$	$(2_1, 1_1)$	$\left(\begin{array}{cc\|c} \kappa_1\psi_2^2 & \kappa_2\phi^1 - \kappa_3\phi^2 & \kappa_5\psi_2^1 \\ \kappa_2\phi^1 + \kappa_3\phi^2 & \kappa_1\psi_1^2 & \kappa_5\psi_1^1 \\ \hline \kappa_4\psi_2^1 & \kappa_4\psi_1^1 & \kappa_6\phi^1 \end{array}\right)$	M_9
$(2_1, 1_2)$	$(2_1, 1_1)$	$\left(\begin{array}{cc\|c} \kappa_1\psi_2^2 & \kappa_2\phi^1 - \kappa_3\phi^2 & \kappa_5\psi_2^1 \\ \kappa_2\phi^1 + \kappa_3\phi^2 & \kappa_1\psi_1^2 & \kappa_5\psi_1^1 \\ \hline \kappa_4\psi_2^1 & -\kappa_4\psi_1^1 & \kappa_6\phi^2 \end{array}\right)$	M_{10}
$(2_1, 1_2)$	$(2_1, 1_2)$	$\left(\begin{array}{cc\|c} \kappa_1\psi_2^2 & \kappa_2\phi^1 - \kappa_3\phi^2 & \kappa_5\psi_2^1 \\ \kappa_2\phi^1 + \kappa_3\phi^2 & \kappa_1\psi_1^2 & -\kappa_5\psi_1^1 \\ \hline \kappa_4\psi_2^1 & -\kappa_4\psi_1^1 & \kappa_6\phi^1 \end{array}\right)$	M_{11}

L	L^C	Mass Matrix	Denomination
$(2_2, 1_1)$	$(2_2, 1_1)$	$\left(\begin{array}{cc\|c} \kappa_1\psi_1^1 & \kappa_2\phi^1 - \kappa_3\phi^2 & \kappa_5\psi_2^2 \\ \kappa_2\phi^1 + \kappa_3\phi^2 & \kappa_1\psi_2^1 & \kappa_5\psi_1^2 \\ \hline \kappa_4\psi_2^2 & \kappa_4\psi_1^2 & \kappa_6\phi^1 \end{array}\right)$	M_{12}
$(2_2, 1_2)$	$(2_2, 1_1)$	$\left(\begin{array}{cc\|c} \kappa_1\psi_1^1 & \kappa_2\phi^1 - \kappa_3\phi^2 & \kappa_5\psi_2^2 \\ \kappa_2\phi^1 + \kappa_3\phi^2 & \kappa_1\psi_2^1 & \kappa_5\psi_1^2 \\ \hline \kappa_4\psi_2^2 & -\kappa_4\psi_1^2 & \kappa_6\phi^2 \end{array}\right)$	M_{13}
$(2_2, 1_2)$	$(2_2, 1_2)$	$\left(\begin{array}{cc\|c} \kappa_1\psi_1^1 & \kappa_2\phi^1 - \kappa_3\phi^2 & \kappa_5\psi_2^2 \\ \kappa_2\phi^1 + \kappa_3\phi^2 & \kappa_1\psi_2^1 & -\kappa_5\psi_1^2 \\ \hline \kappa_4\psi_2^2 & -\kappa_4\psi_1^2 & \kappa_6\phi^1 \end{array}\right)$	M_{14}
$(2_2, 1_1)$	$(2_1, 1_1)$	$\left(\begin{array}{cc\|c} \kappa_1\psi_1^2 & \kappa_2\psi_2^1 & \kappa_4\psi_2^2 \\ \kappa_2\psi_1^1 & \kappa_1\psi_2^2 & \kappa_4\psi_1^2 \\ \hline \kappa_3\psi_2^1 & \kappa_3\psi_1^1 & \kappa_5\phi^1 \end{array}\right)$	M_{15}
$(2_2, 1_2)$	$(2_1, 1_1)$	$\left(\begin{array}{cc\|c} \kappa_1\psi_1^2 & \kappa_2\psi_2^1 & \kappa_4\psi_2^2 \\ \kappa_2\psi_1^1 & \kappa_1\psi_2^2 & \kappa_4\psi_1^2 \\ \hline \kappa_3\psi_2^1 & -\kappa_3\psi_1^1 & \kappa_5\phi^2 \end{array}\right)$	M_{16}
$(2_2, 1_1)$	$(2_1, 1_2)$	$\left(\begin{array}{cc\|c} \kappa_1\psi_1^2 & \kappa_2\psi_2^1 & \kappa_4\psi_2^2 \\ \kappa_2\psi_1^1 & \kappa_1\psi_2^2 & -\kappa_4\psi_1^2 \\ \hline \kappa_3\psi_2^1 & \kappa_3\psi_1^1 & \kappa_5\phi^2 \end{array}\right)$	M_{17}
$(2_2, 1_2)$	$(2_1, 1_2)$	$\left(\begin{array}{cc\|c} \kappa_1\psi_1^2 & \kappa_2\psi_2^1 & \kappa_4\psi_2^2 \\ \kappa_2\psi_1^1 & \kappa_1\psi_2^2 & -\kappa_4\psi_1^2 \\ \hline \kappa_3\psi_2^1 & -\kappa_3\psi_1^1 & \kappa_5\phi^1 \end{array}\right)$	M_{18}

Table C.6: Mass Matrices of D_5 (CG for complex generators used)

Appendix D

Three Higgs Model - Higgs Masses

In the following the characteristic polynomials, the eigenvalues and the eigenvectors for the different cases of the 3 Higgs potential are given. Thereby the eigenvectors are not normalized and constant factors in the characteristic polynomials are omitted (because they have to be zero and therefore they would not change anything).

Aside from this, solutions which would be complex are solved for special cases and/or written in a more convenient way.

Furthermore it should be noticed that the mass squared eigenvalues have to be positive which gives us more restrictions on our parameters which we will show for cases the reader can benefit of it.

One $VEV \neq 0$

For $w \neq 0$

The characteristic polynomial with x as variable is for M_{rr}:

$$\left(x - 2\,\mu_2^2\right)\left[x + \mu_1^2 - \tfrac{1}{2}\left(\sigma_1 - \sigma_2 + \sigma_3\right)\,w^2\right]\left[x + \mu_1^2 - \tfrac{1}{2}\left(\sigma_1 + \sigma_2 + \sigma_3\right)\,w^2\right]$$

and for M_{ii}:

$$x\left[x + \mu_1^2 - \tfrac{1}{2}\left(\sigma_1 - \sigma_2 + \sigma_3\right)\,w^2\right]\left[x + {\mu_1}^2 - \tfrac{1}{2}\left(\sigma_1 + \sigma_2 + \sigma_3\right)\,w^2\right]$$

