Physik-Department

Leptogenesis, Boltzmann Equation and Non-Equilibrium Thermodynamics

Diploma Thesis by

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TO MY PARENTS

Abstract

The universe consists predominantly of matter. Only a tiny fraction is made up by antimatter. This disproportion can be expressed by a number called the baryon asymmetry. It has been measured accurately by the WMAP satellite experiment. Conventional theories trying to explain this observation in a dynamical way are referred to as baryogenesis. A newer class of mechanisms are the leptogenesis theories. They predict that the asymmetry has first been created in the lepton sector and has then been converted partly into a baryon asymmetry. The observation that neutrinos are massive particles makes such mechanisms attractive.
Every theory that hopes to explain the baryon asymmetry needs to satisfy the Sakharov conditions. Among others there must be a deviation from thermal equilibrium. These non-equilibrium scenarios are usually described by means of kinetic theory. However leptogenesis takes place at early times and very high temperatures. It is unclear whether this description is adequate under such extreme conditions.

Zusammenfassung

Das Universum besteht überwiegend aus Materie und nur zu einem sehr kleinen Anteil aus Antimaterie. Dieses Missverhältnis kann durch eine Zahl zum Ausdruck gebracht werden, die man Baryonasymmetrie nennt. Diese ist durch das WMAP Satelliten-Experiment genau bestimmt worden. Herkömmliche Theorien, die diese Beobachtung auf eine dynamische Weise zu erklären versuchen werden als Baryogenese bezeichnet. Eine neuere Klasse von Theorien bilden die so genannten Leptogenese Theorien. Sie gehen davon aus, dass die Asymmetrie ursprünglich im Lepton-Sektor erzeugt, und schließlich teilweise in eine Baryonasymmetrie konvertiert wurde. Die Beobachtung, dass Neutrinos eine endliche Masse besitzen macht solche Mechanismen attraktiv. Jede Theorie, welche die Baryonasymmetrie zu erklären versucht muss notwendigerweise die Sakharov-Bedingungen erfüllen. Unter anderem verlangen diese, dass es eine Abweichung vom thermodynamischen Gleichgewicht gegeben haben muss. Diese Nichtgleichgewicht-Szenarien werden üblicherweise mithilfe von kinetischer Theorie beschrieben. Leptogenese findet jedoch zu sehr frühen Zeiten und bei sehr hohen Temperaturen statt. Es ist unklar, ob dieser Ansatz unter solch extremen Bedingungen richtig ist.

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Chapter 1

Introduction

Our world consists predominantly of matter and only a tiny fraction is made up by antimatter. This is not only an observation which we can make in our every days life.

Planetary probes have visited eight of the nine planets, and the one to visit the ninth has just been brought on its way.[1] None of these probes has found evidence for antimatter. We can conclude, that our solar system is made up of matter. We will see that there is compelling evidence, that this statement also holds for the universe as a whole.

This experimental observation can be expressed by a number called baryon asymmetry η. Namely the difference between the number density of baryons and that of antibaryons divided by the number density of photons. Its numerical value has been determined to good accuracy by the WMAP satellite which explores the anisotropies in the cosmic microwave background radiation.

$$\eta = \frac{n_B - n_{\bar{B}}}{n_\gamma} \simeq \frac{n_B}{n_\gamma} = (6,01 \pm 0,3) \cdot 10^{-10}$$

This number tells us, that if there were $10^{10} + 1$ baryons versus 10^{10} antibaryons at the time when the temperature of the universe dropped below the rest-mass of the nucleon and therefore the baryons and antibaryons started to annihilate, they would leave 1 baryon and $(2)10^{10}$ photons behind giving a baryon to photon ratio of 10^{-10}.

On the one hand one can assume, that this asymmetry is an initial condition of the universe. This approach is somewhat unsatisfactory, because there is no reason why nature should prefer the dominance of particles over anti-particles or vice versa.

Another possibility would be, that the asymmetry has developed dynamically during the evolution of the universe. To find such an explanation has indeed been the goal of many scientists in the past decades.

There is broad agreement, that any theory trying to explain the baryon asymmetry has to satisfy at least three conditions, originally established by Andrej Sakharov. These conditions are in detail: baryon number violation, C and CP violation and finally the deviation from thermal equilibrium.

Conventional mechanisms which manage to accomplish this task can be found in the literature under the name baryogenesis. They can be distinguished by how they realize the Sakharov conditions. Examples are GUT baryogenesis and electroweak baryogenesis. However the theoretical and experimental advance has made these theories seem increasingly unlikely.

[1] The NASA mission New Horizons has been launched on January 19, 2006. However at the moment it is unclear whether Pluto might loose its planetary status, when the probe arrives there by 2015.

The observation that neutrinos perform flavour oscillations and therefore must be massive particles makes another class of scenarios thinkable. These are the leptogenesis theories. They predict that the asymmetry was first created in the leptonic sector before it has been converted into the baryon asymmetry we observe today. So called sphaleron processes are made responsible for this conversion. They are a non-perturbative aspect of non-abelian gauge theories ascribed to the nontrivial structure of the electroweak vacuum. They are baryon and lepton number violating processes.

As we will see these processes are incorporated in the standard model. However leptogenesis also requires extensions to this very successful compendium of elementary particle physics. In particular it requires the existence of heavy Majorana neutrinos. By virtue of the seesaw mechanism, which relates the light neutrino states to the heavy ones, the existence of these speculative particles would at the same time solve one of the open questions posed by the standard model: Why are neutrinos so light?

The major part of this work is related to the third of the Sakharov conditions, the deviation from thermal equilibrium. In most current theories this criterion is realized by a standard out of equilibrium scenario within which the deviation from equilibrium is caused by the rapid expansion of the universe.

The details of this scenario are usually investigated, solving the so called Boltzmann equations. This central equation of kinetic theory describes the evolution of the single particle distribution function. While it has been successfully applied to many different physical problems, it is far from obvious whether this equation is indeed suitable for the description of the primordial plasma of the early universe.

We will see, that in view of the assumptions made in its derivation this seems to be rather unlikely. Unfortunately in most treatises of leptogenesis this aspect is not payed much attention to.

This thesis consists essentially of three parts.

In the first chapter we discuss the underlying physics such as the necessary extensions to the standard model and we give an overview of baryogenesis scenarios discussed in the literature. As an example we will describe thermal leptogenesis in slightly more detail.

The second part is entirely devoted to kinetic theory and statistical mechanics in cosmology. We will introduce the Boltzmann equation and discuss, in some detail, the limits of its applicability. Then we will successively generalize it in order to obtain the form encountered in cosmology.

In the last chapter we will investigate a simple toy model which has all necessary ingredients for baryon number generation. We will try to solve it by two means. First, we follow a certain standard approximation scheme found in the literature to simplify them for approximate analytical solution. Second, we will employ a more sophisticated numerical method in order to solve the unaltered equations exactly.

Throughout this work we will use natural units where $\hbar = c = k = 1$ which implies that [energy] = [mass] = [temperature] = [length]$^{-1}$ = [time]$^{-1}$.

In this system we have $1\,\text{GeV} \simeq (2.0 \times 10^{-14}\,\text{cm})^{-1} \simeq (6.6 \times 10^{-25}\,\text{s})^{-1} \simeq 1.2 \times 10^{13}\,\text{K}$, and $G = m_{pl}^{-2}$, where G is Newtons constant and $m_{pl} \simeq 1.2 \times 10^{19}\,\text{GeV}$ denotes the Planck mass.

Chapter 2

Leptogenesis

In this first chapter we give an overview of the different possibilities to determine η. Then we will summarize the Sakharov conditions and point out how these are satisfied in different baryogenesis scenarios (i.e. GUT baryogenesis, electroweak baryogenesis and Affleck-Dine leptogenesis). Afterwards we briefly review the prerequisites necessary to understand the details of leptogenesis. Finally we discuss thermal leptogenesis in some detail.[1]

2.1 Overview

The goal of all baryogenesis and leptogenesis theories is to explain the value of η in the universe.

The significance of this number is due to the fact that the ratio n_b/n_γ stays constant during the expansion of the universe as long as it is adiabatic.[2] This means that once the baryon asymmetry has come into existence the value of η is altered only by entropy producing non-equilibrium effects.

At temperatures $T \sim 1\,\text{GeV}$ the rate of the back-reaction of the process $p + \bar{p} \longleftrightarrow \gamma + \gamma$ drops rapidly, because the γ's lack the energy to produce the massive protons, but the reaction proceeds in the forward direction. We will calculate in chapter 3 that at this temperature the baryon number density will have dropped to a value of $n_B/n_\gamma \simeq 10^{-18}$, if the baryons are still in equilibrium at this time. Therefore the baryon asymmetry must have been created at temperatures well above $1\,\text{GeV}$.

Consequently baryogenesis takes place in the early universe, independent of how the mechanism works in detail. Table 2.1 lists a number of important events in the history of the early universe as suggested by the Big Bang model.

[1] See [1], [2], [3].

[2] This is due to $n_b \sim a^{-3}$ and $n_\gamma \sim T^3 \sim a^{-3}$ ($n_\gamma \sim s$, $sa^3 = \text{const}$, $s \sim a^{-3}$) for adiabatic expansion. Cf. section 3.8.

Temperature		Event
10^{19} GeV	m_{pl}	*Big Bang*; current theories fail *Inflation*; exponential growth; preexisting baryon asymmetry is essentially whipped out
10^{16} GeV	M_{GUT}	*GUT phase transition*; (timescale of GUT baryogenesis)
100 GeV	M_W	*Electroweak phase transition*; (timescale of electroweak baryogenesis)
1 GeV	m_N	*Hadron freeze-out*; quarks condense into hadrons; baryons and antibaryons annihilate
0.1 MeV	$\lesssim E_D$	*Primordial Nucleosynethesis*; origin of the light elements
0.1 eV	$\lesssim R_Y$	*Recombination* and *CMB Decoupling*

Figure 2.1: Thermal history of the universe in the Big Bang model. Note that all statements about physics before primordial nucleosynethesis are highly speculative. In general the uncertainty increases when going back in time. Conversion between energy and temperature: $1\,\text{GeV} = 1.16 \times 10^{13}\,\text{K}$ (deuteron binding energy $E_D \simeq 2.2\,\text{MeV}$, Rydberg energy $R_Y \simeq 13.6\,\text{eV}$). See e.g. [4], [5].

Evidence for the Baryon Asymmetry

As advertised there is compelling experimental evidence that the universe contains much more matter than antimatter. We will now see where this evidence comes from in detail.

On earth antimatter can only be found in the storage rings of the big collider experiments in quantities of fractions of grams.

Cosmic Rays exhibit an admixture of anti-protons of 10^{-4}. This number can however be explained by secondary processes such as $p + p \to 3p + \overline{p}$, induced by high-energetic particles colliding with interstellar matter.

Also we can infer from the absence of annihilation products (i.e. γ-radiation) that our galaxy contains no antimatter. If there were significant amounts of antimatter present in the universe it would be separated from the matter parts on large scales. However theoretical estimates show that the patches at times when the separation must have been present could not have contained more particles than about $0.1 \cdot M_{\text{sol}}$, which is of course not sufficient to form galaxies.[3] By this argument the existence of such patches can be excluded.

If we set $n_{\bar{B}} = 0$ the most obvious way to determine η is to compute the ratio n_B/n_γ based on astronomical data from galaxy surveys. The photon number density at temperature $T = 2.728$ (CMBR) is $n_\gamma \sim 4 \times 10^8\,\text{m}^{-3}$.[4] A current estimate for the density of baryonic matter from measurements of visible matter in galaxies is $n_b \sim 1/20\,\text{m}^{-3}$. This yields the ratio $n_b/n_\gamma \sim 10^{-10}$, which has already the correct order of magnitude.

Primordial Nucleosynethesis

[3] See also [6]

[4] The bulk of photons today belongs to the CMBR

Primordial nucleosynethesis or Big Bang nucleosynethesis (BBN) predicts very successful the abundance of the light elements D, ^{3}He, ^{4}He and ^{7}Li. For this reason BBN is the most important affirmation of the Big Bang model.[5] The formation of elements proceeds over a chain of reactions which starts with the creation of (stable) deuterons via

$$p + n \to d + \gamma\,.$$

The deuterons are then processed into He by the following reactions:

$$d + d \to^3 \text{He} + n, \quad ^3\text{He} + d \to^4 \text{He} + p$$

Further reactions convert a small fraction of these elements to ^{7}Li. The precise relative abundance of He depends sensitively on the temperature or time when nucleosynethesis starts, because the weak process

$$p + e^- \longleftrightarrow n + \overline{\nu_e}\,, \tag{2.1}$$

which governs the size of n_n/n_p produces preferable (stable) neutrons at low energies. Based on the assumption that essentially all neutrons end up in ^{4}He nuclei one finds for the ratio of the He mass density and the total mass density at the time of nucleosynethesis:

$$\frac{\rho_{\text{He}}}{\rho} = \frac{2(n_n/n_p)}{1 + (n_n/n_p)} \tag{2.2}$$

Now, the crucial reaction, which determines the beginning of nucleosynethesis is the photo-disintegration of the deuteron

$$d + \gamma \to p + n\,.$$

Therefore, a naive estimate would be to use the deuteron binding energy $\simeq 2.2\,\text{MeV}$ as energy scale for nucleosynethesis. However this approach fails, because the photons in the high energy tail of the distribution function prevent the creation of deuterium till temperatures of $T \simeq 0.1\,\text{MeV}$. A sophisticated analysis also has to take the energy dependence of the weak reaction (2.1) into account. In order to obtain a reliable result one needs to solve the network of coupled Boltzmann equations of all involved particle species.[6]

In figure 2.2 the relative abundance of the light elements is drawn as a function of η.