$Eigenvalues\,of\,M_{ri}^2$ $(1-3: M_{rr},\, 4-6: M_{ii})$	$Eigenvectors$
$2\mu_2^2$	$(1,0,0)^T$
$-\mu_1^2+\frac{1}{2}(\sigma_1+\sigma_2+\sigma_3)w^2$	$(0,1,1)^T$
$-\mu_1^2+\frac{1}{2}(\sigma_1-\sigma_2+\sigma_3)w^2$	$(0,-1,1)^T$
0	$(1,0,0)^T$
$-\mu_1^2+\frac{1}{2}(\sigma_1-\sigma_2+\sigma_3)w^2]$	$(0,1,1)^T$
$-\mu_1^2+\frac{1}{2}(\sigma_1+\sigma_2+\sigma_3)w^2$	$(0,-1,1)^T$

The characteristic polynomial with x as variable is for M_{pm}:

$x\left(x+\mu_1^2-\frac{\sigma_1}{2}\,w^2\right)^2$

$Eigenvalues\,of\,M_{pm}^2$	$Eigenvectors$
0	$(1,0,0)^T$
$-\mu_1^2+\frac{\sigma_1}{2}w^2$	$(0,1,0)^T$ [1]
$-\mu_1^2+\frac{\sigma_1}{2}w^2$	$(0,0,1)^T$ [1]

Parameter restrictions through the requirement of positive eigenvalues:

[1] An important circumstance is that if some eigenvalues are degenerate every linear combination of their eigenvectors will be a solution.

$$\lambda_4 > 0 \ ,$$

$$\sigma_1 > 2\frac{\mu_1^2}{w^2} \ ,$$

$$\sigma_3 > \sigma_2 \ ,$$

$$\sigma_2 > 0 \ .$$

For $v_\pm \neq 0$

The characteristic polynomial with x as variable is for M_{rr}:

$$\left[x - \tfrac{1}{2}(\sigma_1 + \sigma_3)\, v_\pm^2 + \mu_2^2\right] \left[\tfrac{1}{2}x - (\lambda_1 + \lambda_2)\, v_\pm^2\right] \left[2\,x + (4\,\lambda_2 - \lambda_3)\, {v_\pm}^2\right]$$

and for M_{ii}:

$$x \left[x - \tfrac{1}{2}(\sigma_1 + \sigma_3)\, v_\pm^2 + \mu_2^2\right] \left[2\,x + (4\,\lambda_2 - \lambda_3)\, {v_\pm}^2\right]$$

$Eigenvalues\, of\, M_{ri}^2$ $(1-3: M_{rr},\, 4-6: M_{ii})$	$Eigenvectors$ $for\, v_+ \neq 0$	$Eigenvectors$ $for\, v_- \neq 0$
$-\mu_2^2 + \frac{1}{2}(\sigma_1+\sigma_3)v_\pm^2$	$(1,0,0)^T$	$(1,0,0)^T$
$2(\lambda_1+\lambda_2)v_\pm^2$	$(0,1,0)^T$	$(0,0,1)^T$
$-\frac{1}{2}(4\lambda_2-\lambda_3)v_\pm^2$	$(0,0,1)^T$	$(0,1,0)^T$
$-\mu_2^2 + \frac{1}{2}(\sigma_1+\sigma_3)v_\pm^2$	$(1,0,0)^T$	$(1,0,0)^T$
0	$(0,1,0)^T$	$(0,0,1)^T$
$-\frac{1}{2}(4\lambda_2-\lambda_3)v_\pm^2$	$(0,0,1)^T$	$(0,1,0)^T$

The characteristic polynomial with x as variable is for M_{pm}:

$$\left(x + \mu_2^2 - \tfrac{\sigma_1}{2}\, {v_\pm}^2\right) x \left(x + 2\,\lambda_2\, {v_\pm}^2\right)$$

$Eigenvalues\, of\; M_{pm}^2$	$Eigenvectors$ $for\, v_+ \neq 0$	$Eigenvectors$ $for\, v_- \neq 0$
$-\mu_2^2 + \frac{\sigma_1}{2} v_\pm^2$	$(1,0,0)^T$	$(1,0,0)^T$
0	$(0,1,0)^T$	$(0,0,1)^T$
$-2\lambda_2 v_\pm^2$	$(0,0,1)^T$	$(0,1,0)^T$

Parameter restrictions through the requirement of positive eigenvalues:

$$\lambda_2 < 0\ ,$$

$$\sigma_1 > 2\frac{\mu_2^2}{v_\pm^2}\ ,$$

$$\lambda_1 + \lambda_2 > 0\ ,$$

$$4\lambda_2 - \lambda_3 < 0\ ,$$

$$\sigma_3 \geq 0\ .$$

Two $VEV \neq 0$

For $w = \alpha = 0$

and $4\lambda_2 = \lambda_3$

The characteristic polynomial with x as variable is for M_{ri}:

$$x^3\left[x - 2(\lambda_1+\lambda_2)[v_+^2+v_-^2]\right]\cdot$$
$$\cdot\left[x+\mu_2^2-\tfrac{1}{2}(\sigma_1+\sigma_3)[v_+^2+v_-^2]-\sigma_2 v_+ v_-\right]\left[x+\mu_2^2-\tfrac{1}{2}(\sigma_1+\sigma_3)[v_+^2+v_-^2]+\sigma_2 v_+ v_-\right]$$

$Eigenvalues\, of\ M_{ri}^2$	$Eigenvectors$
$-\mu_2^2+\frac{1}{2}(\sigma_1+\sigma_3)[v_+^2+v_-^2]+\sigma_2 v_+ v_-$	$(\sin\beta,0,0,1-\cos\beta,0,0)^T$
0	$(0,\cos\beta v_-,-v_+,0,0,0)^T$
0	$(0,v_-\sin\beta,0,0,0,-v_+)^T$
$2(\lambda_1+\lambda_2)[v_+^2+v_-^2]$	$(0,v_+,\cos\beta v_-,0,0,\sin\beta v_-)^T$
$-\mu_2^2+\frac{1}{2}(\sigma_1+\sigma_3)[v_+^2+v_-^2]-\sigma_2 v_+ v_-$	$(-\sin\beta,0,0,1+\cos\beta,0,0)^T$
0	$(0,0,0,0,1,0)^T$

The characteristic polynomial with x as variable is for M_{pm}:

$x\left[x+\mu_2^2-\frac{\sigma_1}{2}[v_+^2+v_-^2]\right]\left[x+2\lambda_2[v_+^2+v_-^2]\right]$

$Eigenvalues\, of\ M_{pm}^2$	$Eigenvectors$
$-\mu_2^2+\frac{\sigma_1}{2}[v_+^2+v_-^2]$	$(1,0,0)^T$
0	$(0,v_+e^{i\beta},v_-)^T$
$-2\lambda_2[v_+^2+v_-^2]$	$\begin{pmatrix} 0 \\ v_-e^{i\beta} \\ -v_+ \end{pmatrix}$