[5] See [7] for an introduction.
[6] See [8] for a broad analysis of the network of Boltzmann equations.

Figure 2.2: Abundance of the light elements as predicted by primordial nucleosynethesis as a function of η. The vertical line gives the observed values. We see that He makes up the biggest part of the light elements in the universe.

The comparison of the theoretical prediction with the measured abundances yields the BBN result

$$\eta^{BBN} = (2.6 - 6.2) \times 10^{-10} \, .$$

Cosmic Microwave Background

A newer value for η can be found based on the cosmic microwave background radiation (CMB). The CMB is to a very good approximation Planck distributed:

$$P(\nu, T)\, d\nu = 8\pi h \left(\frac{\nu}{c}\right)^3 \frac{d\nu}{e^{h\nu/kT} - 1} \tag{2.3}$$

However there exist deviations from perfect isotropy at a level of 10^{-5}. These anisotropies have been measured with high accuracy by the WMAP satellite experiment [9]. The deviations, which are attributed to small density fluctuations in the photon baryon plasma at the time of decoupling are commonly analyzed by decomposition of the spectrum into spherical harmonics:

$$\frac{\Delta T}{T} = \sum_{lm} a_{lm} Y_{lm}(\theta, \phi) \tag{2.4}$$

with expansion coefficients a_{lm} defining the so called power spectrum by

$$l(l+1)C_l = l(l+1)\left\langle |a_{lm}|^2 \right\rangle . \tag{2.5}$$

Later on the fluctuations grew driven by gravitational instability and eventually caused the formation of structures like stars and galaxies.

Many important cosmological entities can be inferred from the WMAP data. For instance one expects a peak at multipole moment $l = 220$ for a flat universe. This peak does indeed exist. See figure 2.3.[7]

Figure 2.3: The famous CMB power spectrum, measured by the WMAP satellite. The solid lines represent the best-fit cosmological model.

Also the baryon asymmetry η has been be determined in this way.

The WMAP result is [9]:

$$\eta^{CMB} = (6.1^{+0.3}_{-0.2}) \times 10^{-10} \tag{2.6}$$

[7] Another important result is the partition of the total density into dark energy, dark matter and baryonic matter contributions.

Now that we are convinced that the universe is baryon dominated we can think of mechanisms which could account for this fact. As outlined above all candidates have to be in agreement with three necessary conditions.

The Sakharov Conditions

Sakharov realized (1968) that a baryon asymmetry could have been dynamically produced in the early universe if three necessary conditions are satisfied[8]

In order to see what these conditions are, suppose we know a certain process, which we suspect to account for the asymmetry. Let us write $\mathcal{M}(i \to j)$ for the amplitude of a transition from the state i to a state j and let $\bar{i}$ be the CP conjugated state to i.[9] CPT invariance then implies that

$$\mathcal{M}(i \to j) = \mathcal{M}(\bar{j} \to \bar{i}) \,. \tag{2.7}$$

Unitarity of the S Matrix implies[10]

$$\sum_j |\mathcal{M}(i \to j)|^2 = \sum_j |\mathcal{M}(j \to i)|^2 \ . \tag{2.8}$$

Combining these two yields

$$\sum_j |\mathcal{M}(i \to j)|^2 = \sum_j |\mathcal{M}(j \to \bar{i})|^2 = \sum_j |\mathcal{M}(j \to i)|^2 \tag{2.9}$$

$$= \sum_j |\mathcal{M}(\bar{i} \to j)|^2 \ , \tag{2.10}$$

because the sum is over all states and anti-states.

And if we have CP invariance

$$|\mathcal{M}(i \to j)|^2 = |\mathcal{M}(\bar{i} \to \bar{j})|^2 = |\mathcal{M}(j \to i)|^2 \tag{2.11}$$

- *Baryon number violation* $\Delta B \neq 0$

  This condition is obvious, since otherwise the universe could never develop a net non-zero baryon number starting from $\eta = 0$.

  Baryon number violation is a generic feature of GUT theories (e.g. $SU(5)$, $SO(10)$, $E6$), because both quarks and fermions are contained in the same irreducible representation of the gauge group G and thus it is possible for scalar particles and gauge bosons to mediate interactions between them [12].

  Within the standard model baryon number can be violated by non-perturbative effects (instantons).

[8] See [10]; Independently Wadim Kuzmin (1970).

[9] For simplicity we assume Maxwell-Boltzmann statistics here, i.e. statistical factors are equal to 1; see [11].

[10] The sum of all transition probabilities from and to a state i are equal, see appendix C.

- *C and CP violation*

  We can infer from (2.11) that without CP violation equal numbers of a state i and its anti-state $\bar{i}$ give equal numbers of final states j and its anti-states $\bar{j}$. Therefore no baryon asymmetry can develop when CP is conserved.

  CP violation in GUT theories arises from loop corrections to the baryon number violating bosonic decay processes. CP -violation is also incorporated in super-symmetric models.

  In the standard model C is maximally broken because only left-handed quarks and leptons are gauge coupled. CP violation has first been observed in 1964 in the neutral Kaon system.[11] CP violation also shows up in the more recent $B^0 - \bar{B}^0$ experiments at BaBaR at SLAC and BELLE at KEK. However the CP -violating complex phase in the CKM-Matrix seems to be to small in order to account for the observed baryon asymmetry.

- *Departure from thermal equilibrium*

  of the respective (i.e. the baryon number violating) process.

  This is a direct consequence of CPT invariance: If we write $\theta = CPT$ and ρ for the density matrix $\rho(t) = e^{-\beta(t)H(t)}$ with Hamiltonian H, then

$$\begin{aligned}\langle B\rangle_T &= \mathrm{Tr}(e^{-\beta H}B) = \mathrm{Tr}(\theta^{-1}\theta e^{-\beta H}B)\\ &= \mathrm{Tr}(\theta e^{-\beta H}B\theta^{-1}) = \mathrm{Tr}(\theta e^{-\beta H}\theta^{-1}\theta B\theta^{-1}) = \mathrm{Tr}(e^{-\beta H}(-B)) = -\langle B\rangle_T\ .\end{aligned} \tag{2.12}$$

  Here we exploited the fact that θ and H commute, when CPT is preserved. We can also infer this condition from eqn. (2.9), which tells us, that when all states are equally populated, transitions from these states must produce i and $\bar{i}$ in equal numbers. Moreover eqn. (2.10) tells us that any initial difference in the particles abundance is erased, because the total amplitudes from all states j to i and $\bar{i}$ are equal and the rates will therefore be proportional to the number densities.

  Therefore, whenever a system is in equilibrium no net baryon number can be generated.

We conclude that when C and CP violation are present such that eqn. (2.11) does not hold, a system containing equal numbers of initial states i and $\bar{i}$ could indeed produce unequal numbers of final states j and $\bar{j}$ provided that the transitions occur in a non-equilibrium situation.

Note that the presence of B-violating interactions (at very high energies) also almost destroys any initial baryon asymmetry. We can therefore assume, that any possibly preexisting baryon number is wiped out and that it is equal to zero when the processes of baryogenesis sets in.