Parameter restrictions through the requirement of positive eigenvalues:

$$\sigma_3 \geq 0 \ ,$$

$$\sigma_1 > 2\frac{\mu_2^2}{v_+^2+v_-^2} \ ,$$

$$\lambda_1 + \lambda_2 > 0 \ ,$$

$$\lambda_2 < 0 \ .$$

For $w = \alpha = 0$ and $v_+^2 = v_-^2$

The characteristic polynomial with x as variable is for M_{ri}:

$$x^2 \left[x - (4\lambda_1 + \lambda_3)v_-^2\right] \left[x - (4\lambda_2 - \lambda_3)v_-^2\right] \cdot$$

$$\cdot \left[x + \mu_2^2 - (\sigma_1 + \sigma_2 + \sigma_3)v_-^2\right] \left[x + \mu_2^2 - (\sigma_1 - \sigma_2 + \sigma_3)v_-^2\right]$$

$Eigenvalues\, of\, M_{ri}^2$	$Eigenvectors$
$(4\lambda_1 + \lambda_3)v_-^2$	$(0, 1, \cos\beta, 0, 0, \sin\beta)^T$
$(4\lambda_2 - \lambda_3)v_-^2$	$(0, -1, \cos\beta, 0, 0, \sin\beta)^T$
0	$(0, 0, \sin\beta, 0, 0, -\cos\beta)^T$
0	$(0, 0, 0, 0, 1, 0)^T$
$-\mu_2^2 + (\sigma_1 + \sigma_2 + \sigma_3)v_-^2$	$(1 + \cos\beta, 0, 0, \sin\beta, 0, 0)^T$
$-\mu_2^2 + (\sigma_1 - \sigma_2 + \sigma_3)v_-^2$	$(-1 + \cos\beta, 0, 0, \sin\beta, 0, 0)^T$

The characteristic polynomial with x as variable is for M_{pm}:

$$x \left[x + \mu_2^2 - \sigma_1 v_-^2\right] \left[x + \lambda_3\, v_-^2\right]$$

$Eigenvalues\, of\, M_{pm}^2$	$Eigenvectors$
$-\mu_2^2+\sigma_1 v_-^2$	$(1,0,0)^T$
$-\lambda_3\, v_-^2$	$(0,e^{i\beta},-1)^T$
0	$(0,1,e^{-i\beta})^T$

Parameter restrictions through the requirement of positive eigenvalues:

$$\lambda_3 - 4\lambda_1 > 0 \ ,$$

$$\lambda_3 + 4\lambda_2 < 0 \ ,$$

$$\sigma_1 > \frac{\mu_2^2}{v_-^2} \ ,$$

$$\sigma_3 + \sigma_2 \geq 0 \ ,$$

$$\sigma_3 - \sigma_2 \geq 0 \ ,$$

$$\lambda_3 < 0 \ .$$

For $v_\pm = 0$ and $\sigma_2 = 0$

The characteristic polynomial with x as variable is for M_{ri}:

$$x^2\left[(\lambda_3 - 4\,\lambda_2)v_\pm{}^2 - 2\,x\right]^2 \cdot$$
$$\cdot\left[-x^2 + 2(\lambda_1+\lambda_2)v_\pm^2 x - 2\lambda_4 w^2 x + \left[(\sigma_1+\sigma_3)^2 - 4\lambda_4(\lambda_1+\lambda_2)\right] w^2 v_\pm^2\right]$$

$Eigenvalues\, of\, M_{ri}^2$	$Eigenvectors$ $for\, v_- \neq 0$	$Eigenvectors$ $for\, v_+ \neq 0$
$-\frac{1}{2}(4\lambda_2-\lambda_3)v_\pm^2$	$(0,1,0,0,0,0)^T$	$(0,0,1,0,0,0)^T$
0	$(0,0,0,1,0,0)^T$	$(0,1,0,0,0,0)^T$
$-\frac{1}{2}(4\lambda_2-\lambda_3)v_\pm^2$	$(0,0,0,0,1,0)^T$	$(0,0,0,0,0,1)^T$

The rest of the eigenvalues and eigenvectors we get for $v_+ \neq 0$ from the following matrix which have its seeds in M_{ri} by deleting the 1^{st}, 2^{nd}, 4^{th} and 5^{th} row as well as column. For $v_- \neq 0$ we just have to replace v_+ with v_- and α with β. But this time are the 1^{st}, 3^{rd}, 4^{th} and 6^{th} row and column from M_{ri} deleted.

$$\begin{bmatrix} 2\,\lambda_4\,w^2 & w\,v_+\,\cos(\alpha)\,(\sigma_1+\sigma_3) & w\,v_+\,\sin(\alpha)\,(\sigma_1+\sigma_3) \\ w\,v_+\,\cos(\alpha)\,(\sigma_1+\sigma_3) & (2\,\lambda_1+2\,\lambda_2)\,{v_+}^2\,(\cos(\alpha))^2 & (2\,\lambda_1+2\,\lambda_2)\,{v_+}^2\sin(\alpha)\cos(\alpha) \\ w\,v_+\,\sin(\alpha)\,(\sigma_1+\sigma_3) & (2\,\lambda_1+2\,\lambda_2)\,{v_+}^2\sin(\alpha)\cos(\alpha) & (2\,\lambda_1+2\,\lambda_2)\,{v_+}^2\,(\sin(\alpha))^2 \end{bmatrix}$$

where the eigenvalue with its eigenvector

0	$(0,0,\sin\beta,0,0,-\cos\beta)^T$	$(0,\sin\alpha,0,0,-\cos\alpha,0)^T$

is contained.

The characteristic polynomial with x as variable is for M_{pm}:

$x\left[2\,x+\sigma_3\,(w^2+{v_\pm}^2)\right]\left(2\,x+\sigma_3\,w^2+4\,\lambda_2\,{v_\pm}^2\right)$

$Eigenvalues\,of\,M_{pm}^2$	$Eigenvectors\,for\,v_- \neq 0$	$Eigenvectors\,for\,v_+ \neq 0$
$-\frac{\sigma_3}{2}w^2 - 2\lambda_2 v_\pm^2$	$(0,1,0)^T$	$(0,0,1)^T$
0	$(w,0,v_- e^{-i\beta})^T$	$(w,-v_+e^{-i\alpha},0)^T$
$-\frac{\sigma_3}{2}(v_\pm^2+w^2)$	$(-v_-e^{i\beta},0,w)^T$	$(-v_+e^{i\alpha},w,0)^T$

Parameter restrictions through the requirement of positive eigenvalues:

$$\lambda_3 - 4\lambda_2 > 0\ ,$$

$$-4\frac{\lambda_2}{\sigma_3} < \frac{w^2}{v_\pm^2}\ ,$$

$$\sigma_3 < 0\ ,$$

$$\lambda_4 w^2 + (\lambda_1+\lambda_2)v_\pm^2 > \sqrt{\lambda_4^2 w^4 + (\lambda_1+\lambda_2)^2 v_\pm^4 - [2\lambda_4(\lambda 1+\lambda_2) - (\sigma_1+\sigma_3)^2](wv_\pm)^2}\ ,$$

where the radiant has to be positive.