There are several scenarios for baryogenesis. They can be classified based on how the Sakharov conditions are realized. We will now mention some of them.

GUT Baryogenesis

GUT baryogenesis was the first scenario discussed to account for the baryon asymmetry. In $SU(5)$ GUT[12] the fermions belong to the irreducible representations

$$\overline{5}_f = [d_L^c, l_L] \text{ and } 10_f = \{d_l, u_L^c, e_L^c\}\ .$$

[11] The semileptonic decays of K^0 and $\overline{K^0}$ to $\pi^- l^+ \nu_l$ and $\pi^+ l^- \overline{\nu_l}$ are found to have a slight preference for the former.

[12] $SU(5)$ GUT is essentially ruled out by the experimental limits on proton lifetime. Nevertheless it is often used for illustrational purposes.

The fermions are coupled to the gauge bosons by

$$\frac{g}{\sqrt{2}} 24_V \left[(\overline{5}_f)^\dagger (\overline{5}_f) + (10_f)^\dagger (10_f) \right] .$$

$SU(5)$ GUT contains 24 gauge bosons: $W^\pm$, Z, γ, 8 gluons and 12 lepto-quarks X, $\overline{X}$ belonging to 24_V.

Baryon number violation is naturally satisfied in GUT theories, because quarks and leptons belong to the same irreducible representation. The heavy X and $\overline{X}$ particles undergo lepton and baryon number violating decays to quark lepton pairs.[13]

When the reaction rate of these particles drops below the expansion rate of the universe the particles decay out of equilibrium (see below). If the decays violate C and CP at the same time, then all the Sakharov conditions are satisfied.

However, until today there are no experimental indications for the supposed particle decays. Furthermore, any baryon asymmetry created before the electroweak phase transition is exposed to the washout effect by so called sphaleron processes if no nonzero $B - L$ asymmetry exists. we will discuss these effects in some detail.

Electroweak Baryogenesis

In electroweak baryogenesis the property of sphalerons to violate B is exploited to produce baryon number. The third Sakharov condition is satisfied by boundary effects between bubbles of coexisting broken and unbroken electroweak phase, which could form at the electroweak phase transition. The prerequisite of this scenario is that the electroweak phase transition must be strictly first order. This requires the Higgs mass to be significantly smaller than $80\,$GeV. ($m_H > 115\,$GeV imposed by LEP)

This scenario is therefore viable only in certain super symmetric extensions in which the stop[14] mass could be smaller than the top-quark mass.

Affleck-Dine Baryogenesis

Affleck-Dine baryogenesis[15] is a mechanism suggested for SUSY theories. It is based on the existence of many flat directions in the potential of the scalar quarks and scalar leptons. After SUSY breaking the flat directions acquire soft SUSY-breaking masses of order of $1\,$TeV.

The expectation values of the scalar fields could get values of the order of the Planck mass. These scalar fields then start to oscillate in the flat directions, when their mass is comparable to the expansion rate of the universe. If there exist any baryon number violating nonrenormalizable operators, these would induce baryon number violation.

Baryon asymmetry is generated at the end of the inflationary period. This leads to an asymmetry between quarks and antiquarks after reheating.

This mechanism is still viable if the dilution during expansion is not greater than by a factor of 10^{10}.

[13] See e.g. [13], [14], [15]

[14] The stop is the supersymmetric partner of the top particle.

[15] See [16].

Leptogenesis

We have seen that the different baryogenesis theories suffer from various problems. It makes therefore sense to seek for alternative mechanisms.

The basic idea of leptogenesis is, that first a lepton asymmetry is created, which is then converted (partially) to a baryon asymmetry.

The conversion between baryon number and lepton number is then accomplished by sphaleron processes.

As for baryogenesis there exist numerous candidates. Examples are thermal leptogenesis, non-thermal leptogenesis and neutrinogenesis [17]. In this thesis we will mainly address the first one.

Before we begin with the detailed discussion of thermal leptogenesis, we outline the underlying physics.

2.2 The Standard Model

In this section we will briefly summarize the standard model.[16]

The standard model is a gauge theory based on the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ symmetry group. The fermionic particle content is made up by fundamental quarks and leptons. They can be arranged in a scheme of three families, each of which contains a charged lepton and the corresponding neutrino and an up and a down like quark. In the standard model the charged lepton and its neutral partner as well as the quarks appear in left-handed $SU(2)_L$ doublets. Furthermore, there is a right-handed singlet field for each charged lepton and quark. There are no right-handed neutrino fields:

$$l_L = \begin{pmatrix} \nu_l \\ e \end{pmatrix}_L , \; q_L = \begin{pmatrix} q_u \\ q_d \end{pmatrix}_L , \; e_R, \; q_{uR}, \; q_{dR} \, . \tag{2.13}$$

Within this model strong, weak and electromagnetic interactions between the fermions are described by the exchange of gauge bosons (8 gluons $G^a_{\mu\nu}$, the $W^\pm$ and Z bosons and the photon)

The full SM-Lagrangian can be written as

$$\mathscr{L} = \mathscr{L}_\mathcal{F} + \mathscr{L}_\mathcal{H} + \mathscr{L}_Y + \mathscr{L}_\mathcal{G} \, . \tag{2.14}$$

The fermion Lagrangian $\mathscr{L}_\mathcal{F}$ describes the coupling of the gauge fields to the fermions.

$$\mathscr{L}_\mathcal{F} = i\overline{\psi}\gamma^\mu D_\mu \psi \tag{2.15}$$

where we combined all fermion fields to one spinor $\psi = (\nu_{eL}, \, e_L, \, e_R, \, \ldots, \, b_R)^T$. The gauge fields show up in the covariant derivatives

$$D_\mu \psi = \left(\partial_\mu + i g_s G^a_\mu F_a + i g W^a_\mu T_a + i g' B_\mu Y\right) \psi \, .$$

F_a applied to ψ multiplies all lepton fields with 0 and all quark fields with $\lambda_a/2$ (λ_a are the Gell-Mann matrices).

[16] See e.g. [18], [19]

The gauge part of the Lagrangian reads

$$\mathscr{L}_{\mathcal{G}} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}\mathrm{Tr}W_{\mu\nu}W^{\mu\nu} - \frac{1}{2}\mathrm{Tr}G_{\mu\nu}G^{\mu\nu} \tag{2.16}$$

with electroweak and gluon field strength tensors $W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ig\,[W_\mu, W_\nu]$, $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and $G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu + ig_s\,[G_\mu, G_\nu]$ respectively.

Spontaneous Symmetry Breaking

Within the SM the fermion masses are generated by the Higgs mechanism of spontaneous symmetry breaking.[17] The full gauge symmetry of the standard model is not exact below the electroweak scale (246 GeV). Rather it is broken to the subgroup $SU(3)_C \otimes U(1)_{QED}$. This is due to the existence of flat directions connecting the degenerate states of minimal energy in the Higgs potential $V(\phi^\dagger\phi) = -\mu^2(\phi^\dagger\phi) + \lambda(\phi^\dagger\phi)^2$, which appears in the Higgs part of the Lagrangian.

$$\mathscr{L}_{\mathcal{H}} = (D_\mu\phi^\dagger)(D^\mu\phi) - V(\phi^\dagger\phi) \tag{2.17}$$

Here we introduced the Higgs-doublet

$$\phi = \begin{pmatrix} \phi^\dagger \\ \phi^0 \end{pmatrix} .$$

The Yukawa interaction Lagrangian couples the fermion fields to the Higgs-field:

$$-\mathscr{L}_Y = h^u_{ij}\bar{q_{Li}}\tilde{\phi}q_{uRj} + h^d_{ij}\bar{q_{Li}}\phi q_{dRj} + f^e_{ij}\bar{l_{Li}}\phi e_{Rj} + h.c.\,. \tag{2.18}$$

Here h^u_{ij}, h^d_{ij}, f^e_{ij} are constant 3×3 coupling matrices, $\tilde{\phi}$ is related to the Higgs doublet by $\tilde{\phi} = i\tau^2\phi^*$ and i, j are generation indices.

When the electroweak symmetry is broken, the Higgs field acquires a nonzero vacuum expectation value (VEV):

$$\langle\phi\rangle_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix},\ v = \left(\sqrt{2}G_F\right)^{-\frac{1}{2}} \simeq 246\,\mathrm{GeV}\,.$$

The Yukawa terms in (2.18) then exhibit the structure of mass terms with mass matrices

$$(m_u)_{ij} = h^u_{ij}v,\ (m_d)_{ij} = h^d_{ij}v,\ (m_e)_{ij} = f^e_{ij}v\,. \tag{2.19}$$

We recall that there exist no right handed neutrino fields in the SM which could be coupled by Yukawa interactions. Consequently neutrinos are massless within the SM .

The Higgs mechanism also predicts the masses of the heavy gauge bosons $W^\pm$ and Z by

$$M_Z\cos\theta_W = M_W = \frac{1}{2}vg\,,$$

and a massive scalar gauge boson, the Higgs particle.[18] The photon stays massless.

In the next section we will describe a less popular facet of the standard model, the so called sphaleron processes.

[17] Fermionic mass terms are not allowed, because they would break the gauge symmetry. See e.g. [20], [21].

[18] This is the only particle predicted by the standard model which has not been observed yet. However one can be optimistic that it will be found by upcoming experiments at LHC. The current lower bound for its mass is 114 GeV.

2.3 Sphaleron Processes

Anomalies

Before we start to explain what sphalerons are, we turn our attention to an important aspect of quantum field theory. Anomalies can occur, whenever a classical field theory is quantized. When the classical Lagrangian exhibits a certain symmetry this implies the existence of conserved Noether currents. During the process of quantization this conservation can be lost.

Consider as an example the chiral triangle anomaly.[19]

We start with the QED Lagrangian

$$\mathscr{L}_{\text{QED}} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{2.20}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and covariant derivative $D_\mu\psi = (\partial_\mu - ieA_\mu)\psi$.

It posesses a global $U(1)$ symmetry and a local chiral symmetry $\psi \to e^{i\alpha\gamma_5}\psi$ and $\bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_5}$, i.e. it is invariant with respect to the following infinitesimal symmetry transformations:

$$\begin{aligned} U(1)_V: \quad & \delta_\theta\psi = ie\theta(x)\psi \\ & \delta_\theta A_\mu = \partial_\mu\theta(x) \\ U(1)_A: \frac{1}{2} \quad & \delta_\alpha\psi = i\alpha\gamma_5\psi\,. \end{aligned} \tag{2.21}$$

The Noether currents corresponding to this symmetries are given by:

$$\begin{aligned} j_\mu &= \bar{\psi}\gamma_\mu\psi \quad \text{(vector current)}\,, \\ j^5_\mu &= \bar{\psi}\gamma_\mu\gamma_5\psi \quad \text{(axial vector current)}\,. \end{aligned} \tag{2.22}$$

In the massless case both currents are conserved:

$$\begin{aligned} \partial^\mu j_\mu &= 0\,, \\ \partial^\mu j^5_\mu &= 2imP = 2im\bar{\psi}\gamma_5\psi \underset{m=0}{=} 0\,. \end{aligned}$$

At quantum level these conservation laws are replaced by so called Ward-Identities. These can be expressed in terms of three point functions:

$$\begin{aligned} \mathcal{T}^{\mu\nu\lambda}_c(x,y,z) &= i\langle 0|\,T\big(j^\mu(x)j^\nu(y)j^\lambda_5(z)\big)\,|0\rangle\ , \\ \mathcal{P}^{\mu\nu}_c(x,y,z) &= i\langle 0|\,T\big(j^\mu(x)j^\nu(y)P(z)\big)\,|0\rangle\ . \end{aligned}$$

Here T denotes time ordering.

According to a theorem by Adler and Bardeen the chiral anomaly in QED is given entirely by the triangle diagrams shown in figure 2.4. Higher order corrections are irrelevant, hence the name triangle anomaly.

[19] The chiral anomaly makes an important contribution to the decay of the neutral pion $\pi^0 \to \gamma\gamma$. The triangle anomaly is also called abelian anomaly. See [22].

$T_c^{\mu\nu\lambda}(k_1, k_2; a) =$

Figure 2.4: Triangle diagrams of the chiral anomaly. a denotes the ambiguity in the regularization by shifting the loop momentum by an amount of a. The diagrams are drawn for $a = 0$. In order to obtain conservation of the vector current $\partial_\mu j^\mu = 0$ we must make the choice $a = -1$.

Fermion triangle diagrams are divergent. Therefore one needs a regulator in order to compute them. The trouble is that there exists no regulator which conserves both vector current and axial vector current. Choosing the regulator such that the vector Ward identity is non-anomalous one finds in coordinate space for the Ward identities (vector Ward identity and axial vector Ward identity)

$$\begin{aligned} i\partial_\mu^x T_c^{\mu\nu\lambda}(x,y,z) &= 0\,, \\ -i\partial_\lambda^x T_c^{\mu\nu\lambda}(x,y,z) &= 2m\mathcal{P}_c^{\mu\nu}(x,y,z) + \frac{1}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha^x\partial_\beta^y\big(\delta^{(4)}(x-z)\delta^{(4)}(y-z)\big)\,. \end{aligned} \tag{2.23}$$

Calculating

$$\langle 0|\, T(\partial_\mu j_5^\mu)\,|0\rangle = \frac{i}{2}e^2\int A_\mu(y)A_\nu(z)\partial_\lambda^x T_c^{\mu\nu\lambda}(y,z,x) + \mathcal{O}(e^2)\,, \tag{2.24}$$

one can employ the Ward identity to find the (non-) conservation laws for j^μ and j_5^μ:

$$\begin{aligned} \partial_\mu j^\mu &= 0\,, \\ \partial_\mu j_5^\mu &= 2imP + \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\,. \end{aligned} \tag{2.25}$$

The quantized axial current is not conserved, even in the limit $m = 0$.