All Three $VEV \neq 0$

For $\sigma_2 = 0$

and $4\,\lambda_2 = \lambda_3$

The characteristic polynomial with x as variable is for M_{ri}:

$x^4\left(x - \lambda_4 w^2 - (\lambda_1+\lambda_2)(v_+^2+v_-^2) - X\right)\left(x - \lambda_4 w^2 - (\lambda_1+\lambda_2)(v_+^2+v_-^2) + X\right)$

$$X = \sqrt{\lambda_4^2 w^4 + (\lambda_1+\lambda_2)^2(v_+^2+v_-^2)^2 + \left[(\sigma_1+\sigma_3)^2 - 2\lambda_4(\lambda_1+\lambda_2)\right](v_+^2+v_-^2)w^2}$$
$$Y = (4\,\lambda_1 + \lambda_3)$$

The eigenvalues and eigenvectors we get from $M_{ri} =$

$$\left[\begin{array}{cccccc} 2\,\lambda_4\,w^2 & w\,v_+\,\cos(\alpha)\,(\sigma_1+\sigma_3) & w\,v_-\,\cos(\beta)\,(\sigma_1+\sigma_3) & 0 & w\,v_+\,\sin(\alpha)\,(\sigma_1+\sigma_3) & w\,v_-\,\sin(\beta)\,(\sigma_1+\sigma_3) \\ w\,v_+\,\cos(\alpha)\,(\sigma_1+\sigma_3) & \frac{1}{2}\,{v_+}^2\,(\cos(\alpha))^2\,Y & \frac{1}{2}\,v_-\,\cos(\beta)\,v_+\,\cos(\alpha)\,Y & 0 & \frac{1}{2}\,{v_+}^2\cos(\alpha)\sin(\alpha)\,Y & \frac{1}{2}\,v_+\,v_-\,\cos(\alpha)\sin(\beta)\,Y \\ w\,v_-\,\cos(\beta)\,(\sigma_1+\sigma_3) & \frac{1}{2}\,v_-\,\cos(\beta)\,v_+\,\cos(\alpha)\,Y & \frac{1}{2}\,{v_-}^2\,(\cos(\beta))^2\,Y & 0 & \frac{1}{2}\,v_+\,v_-\,\sin(\alpha)\cos(\beta)\,Y & \frac{1}{2}\,{v_-}^2\cos(\beta)\sin(\beta)\,Y \\ 0 & 0 & 0 & 0 & 0 & 0 \\ w\,v_+\,\sin(\alpha)\,(\sigma_1+\sigma_3) & \frac{1}{2}\,{v_+}^2\cos(\alpha)\sin(\alpha)\,Y & \frac{1}{2}\,v_+\,v_-\,\sin(\alpha)\cos(\beta)\,Y & 0 & \frac{1}{2}\,(\sin(\alpha))^2\,{v_+}^2 Y & \frac{1}{2}\,v_+\,\sin(\alpha)\,v_-\,\sin(\beta)\,Y \\ w\,v_-\,\sin(\beta)\,(\sigma_1+\sigma_3) & \frac{1}{2}\,v_+\,v_-\,\cos(\alpha)\sin(\beta)\,Y & \frac{1}{2}\,{v_-}^2\cos(\beta)\sin(\beta)\,Y & 0 & \frac{1}{2}\,v_+\,\sin(\alpha)\,v_-\,\sin(\beta)\,Y & \frac{1}{2}\,(\sin(\beta))^2\,{v_-}^2 Y \end{array}\right],$$

where the following eigenvalues and eigenvectors are included.

$Eigenvalues\,of\;M_{ri}^2$	$Eigenvectors$
0	$(0,0,0,1,0,0)^T$
0	$(0,0,\sin\beta,0,0,-\cos\beta)^T$
0	$(0,0,v_+\sin\alpha,0,-v_-\cos\beta,0)^T$
0	$(0,-v_-\cos\beta,v_+\cos\alpha,0,0,0)^T$

The characteristic polynomial with x as variable is for M_{pm}:

$x\left[x + \frac{\sigma_3}{2}(w^2+v_+^2+v_-^2)\right]\left[x + \frac{\sigma_3}{2}w^2 + 2\lambda_2(v_+^2+v_-^2)\right]$

$Eigenvalues\, of\ M_{pm}^2$	$Eigenvectors$
0	$\left(we^{i\alpha}, v_+, v_- e^{i(\alpha-\beta)}\right)^T$
$-\frac{\sigma_3}{2}(w^2+v_+^2+v_-^2)$	$\left(-e^{i\alpha}(v_+^2+v_-^2), v_+ w, v_- w e^{i(\alpha-\beta)}\right)^T$
$-\frac{\sigma_3}{2}w^2 - 2\lambda_2(v_+^2+v_-^2)$	$\left(0, v_- e^{i\beta}, -v_+ e^{i\alpha}\right)^T$

Parameter restrictions through the requirement of positive eigenvalues:

$$-4\frac{\lambda_2}{\sigma_3} < \frac{w^2}{v_+^2+v_-^2}\ ,$$

$$\sigma_3 < 0\ ,$$

$$\lambda_4 w^2 + (\lambda_1+\lambda_2)(v_+^2+v_-^2) > X\ ,$$

where the radiant of X has to be positive.

For $\sigma_2 = 0$ and $v_+^2 = v_-^2$

The characteristic polynomial with x as variable is for M_{ri}:

$$x^3\left[(\lambda_3 - 4\,\lambda_2)\,{v_-}^2 + x\right]\left[x^2 + \left(-2\,\lambda_4\,w^2 - (\lambda_3 + 4\,\lambda_1)\,{v_-}^2\right)x - 2\left[(\sigma_1+\sigma_3)^2 - \lambda_4(4\lambda_1+\lambda_3)\right]{v_-}^2 w^2\right]$$

$$\begin{gathered} X = [8(4\lambda_1+\lambda_3)\lambda_4 - 16(\sigma_1+\sigma_3)^2]v_-^2 \\ Y = (2\,\lambda_1 - 2\,\lambda_2 + \lambda_3) \end{gathered}$$

The eigenvalues and eigenvectors we get from $M_{ri} =$

$$\left[\begin{array}{cccccc} 2\,\lambda_4\,w^2 & w\,v_+\,\cos(\alpha)\,(\sigma_1+\sigma_3) & \cos(\beta)\,w\,v_+\,(\sigma_1+\sigma_3) & 0 & w\,v_+\,\sin(\alpha)\,(\sigma_1+\sigma_3) & \sin(\beta)\,w\,v_+\,(\sigma_1+\sigma_3) \\ w\,v_+\,\cos(\alpha)\,(\sigma_1+\sigma_3) & 2\,{v_+}^2\,(\cos(\alpha))^2\,(\lambda_1+\lambda_2) & {v_+}^2\cos(\alpha)\cos(\beta)\,Y & 0 & 2\,{v_+}^2\cos(\alpha)\sin(\alpha)\,(\lambda_1+\lambda_2) & {v_+}^2\cos(\alpha)\sin(\beta)\,Y \\ \cos(\beta)\,w\,v_+\,(\sigma_1+\sigma_3) & {v_+}^2\cos(\alpha)\cos(\beta)\,Y & 2\,(\cos(\beta))^2\,{v_+}^2\,(\lambda_1+\lambda_2) & 0 & {v_+}^2\sin(\alpha)\cos(\beta)\,Y & 2\,\cos(\beta)\sin(\beta)\,{v_+}^2\,(\lambda_1+\lambda_2) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ w\,v_+\,\sin(\alpha)\,(\sigma_1+\sigma_3) & 2\,{v_+}^2\cos(\alpha)\sin(\alpha)\,(\lambda_1+\lambda_2) & {v_+}^2\sin(\alpha)\cos(\beta)\,Y & 0 & 2\,(\sin(\alpha))^2\,{v_+}^2\,(\lambda_1+\lambda_2) & {v_+}^2\sin(\alpha)\sin(\beta)\,Y \\ \sin(\beta)\,w\,v_+\,(\sigma_1+\sigma_3) & {v_+}^2\cos(\alpha)\sin(\beta)\,Y & 2\,\cos(\beta)\sin(\beta)\,{v_+}^2\,(\lambda_1+\lambda_2) & 0 & {v_+}^2\sin(\alpha)\sin(\beta)\,Y & 2\,(\sin(\beta))^2\,{v_+}^2\,(\lambda_1+\lambda_2) \end{array}\right],$$

where the following eigenvalues and eigenvectors are included.

$Eigenvalues\,of\;M^2_{ri}$	$Eigenvectors$
0	$(0,-\tan\alpha,0,0,1,0)^T$
0	$(0,0,-\tan\beta,0,0,1)^T$
0	$(0,0,0,1,0,0)^T$
$(4\lambda_2-\lambda_3)v_-^2$	$(0,\cos\alpha,-\cos\beta,0,\sin\alpha,-\sin\beta)^T$

The characteristic polynomial with x as variable is for M_{pm}:

$x\left(x+\frac{\sigma_3}{2}w^2+\lambda_3 v_-^2\right)\left[x+\sigma_3(\frac{w^2}{2}+v_-^2)\right]$

$Eigenvalues\,of\;M^2_{pm}$	$Eigenvectors$
$-\frac{\sigma_3}{2}w^2-\lambda_3 v_-^2$	$(0,-1,e^{i(\alpha-\beta)})^T$
0	$\begin{pmatrix} w\,e^{i\alpha} \\ v_- \\ v_- e^{i(\alpha-\beta)} \end{pmatrix}$
$-\sigma_3[\frac{w^2}{2}+v_-^2]$	$\begin{pmatrix} -2v_- e^{i\alpha} \\ w \\ w\,e^{i(\alpha-\beta)} \end{pmatrix}$

Parameter restrictions through the requirement of positive eigenvalues:

$$-2\frac{\lambda_3}{\sigma_3}<\frac{w^2}{v_-^2}\ ,$$

$$\sigma_3<0\ ,$$

$$4\lambda_2-\lambda_3>0\ ,$$

$$\lambda_4 w^2+\frac{1}{2}(4\lambda_1+\lambda_3)v_-^2+\sqrt{(4\lambda_1+\lambda_3)^2 v_-^4-(\frac{X}{2}+4\lambda_4^2 w^2)w^2}>0\ .$$

where the radiant also have to be positive.

For $\alpha = -\beta$

and $v_+^2 \neq v_-^2$

This case is not analytically solvable and we forbear from showing a particular ”solution” because as we have already seen in 4.5 this case is phenomenologically not possible.

For $\alpha = -\beta$ **and** $v_+^2 = v_-^2$

The characteristic polynomial with x as variable is for M_{ri}:

$$x^2 \left(x + \sigma_2\, w^2 - [4\,\lambda_2 - \lambda_3]\, {v_-}^2\right) \left(x + \sigma_2[w^2 + 2{v_-}^2]\right) \cdot$$
$$\cdot \bigg(\left[x - \lambda_4 w^2 - \tfrac{1}{2}(4\lambda_1 + \lambda_3) v_-^2\right]^2 - \lambda_4^2 w^4 +$$
$$\left[\lambda_4\,(4\lambda_1 + \lambda_3) - 2\,(\sigma_1 + \sigma_2 + \sigma_3)^2\right] v_-^2 w^2 - \tfrac{1}{4}(4\lambda_1 + \lambda_3)^2 v_-^4 \bigg)$$

$$\begin{aligned} Y &= (2\lambda_1 - 2\lambda_2 + \lambda_3) \\ Z &= (\sigma_1 + \sigma_2 + \sigma_3) w v_- \\ B_\pm &= (1 \pm \cos(2\beta)) \\ K &= (4\lambda_2 - \lambda_3 + Y) \end{aligned}$$

The eigenvalues and eigenvectors we get from $M_{ri} =$

$$\begin{bmatrix} 2\lambda_4 w^2 & \cos\beta Z & \cos\beta Z & 0 & -\sin\beta Z & \sin\beta Z \\ \cos\beta Z & \frac{1}{2}[KB_+]{v_-}^2 - \frac{\sigma_2}{2} w^2 & \frac{1}{2} B_+ Y {v_-}^2 + \frac{\sigma_2}{2} w^2 & w\sigma_2 v_- \sin\beta & -\frac{1}{2}{v_-}^2 \sin 2\beta\,(4\lambda_2 - \lambda_3 + Y) & \frac{1}{2}{v_-}^2 Y \sin 2\beta \\ \cos\beta Z & \frac{1}{2} Y B_+ {v_-}^2 + \frac{\sigma_2}{2} w^2 & \frac{1}{2}[KB_+]{v_-}^2 - \frac{\sigma_2}{2} w^2 & -w\sigma_2 v_- \sin\beta & -\frac{1}{2}{v_-}^2 Y \sin 2\beta & \frac{1}{2}{v_-}^2 \sin 2\beta\,(4\lambda_2 - \lambda_3 + Y) \\ 0 & w\sigma_2 v_- \sin\beta & -w\sigma_2 v_- \sin\beta & -2\sigma_2 {v_-}^2 & \sigma_2 w v_- \cos\beta & \sigma_2 w v_- \cos\beta \\ -\sin\beta Z & -\frac{1}{2}{v_-}^2 \sin 2\beta\,(4\lambda_2 - \lambda_3 + Y) & -\frac{1}{2}{v_-}^2 Y \sin 2\beta & \sigma_2 w v_- \cos\beta & \frac{1}{2}[KB_+]{v_-}^2 - \frac{\sigma_2}{2} w^2 & -\frac{1}{2} B_- Y {v_-}^2 - \frac{\sigma_2}{2} w^2 \\ \sin\beta Z & \frac{1}{2}{v_-}^2 Y \sin 2\beta & \frac{1}{2}{v_-}^2 \sin 2\beta\,(4\lambda_2 - \lambda_3 + Y) & \sigma_2 w v_- \cos\beta & -\frac{1}{2} B_- Y {v_-}^2 - \frac{\sigma_2}{2} w^2 & \frac{1}{2} K B_- {v_-}^2 - \frac{\sigma_2}{2} w^2 \end{bmatrix},$$

where the following eigenvalues and eigenvectors are included.