Sphalerons

The most general renormalizable Lagrangian invariant under the SM gauge group (and only containing color singlet fields) is automatically invariant under global abelian symmetries (accidental symmetries) which may be identified with baryonic and leptonic symmetries. Therefore it is not possible to violate B and L at any order of perturbation theory.

However there are ('t Hooft 1976) nonperturbative processes that may give rise to processes which violate $B + L$ but conserve $B - L$.[20] The presence of the triangle anomaly implies the presence of infinitely many vacuua. These vacuua are connected through instantons. At quantum level, the baryon and the lepton symmetries are anomalous so that their respective Noether currents j_B^μ and j_L^μ are no longer conserved, but satisfy (cf. (2.25)):

$$\partial_\mu j_B^\mu = \partial_\mu j_L^\mu = n_f\left(\frac{g^2}{32\pi^2}W_{\mu\nu}^a\tilde{W}^{a\mu\nu} - \frac{g'^2}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}\right)\,,$$

[20] See [23], [24], [25].

where

$$\tilde{W}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}W_{\alpha\beta}$$

is the dual of the $SU(2)_L$ field strength tensor and $\tilde{F}^{\mu\nu}$ accordingly. n_f is the number of families. We see immediately, that

$$\partial_\mu\left(j_B^\mu - j_L^\mu\right) = \partial_\mu j_{B-L}^\mu = 0\,,$$

$$\partial_\mu\left(j_B^\mu + j_L^\mu\right) = \partial_\mu j_{B+L}^\mu \neq 0$$

which means, that $B - L$ is conserved whereas $B + L$ is violated. The change in baryon number can then be written

$$\Delta B = \Delta N_{CS} = n_f\left[N_{CS}(t_f) - N_{CS}(0)\right]$$

where the (integer) Chern-Simons number is defined to be

$$N_{CS}(t) = \frac{g^2}{32\pi^2}\int d^3x\epsilon^{ijk}\mathrm{Tr}\left(A_i\partial_j A_k + \frac{2}{3}igA_iA_jA_k\right)\,.$$

Here we used Gauss's theorem

$$\Delta B = \int_{-\infty}^{\infty} dt\partial_0\int d^3x j_B^0(x) = \int_{-\infty}^{\infty} dt\int d^3x\partial_\mu j_B^0(x) = B(+\infty) - B(-\infty)\,.$$

Thus changes in the Chern-Simons Number result in changes of the baryon number which are integral multiples of the number of families n_f. If the system performs a transition from one vacuum to the closest one, the Chern-Simons number is changed by unity and $\Delta B = \Delta L = n_f$.

So each transition creates 9 left-handed quarks (3 color states for each generation) and 3 left-handed leptons (one per generation).[21]

However adjacent vacuua of the electroweak theory are separated by a ridge of configurations with larger energies.

The probability for this transition to occur is highly suppressed by a Boltzmann factor:

$$\Gamma_{\text{sphaleron}}(T) \propto T^{-3}\exp\left(-\frac{E_{\text{sphaleron}}(T)}{T}\right)\,. \tag{2.26}$$

One distinguishes two kinds of transitions between distinct vacuua of a field configuration.

Transitions at zero temperature are called *instantons*. In QCD instantons are soliton solutions of the Yang-Mills equations in euclidean space which are localised in space and time. The instanton solutions describe the quantum mechanical transition between the different classes of Yang-Mills vacuua.

Transitions at finite temperatures are called *sphaleron processes*. The factor in (2.26) prevents the process from occurring under today's conditions but it is expected that the rate becomes sufficiently large at temperatures above the electroweak phase transition $> 100\,\mathrm{GeV}$ (Kuzmin, Rubakov and Shaposhnikov (KRS)).

[21] Note that the W-field couples only to left-handed fermions.

Figure 2.5: Schematic picture of the electroweak vacuum structure. In order for a sphaleron transition to occur the system has to overcome the barrier between two adjacent vacuua. This process can take place at a high rate, only if the energy is high enough. An instanton would be a transition by tunneling between two vacuua at $T = 0$. The ridge with the lowest energy is a saddle point and is referred to as the *sphaleron*. This picture was taken from [26].

Although sphalerons are nonperturbative phenomena this effective 12-fermion interaction can be described by the following Lagrange term [23]:

$$O_{B+L} = \prod_{i=1..3} (q_{Li} q_{Li} q_{Li} l_{Li}) \tag{2.27}$$

One of the possible processes is depicted in figure 2.6.

Figure 2.6: One of the possible sphaleron processes. Involved are 12 left-handed fermions (3 leptons and 9 quarks). This picture was taken from [27].

In the high temperature plasma of the early universe the Boltzmann factor (2.26) becomes small and the $B + L$ violating but $B - L$ conserving processes might be efficient enough to produce the

observed baryon asymmetry. However sphalerons are more efficient in washing out any existing baryon asymmetry.

Whereas the instantaneous effects described in this section are entirely part of the standard model, the theory of leptogenesis is based on physics beyond the standard model. Namely right-handed neutrinos. We are going to introduce them in the next section.

2.4 Beyond the Standard Model

Neutrino Oscillations and Neutrino Masses

In recent years neutrino experiments like SuperKamiokande, the Sudbury Neutrino Observatory (SNO) and KamLAND showed that neutrinos do flavour oscillate. This observation requires non-zero neutrino masses. It can be seen as the first experimental evidence for physics beyond the standard model. Current best fit values for the mass-squared differences are[22] (at 3σ level):

$$\begin{aligned} \left|\Delta m^2_{23}\right| &= (1.2 - 4.8) \times 10^{-3}\,\text{eV}^2 \quad \text{(athmospheric } \nu\text{'s)} \\ \left|\Delta m^2_{12}\right| &= (5.4 - 9.5) \times 10^{-5}\,\text{eV}^2 \quad \text{(solar } \nu\text{'s)} \end{aligned} \tag{2.28}$$

In particular these values place lower bounds on the neutrino masses of $\sqrt{\Delta m^2_{\text{atm}}} \simeq 0.05\,\text{eV}$ and $\sqrt{\Delta m^2_{\text{sol}}} \simeq 8 \times 10^{-3}\,\text{eV}$.

Neutrino oscillation experiments are only sensitive to mass-squared differences. Direct mass constraints come e.g. from the measurement of the endpoint spectrum of tritium beta decay ${}^3\text{H} \to {}^3\text{He} + \overline{\nu_e} + e^-$ for the electron neutrino and the pion decay $\pi^+ \to \mu^+ + \nu_\mu$ for ν_μ.

$$\begin{aligned} m_{\nu_e} &< 2.5\,\text{eV} \\ m_{\nu_\mu} &< 170\,\text{keV} \\ m_{\nu_\tau} &< 18\,\text{MeV} \end{aligned}$$

Despite of the efforts in this field, the most stringent bounds still come from cosmology:[23]

$$\sum_i m_i < 30h^2\,\text{eV}\,. \tag{2.29}$$

Chiral Decomposition

In order to understand neutrino masses we need some technical knowledge of chiral fields. The left- and right-handed chirality projection operators are defined as:

$$P_L = \frac{1-\gamma_5}{2},\ P_R = \frac{1+\gamma_5}{2}\,. \tag{2.30}$$

[22] Neutrino oscillation experiments also measure the mixing angles $\sin^2 2\theta_{23}$ and $\sin^2 2\theta_{12}$. For these values see e.g. [28].