Eigenvalues of M_{ri}^2	*Eigenvectors*
$-\sigma_2 w^2 + (4\lambda_2 - \lambda_3) v_-^2$	$(0, -\cos\beta, \cos\beta, 0, \sin\beta, \sin\beta)^T$
0	$(0, 0, 2\sin\beta\, v_-, -w, 0, -2\cos\beta\, v_-)^T$
0	$(0, 2\sin\beta\, v_-, 0, w, 2\cos\beta\, v_-, 0)^T$
$-\sigma_2(w^2 + 2v_-^2)$	$(0, \sin\beta\, w, -\sin\beta\, w, -2v_-, w\cos\beta, w\cos\beta)^T$

The characteristic polynomial with x as variable is for M_{pm}:

$$x\left(2x+[\sigma_2+\sigma_3][2{v_-}^2+w^2]\right)\left(2\,x+[\sigma_2+\sigma_3]\,w^2+2\,\lambda_3\,{v_-}^2\right)$$

$Eigenvalues\, of\; M_{pm}^2$	$Eigenvectors$
$-\frac{\sigma_2+\sigma_3}{2}w^2-\lambda_3 v_-^2$	$(0,1,-e^{-2\,i\,\beta})^T$
0	$(w,v_- e^{i\beta},v_- e^{-i\beta})^T$
$-(\sigma_2+\sigma_3)(\frac{1}{2}w^2+v_-^2)$	$(-2v_-,w\,e^{i\beta},w\,e^{-i\beta})^T$

Parameter restrictions through the requirement of positive eigenvalues:

$$\sigma_2<0\ ,$$

$$\sigma_2+\sigma_3<0\ ,$$

$$2\frac{\lambda_3}{\sigma_2+\sigma_3}>\frac{w^2}{v_-^2}\ ,$$

$$\lambda_4 w^2+\frac{1}{2}(4\lambda_1+\lambda_3)v_-^2>Y\ .$$

where the radiant of Y has to be positive.

For $\alpha=\pi-\beta$

This case is similar to the cases before, where $\alpha=-\beta$. We just have to replace σ_2 by $-\sigma_2$ and to calculate the different eigenvectors.

and $v_+^2\neq v_-^2$

This case is not analytically solvable and we forbear from showing a particular ”solution” because as we saw in 4.5 this case is phenomenologically not possible.

For $\alpha=\pi-\beta$ **and** $v_+^2=v_-^2$

The characteristic polynomial with x as variable is for M_{ri}:

$$\begin{gathered}x^2\left(x-\sigma_2\,w^2-[4\,\lambda_2-\lambda_3]\,{v_-}^2\right)\left(x-\sigma_2[w^2+2{v_-}^2]\right)\cdot\\ \cdot\left(\left[x-\lambda_4 w^2-\frac{1}{2}(4\lambda_1+\lambda_3)v_-^2\right]^2-\lambda_4^2 w^4+\right.\\ \left.\left[\lambda_4\,(4\lambda_1+\lambda_3)-2\,(\sigma_1-\sigma_2+\sigma_3)^2\right]v_-^2 w^2-\frac{1}{4}(4\lambda_1+\lambda_3)^2 v_-^4\right)\end{gathered}$$

$$
\begin{aligned}
Y &= (2\lambda_1 - 2\lambda_2 + \lambda_3) \\
Z &= (\sigma_1 + \sigma_2 + \sigma_3) w v_- \\
B_\pm &= (1 \pm \cos(2\beta)) \\
C &= (4\,\lambda_2 - \lambda_3 + Y)
\end{aligned}
$$

The eigenvalues and eigenvectors we get from $M_{ri} =$

$$
\begin{bmatrix}
2\,\lambda_4\,w^2 & (2\,\sigma_2\,w\,v_- - Z)\cos(\beta) & (-2\,\sigma_2\,w\,v_- + Z)\cos(\beta) & 0 & (-2\,w\,\sigma_2\,v_- + Z)\sin(\beta) & (-2\,w\,\sigma_2\,v_- + Z)\sin(\beta) \\
(2\,\sigma_2\,w\,v_- - Z)\cos(\beta) & \frac{1}{2}CB_+{v_-}^2 + \frac{1}{2}\,\sigma_2\,w^2 & -\frac{1}{2}B_+Y{v_-}^2 + \frac{1}{2}\,\sigma_2\,w^2 & w\,\sigma_2\,v_-\,\sin(\beta) & -\frac{1}{2}\,{v_-}^2\sin(2\,\beta)\,C & -\frac{1}{2}\,{v_-}^2Y\sin(2\,\beta) \\
(-2\,\sigma_2\,w\,v_- + Z)\cos(\beta) & -\frac{1}{2}B_+Y{v_-}^2 + \frac{1}{2}\,\sigma_2\,w^2 & \frac{1}{2}CB_+{v_-}^2 + \frac{1}{2}\,\sigma_2\,w^2 & w\,\sigma_2\,v_-\,\sin(\beta) & \frac{1}{2}\,{v_-}^2Y\sin(2\,\beta) & \frac{1}{2}\,{v_-}^2\sin(2\,\beta)\,C \\
0 & w\,\sigma_2\,v_-\,\sin(\beta) & w\,\sigma_2\,v_-\,\sin(\beta) & 2\,\sigma_2\,{v_-}^2 & \sigma_2\,w\,v_-\,\cos(\beta) & -\sigma_2\,w\,v_-\,\cos(\beta) \\
(-2\,w\,\sigma_2\,v_- + Z)\sin(\beta) & -\frac{1}{2}\,{v_-}^2\sin(2\,\beta)\,C & \frac{1}{2}\,{v_-}^2Y\sin(2\,\beta) & \sigma_2\,w\,v_-\,\cos(\beta) & \frac{1}{2}CB_-{v_-}^2 + \frac{1}{2}\,\sigma_2\,w^2 & \frac{1}{2}B_-Y{v_-}^2 - \frac{1}{2}\,\sigma_2\,w^2 \\
(-2\,w\,\sigma_2\,v_- + Z)\sin(\beta) & -\frac{1}{2}\,{v_-}^2Y\sin(2\,\beta) & \frac{1}{2}\,{v_-}^2\sin(2\,\beta)\,C & -\sigma_2\,w\,v_-\,\cos(\beta) & \frac{1}{2}B_-Y{v_-}^2 - \frac{1}{2}\,\sigma_2\,w^2 & \frac{1}{2}CB_-{v_-}^2 + \frac{1}{2}\,\sigma_2\,w^2
\end{bmatrix},
$$

where the following eigenvalues and eigenvectors are included.

$Eigenvalues\,of\,M_{ri}^2$	$Eigenvectors$
$\sigma_2 w^2 + (4\lambda_2 - \lambda_3)v_-^2$	$(0, \cos\beta, \cos\beta, 0, -\sin\beta, \sin\beta)^T$
0	$(0, 0, 2\sin\beta\, v_-, -w, 0, -2\cos\beta\, v_-)^T$
0	$(0, 2\sin\beta\, v_-, 0, -w, 2\cos\beta\, v_-, 0)^T$
$\sigma_2(w^2 + 2v_-^2)$	$(0, \sin\beta\, w, \sin\beta\, w, 2v_-, w\cos\beta, -w\cos\beta)^T$

The characteristic polynomial with x as variable is for M_{pm}:

$$
x\left(2x + [\sigma_3 - \sigma_2][2{v_-}^2 + w^2]\right)\left(2\,x + [\sigma_3 - \sigma_2]\,w^2 + 2\,\lambda_3\,{v_-}^2\right)
$$

$Eigenvalues\,of\,M_{pm}^2$	$Eigenvectors$
$-\frac{\sigma_3 - \sigma_2}{2} w^2 - \lambda_3 v_-^2$	$(0, 1, e^{-2\,i\,\beta})^T$
0	$(w, -v_- e^{i\beta}, v_- e^{-i\beta})^T$
$-(\sigma_3 - \sigma_2)(\frac{1}{2}w^2 + v_-^2)$	$(2v_-, w\,e^{i\beta}, -w\,e^{-i\beta})^T$

Parameter restrictions through the requirement of positive eigenvalues:

$$\sigma_2 > 0 \ ,$$

$$\sigma_3 - \sigma_2 < 0 \ ,$$

$$2\tfrac{\lambda_3}{\sigma_3-\sigma_2} > \tfrac{w^2}{v_-^2} \ ,$$

$$\lambda_4 w^2 + \tfrac{1}{2}(4\lambda_1 + \lambda_3)v_-^2 > Y \ .$$

where the radiant of Y has to be positive.

Appendix E

Four Higgs Model - Higgs Masses

In the following are the eigenvalues and eigenvectors for VEV configuration of the four Higgs potential (6.5) given.

$$v_1^1 = v_2^1 = v_1^2 = v_2^2 \equiv v$$

If we assume the VEVs to be real and allow the other parameters in the Higgs potential to be complex, we obtain (6.34) and (6.35) as solutions of the minimization conditions. For these VEV configurations we show the explicit solutions as well as the Higgs masses with the corresponding eigenvectors.

$$v_1^1 = v_2^1 = v_1^2 = v_2^2 \equiv v \ ,$$
$$\mu_1^2 = \left(\tfrac{1}{2}\,\lambda_3 + 2\,\lambda_{11}^r + \lambda_7 + 2\,\lambda_1 + \lambda_{13} + \lambda_9^r\right)\left(v_1^1\right)^2 \ .$$

Then we get the following masses with its corresponding eigenvector for the charged Higgs:

Eigenvalues	Eigenvectors
0	$(1,1,1,1)^T$
$(-2\,\lambda_{11}^r - 2\,\lambda_{13} - 2\,\lambda_9^r)\,v^2$	$(-1,-1,1,1)^T$
$\left(\lambda_9^i + 2\,\lambda_7 - \lambda_{11}^i + \lambda_{11}^r + \lambda_{13} + \lambda_9^r + 4\,\lambda_1\right)v^2 - 2\,{\mu_1}^2$	$(-1,1,-i,i)^T$
$\left(-\lambda_9^i + 2\,\lambda_7 + \lambda_{11}^i + \lambda_{11}^r + \lambda_{13} + \lambda_9^r + 4\,\lambda_1\right)v^2 - 2\,{\mu_1}^2$	$(-i,i,-1,1)^T$

And for the uncharged Higgs we obtain:

Eigenvalues	Eigenvectors
0	$(0,0,0,0,1,1,1,1)^T$
$2\mu_1^2$	$(1,1,1,1,0,0,0,0)^T$
$-\left(4\,\lambda_{11}^r+4\,\lambda_9^r+2\,\lambda_7+2\,\lambda_{13}\right)v^2+A+{\mu_1}^2$	$\begin{bmatrix} -2(\lambda_{11}^r+2\lambda_9^r)v^2-EV \\ -2(\lambda_{11}^r+2\lambda_9^r)v^2-EV \\ 2(\lambda_{11}^r+2\lambda_9^r)v^2+EV \\ 2(\lambda_{11}^r+2\lambda_9^r)v^2+EV \\ 2\,v^2\lambda_{11}^i \\ 2\,v^2\lambda_{11}^i \\ -2\,v^2\lambda_{11}^i \\ -2\,v^2\lambda_{11}^i \end{bmatrix}$
$-\left(4\,\lambda_{11}^r+4\,\lambda_9^r+2\,\lambda_7+2\,\lambda_{13}\right)v^2-A+{\mu_1}^2$	$\begin{bmatrix} -2(\lambda_{11}^r+2\lambda_9^r)v^2-EV \\ -2(\lambda_{11}^r+2\lambda_9^r)v^2-EV \\ 2(\lambda_{11}^r+2\lambda_9^r)v^2+EV \\ 2(\lambda_{11}^r+2\lambda_9^r)v^2+EV \\ -2\,v^2\lambda_{11}^i \\ -2\,v^2\lambda_{11}^i \\ 2\,v^2\lambda_{11}^i \\ 2\,v^2\lambda_{11}^i \end{bmatrix}$
$\left(\lambda_7+\lambda_{13}-2\,\lambda_{11}^r+2\,\lambda_1+2\,\lambda_2\right)v^2+B-{\mu_1}^2$	$\begin{bmatrix} -10\lambda_{11}^r v^2-2\,EV \\ 10\lambda_{11}^r v^2+2\,EV \\ -5\lambda_{11}^r v^2-EV \\ 5\lambda_{11}^r v^2+EV \\ 0 \\ 0 \\ -5\,v^2\lambda_{11}^i \\ 5\,v^2\lambda_{11}^i \end{bmatrix}, \begin{bmatrix} -5\lambda_{11}^r v^2-EV \\ 5\lambda_{11}^r v^2+EV \\ 10\lambda_{11}^r v^2+2\,EV \\ -10\lambda_{11}^r v^2-2\,EV \\ -5\,v^2\lambda_{11}^i \\ 5\,v^2\lambda_{11}^i \\ 0 \\ 0 \end{bmatrix}$
$\left(\lambda_7+\lambda_{13}-2\,\lambda_{11}^r+2\,\lambda_1+2\,\lambda_2\right)v^2-B-{\mu_1}^2$	$\begin{bmatrix} -10\lambda_{11}^r v^2-2\,EV \\ 10\lambda_{11}^r v^2+2\,EV \\ -5\lambda_{11}^r v^2-EV \\ 5\lambda_{11}^r v^2+EV \\ 0 \\ 0 \\ -5\,v^2\lambda_{11}^i \\ 5\,v^2\lambda_{11}^i \end{bmatrix}, \begin{bmatrix} -5\lambda_{11}^r v^2-EV \\ 5\lambda_{11}^r v^2+EV \\ 10\lambda_{11}^r v^2+2\,EV \\ -10\lambda_{11}^r v^2-2\,EV \\ -5\,v^2\lambda_{11}^i \\ 5\,v^2\lambda_{11}^i \\ 0 \\ 0 \end{bmatrix}$

where

$A=\sqrt{4\,{\lambda_{11}^i}^2v^4+\left[2\,\left(\lambda_{11}^r+\lambda_{13}+\lambda_7\right)v^2-{\mu_1}^2\right]^2}$

$B=\sqrt{5\,{\lambda_{11}^i}^2v^4+\left[\left(\lambda_{13}+2\,\lambda_2+\lambda_7+2\,\lambda_1+3\,\lambda_{11}^r\right)v^2-{\mu_1}^2\right]^2}$

and EV is the corresponding eigenvalue.