[23] The Big Bang model predicts that there is a fixed ratio between the number of neutrinos ($n_i \simeq 110\,\text{cm}^{-3}$) and the number of photons in the cosmic microwave background. If the total mass of all three types of neutrinos would be to large, there would be so much mass in the universe that it would collapse. (see below)

With help of these operators we can decompose any Dirac field in its chiral components:

$$\psi = P_L\psi + P_R\psi = \psi_L + \psi_R \,. \tag{2.31}$$

We denote charge conjugation by

$$\psi \to (\psi)^c = C\overline{\psi}^T = i\gamma_2\gamma_0\overline{\psi}^T \,. \tag{2.32}$$

When applied to a chiral field, the charge operator flips its chirality:

$$(\psi_L)^c = (\psi^c)_R, \;\; (\psi_R)^c = (\psi^c)_L \,. \tag{2.33}$$

It follows, that we may write

$$\overline{\psi_R} = (\psi_L^c)^T C, \;\; \overline{\psi_L} = (\psi_R^c)^T C \,. \tag{2.34}$$

Right Handed Neutrinos and the Seesaw Mechanism

In principle one could account for neutrino masses by simply adding a Dirac mass term for the neutrinos (i.e. adding adequate right-handed neutrino fields) to the standard model Lagrangian [29], [30]. However this would leave the puzzle of the existence of different mass hierarchies within the standard model unresolved. All charged fermion masses are within a range of two orders of magnitude, but the masses of the neutrinos are smaller by several orders of magnitude.

As we have seen, within the standard model the masses of the charged quarks and leptons arise from Yukawa couplings of the form

$$\mathscr{L}_Y = -f\phi\overline{l_L}l_R + h.c. \tag{2.35}$$

The coupling constant for such a Yukawa term would be $f \simeq \frac{m_f}{\langle\phi\rangle_0} \backsim 10^{-14}$ for a neutrino of mass $m_\nu \simeq 0.01\,\text{eV}$, whereas the Yukawa coupling for the electron is $f_e \backsim 10^{-6}$. One could of course assume, that this hierarchy is accidental.

The option favored by most theorists to circumvent this unattractive scenario is the so called see-saw mechanism. It explains the smallness of light neutrino masses by the largeness of heavy Majorana neutrino masses.

In contradiction to Dirac mass terms a Majorana mass term can be constructed out of the left-handed fields alone (and for the right-handed fields accordingly):

$$\mathscr{L}_{m_L} = -\frac{m_L}{2}\overline{(\nu_L)^c}\nu_L + h.c. \tag{2.36}$$

We can see that the Majorana mass terms are not invariant under $U(1)$ transformations

$$\psi \to e^{i\alpha}\psi, \; \overline{\psi} \to \overline{\psi}e^{-i\alpha} \,, \tag{2.37}$$

like the Dirac mass terms. This means that the Majorana mass term will break the associated charges (electric charge, lepton number). Therefore charged particles cannot have a Majorana like mass term.

Figure 2.7: A Dirac mass term converts a neutrino into a neutrino (above), i.e it absorbs a neutrino and emits a neutrino, or an anti-neutrino into an anti-neutrino. On the other hand Majorana mass terms convert neutrinos into anti-neutrinos (below), or vice versa. The cross denotes mass insertion.

In the seesaw picture neutrinos are assigned both, a Dirac mass and a right-handed Majorana mass (we switch to three leptonic generations now):

$$\mathscr{L}_{m_\nu} = -\frac{1}{2}\left[\overline{(\nu_L)^c}, \overline{(\nu_L)^c}\right] \mathcal{M} \begin{bmatrix} \nu_L \\ (\nu_R)^c \end{bmatrix} + h.c. \tag{2.38}$$

with 6×6 mass matrix $\mathcal{M}$. ν_L and ν_R are vectors in flavour space:

$$\nu_L = \begin{bmatrix} \nu_{Le} \\ \nu_{L\mu} \\ \nu_{L\tau} \end{bmatrix}, \quad \nu_R = \begin{bmatrix} \nu_{Re} \\ \nu_{R\mu} \\ \nu_{R\tau} \end{bmatrix} .$$

Because of $\overline{(\nu_{R\alpha})}(\nu_{R\beta})^c = \overline{(\nu_{R\beta})}(\nu_{R\alpha})^c$, $\mathcal{M}$ can be taken to be symmetric:

$$\mathcal{M} = \begin{bmatrix} 0 & m_D^T \\ m_D & m_R \end{bmatrix} . \tag{2.39}$$

m_D and m_R are 3×3 matrices.

Therefore $\mathcal{M}$ can be brought into block diagonal form with the help of an orthogonal transformation

$$\mathcal{M}_D = U^T \mathcal{M} U = \begin{bmatrix} m_n' & 0 \\ 0 & m_N \end{bmatrix} . \tag{2.40}$$

Since we want m_D to be of the same order as the other leptonic masses and since m_R must be large enough not to induce any processes, which would have been observed in the laboratories, we have $m_D \ll m_R$.[24] This allows us to set

$$U = \begin{bmatrix} 1 & \rho \\ -\rho^\dagger & 1 \end{bmatrix} . \tag{2.41}$$

This means, that

$$U^\dagger U = 1 + \mathcal{O}(\rho^2) , \tag{2.42}$$

to first order in the small parameter ρ.

In order to express $\mathscr{L}_{m_\nu}$ in terms of mass-eigenstates, we write

$$\chi_L = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = U^{-1} \begin{bmatrix} \nu_L \\ (\nu_R)^c \end{bmatrix} . \tag{2.43}$$

[24] Since the right handed neutrino is an electroweak singlet, its mass is not protected by the electroweak symmetry. One can assume that its mass is of order of the GUT scale, or some intermediate scale $\sim 10^{10} - 10^{15}$.

Now we have to diagonalize m'_n (m_N is already diagonal). We do so by performing another orthogonal transformation with 3×3 matrix V. The physical neutrino states of definite mass are then defined by

$$\begin{aligned} n &= V^\dagger \chi_1 + V^T \chi_1^c \,, \\ N &= \chi_2 + \chi_2^c \,. \end{aligned} \tag{2.44}$$

And the diagonal mass matrices are

$$\begin{aligned} m_n &= V^T m_D M^{-1} m_D^T V \,, \\ m_N &= m_R \,. \end{aligned} \tag{2.45}$$

In terms of these expressions (2.38) can be written as

$$-\mathscr{L}_{m_\nu} = \frac{1}{2}\overline{n} m_n n + \frac{1}{2}\overline{N} m_N N \,. \tag{2.46}$$

Furthermore, we can see that these states are transformed into themselves under charge conjugation $n^c = n$ and $N^c = N$. Neutrinos which are their own antiparticles are known as *Majorana neutrinos*.