$v_1^1 = v_2^1 = -v_1^2 = -v_2^2 \equiv v$

The other mentioned solution is

$$v_1^1 = v_2^1 = -v_1^2 = -v_2^2 \equiv v \ ,$$
$$\mu_1^2 = \left(\tfrac{1}{2}\,\lambda_3 - 2\,\lambda_{11}^r + \lambda_7 + 2\,\lambda_1 + \lambda_{13} + \lambda_9^r\right)\left(v_1^1\right)^2 \ ,$$

There we get the following masses for the scalar and pseudo-scalar Higgs:

Eigenvalues	Eigenvectors
0	$(0,0,0,0,1,1,1,1)^T$
$2\mu_1^2$	$(1,1,1,1,0,0,0,0)^T$
$-\left(4\,\lambda_{11}^r + 4\,\lambda_9^r + 2\,\lambda_7 + 2\,\lambda_{13}\right)v^2 + A + {\mu_1}^2$	$\begin{bmatrix} -2(\lambda_{11}^r + 2\lambda_9^r)v^2 - EV \\ -2(\lambda_{11}^r + 2\lambda_9^r)v^2 - EV \\ 2(\lambda_{11}^r + 2\lambda_9^r)v^2 + EV \\ 2(\lambda_{11}^r + 2\lambda_9^r)v^2 + EV \\ -2\,v^2\lambda_{11}^i \\ -2\,v^2\lambda_{11}^i \\ 2\,v^2\lambda_{11}^i \\ 2\,v^2\lambda_{11}^i \end{bmatrix}$
$-\left(4\,\lambda_{11}^r + 4\,\lambda_9^r + 2\,\lambda_7 + 2\,\lambda_{13}\right)v^2 - A + {\mu_1}^2$	$\begin{bmatrix} -2(\lambda_{11}^r + 2\lambda_9^r)v^2 - EV \\ -2(\lambda_{11}^r + 2\lambda_9^r)v^2 - EV \\ 2(\lambda_{11}^r + 2\lambda_9^r)v^2 + EV \\ 2(\lambda_{11}^r + 2\lambda_9^r)v^2 + EV \\ -2\,v^2\lambda_{11}^i \\ -2\,v^2\lambda_{11}^i \\ 2\,v^2\lambda_{11}^i \\ 2\,v^2\lambda_{11}^i \end{bmatrix}$
$\left(\lambda_7 + \lambda_{13} - 2\,\lambda_{11}^r + 2\,\lambda_1 + 2\,\lambda_2\right)v^2 + B - {\mu_1}^2$	$\begin{bmatrix} -10\lambda_{11}^r v^2 - 2\,EV \\ 10\lambda_{11}^r v^2 + 2\,EV \\ -5\lambda_{11}^r v^2 - EV \\ 5\lambda_{11}^r v^2 + EV \\ 0 \\ 0 \\ -5\,v^2\lambda_{11}^i \\ 5\,v^2\lambda_{11}^i \end{bmatrix}, \begin{bmatrix} -5\lambda_{11}^r v^2 - EV \\ 5\lambda_{11}^r v^2 + EV \\ 10\lambda_{11}^r v^2 + 2\,EV \\ -10\lambda_{11}^r v^2 - 2\,EV \\ -5\,v^2\lambda_{11}^i \\ 5\,v^2\lambda_{11}^i \\ 0 \\ 0 \end{bmatrix}$
$\left(\lambda_7 + \lambda_{13} - 2\,\lambda_{11}^r + 2\,\lambda_1 + 2\,\lambda_2\right)v^2 - B - {\mu_1}^2$	$\begin{bmatrix} -10\lambda_{11}^r v^2 - 2\,EV \\ 10\lambda_{11}^r v^2 + 2\,EV \\ -5\lambda_{11}^r v^2 - EV \\ 5\lambda_{11}^r v^2 + EV \\ 0 \\ 0 \\ -5\,v^2\lambda_{11}^i \\ 5\,v^2\lambda_{11}^i \end{bmatrix}, \begin{bmatrix} -5\lambda_{11}^r v^2 - EV \\ 5\lambda_{11}^r v^2 + EV \\ 10\lambda_{11}^r v^2 + 2\,EV \\ -10\lambda_{11}^r v^2 - 2\,EV \\ -5\,v^2\lambda_{11}^i \\ 5\,v^2\lambda_{11}^i \\ 0 \\ 0 \end{bmatrix}$

where

$A = \sqrt{4\,{\lambda_{11}^i}^2 v^4 + \left[2\,\left(\lambda_{11}^r + \lambda_{13} + \lambda_7\right)v^2 - {\mu_1}^2\right]^2}$

$B = \sqrt{5\,{\lambda_{11}^i}^2 v^4 + \left[\left(\lambda_{13} + 2\,\lambda_2 + \lambda_7 + 2\,\lambda_1 + 3\,\lambda_{11}^r\right)v^2 - {\mu_1}^2\right]^2}$

and EV is the corresponding eigenvalue.

The masses for the charged Higgs are shown below.

Eigenvalues	Eigenvectors
0	$(1,1,-1,-1)^T$
$(2\,\lambda_{11}^r - 2\,\lambda_{13} - 2\,\lambda_9^r)\,v^2$	$(1,1,1,1)^T$
$\left(\lambda_9^i + 2\,\lambda_7 + \lambda_{11}^i - \lambda_{11}^r + \lambda_{13} + \lambda_9^r + 4\,\lambda_1\right) v^2 - 2\,{\mu_1}^2$	$(-i,i,-1,1)^T$
$\left(-\lambda_9^i + 2\,\lambda_7 - \lambda_{11}^i - \lambda_{11}^r + \lambda_{13} + \lambda_9^r + 4\,\lambda_1\right) v^2 - 2\,{\mu_1}^2$	$(-1,1,-i,i)^T$

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Nomenclature

CG Coefficient	Clebsch-Gordan Coefficient
CKM Matrix	Cabibbo-Kobayashi-Maskawa Matrix
EV	Eigenvalue
FCNC	Flavor Changing Neutral Currents
GB	Goldstone Boson
GUT	Grand Unified Theory
IH	Inverted Hierarchy
IO	Inverted Ordering
LFV	Lepton Flavor Violation
LH	Left-Handed
LR Model	Left-Right Symmetric Model
NH	Normal Hierarchy
NO	Normal Ordering
PMNS Matrix	Pontecorvo-Maki-Nakagawa-Sakata Matrix
QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
RH	Right-Handed
SB	Soft-Breaking
SM	Standard Model
SSB	Spontaneous Symmetry Breaking
SUSY	Supersymmetry
THDM	Two Higgs Doublet Model
VEV	Vacuum Expectation Value

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