Carrying out the block diagonalization (2.39) we find

$$\rho \simeq m_D m_R^{-1} \,. \tag{2.47}$$

We see that ρ is indeed a small parameter, just as we have assumed.

We conclude that the general form of the Lagrangian (2.38) yields three light and three heavy Majorana neutrino states. According to (2.45) the large masses of the heavy states suppress the masses of the light states, hence the name seesaw.

Despite of the beauty of the seesaw mechanism it is still unclear whether neutrinos are Majorana particles or not. A direct prove for the Majorana character would be neutrinoless double beta decay Nuc $\rightarrow$ Nuc$' + 2e^-$. Current experimental results on this field, however give no convincing evidence for this process.

2.5 Thermal Leptogenesis

As we have seen above, sphaleron processes could convert any (in the high temperature phase) existing $B - L$ asymetry to a non zero value of $B + L$. In thermal leptogenesis the initial $B - L$ asymmetry is generated by the out of equilibrium decay of the lightest (heavy) Majorana neutrino N_1 into lepton-Higgs pairs.[25] One can satisfy Sakharov's conditions for lepton number in the same way as for baryon number in the GUT scenario.

We begin with the SM extended by 3 right-handed neutrino fields. The Yukawa part of the Lagrangian, inducing the CP violating decays is

$$\mathscr{L}_Y = -\overline{l_{Li}}\phi h_{eij} e_{Rj} + \overline{l_{Li}}\tilde{\phi} h^*_{\nu ij}\nu_{Rj} - \frac{1}{2}\overline{(\nu_R)^c} M \nu_R + h.c. \tag{2.48}$$

[25] See [31], [1], [32], [33].

Spontaneous symmetry breaking generates the Dirac mass matrices

$$(m_D)_{ij} = h_{ij} v \,, \tag{2.49}$$

where the physical Majorana neutrino states with definite mass M_i are given by

$$N_i = \nu_{Ri} + \nu_{Ri}^c \,. \tag{2.50}$$

For simplicity one assumes hierarchical Majorana neutrino masses $M_1 \ll M_2,\, M_3$.
The lepton asymmetry is then generated by the CP -violating decays

$$\begin{aligned} N_1 &\to l\phi \,, \\ N_1 &\to \bar{l}\phi^* \,. \end{aligned} \tag{2.51}$$

These decays obviously violate lepton number. The relevant temperature scale will be $T \sim M_1$. That means that any lepton asymmetry possibly produced by N_2 and N_3 decays would be depleted by processes involving N_1, and that the observed asymmetry eventually is created by the out of equilibrium decay of N_1.

CP Asymmetry

The CP asymmetry ϵ_1 in the Majorana neutrino decays is caused by interference between the tree level and the one-loop diagrams in figure 2.8.[26]

Figure 2.8: Tree level and one-loop contributions to the heavy Majorana neutrino decay $N_j \to l\phi$.

$$\begin{aligned} \epsilon_1 &= \frac{\Gamma(N_1 \to l_i\phi) - \Gamma(N_1 \to \bar{l}_i\bar{\phi})}{\Gamma(N_1 \to l_i\phi) + \Gamma(N_1 \to \bar{l}_i\bar{\phi})} \\ &= \frac{1}{8\pi}\frac{1}{|h_{1i}|^2}\sum_{l=2,3} \mathrm{Im}\left[h_{1i}h_{1k}h_{li}^*h_{lk}^*\right]\left[f\left(\frac{M_l^2}{M_1^2}\right) + g\left(\frac{M_l^2}{M_1^2}\right)\right] \,, \end{aligned} \tag{2.52}$$

where

$$\begin{aligned} f(x) &= \sqrt{x}\left[1 - (1+x)\ln\left(\frac{1+x}{x}\right)\right] \underset{x^{-1}\to 0}{\longrightarrow} -\frac{1}{2\sqrt{x}} \\ g(x) &= \frac{\sqrt{x}}{1-x} \underset{x^{-1}\to 0}{\longrightarrow} -\frac{1}{\sqrt{x}} \,. \end{aligned} \tag{2.53}$$

[26] This is similar to GUT baryogenesis; See for instance [34], [1].

Therefore, we have, using $M_1 \ll M_2,\ M_3$:

$$\epsilon_1 = -\frac{3}{16\pi}\frac{1}{|h_{1i}|^2}\left[\mathrm{Im}\,[h_{1i}h_{1k}h^*_{2i}h^*_{2k}]\,\frac{M_1}{M_2} + \mathrm{Im}\,[h_{1i}h_{1k}h^*_{3i}h^*_{3k}]\,\frac{M_1}{M_3}\right]. \tag{2.54}$$

Using

$$\begin{aligned}
&(m_D)_{ij} = h_{ij}v\,,\\
&(m_D m_D^\dagger)_{11} = (m_D)_{1i}(m_D^\dagger)_{i1} = (h_{1i}h^*_{1i})v^2\,,\\
&(m_D m_D^\dagger)_{12} = (m_D)_{1i}(m_D^\dagger)_{i2} = (h_{1i}h^*_{2i})v^2\,,
\end{aligned} \tag{2.55}$$

one finds:

$$\begin{aligned}
&\frac{1}{|h_{1i}|^2}\mathrm{Im}\,[h_{1i}h_{1k}h^*_{2i}h^*_{2k}] =\\
&\quad = \frac{1}{(m_D m_D^\dagger)_{11}v^2}\left[\mathrm{Im}\left[\left((m_D m_D^\dagger)_{12}\right)^2\right]\frac{M_1}{M_2} + \mathrm{Im}\left[\left((m_D m_D^\dagger)_{13}\right)^2\right]\frac{M_1}{M_3}\right]
\end{aligned} \tag{2.56}$$

A rough estimate for ϵ_1 is given by

$$\epsilon_1 \sim 0.1\frac{M_1}{M_3} \sim 10^{-6}\,. \tag{2.57}$$

This means that the order of magnitude of CP asymmetry is given by the mass hierarchy of the heavy Majorana Neutrinos i.e. $\frac{M_1}{M_3} \sim 10^{-5}$

Evolution Equations

Buchmï¿½ller et al. derive the following equations for the evolution of the heavy neutrino abundance $N_{N_1} = n_{N_1}/s$ (cf. section 3.8) and the $B-L$ asymmetry $N_{B-L} = n_{B-L}/s$ [28], [35]:[27]

$$\frac{dN_{N_1}}{dx} = -(\Gamma_D + \Gamma_S)(N_{N_1} - N^{EQ}_{N_1})\,, \tag{2.58}$$

$$\frac{dN_{B-L}}{dx} = -\epsilon_1\Gamma_D(N_{N_1} - N^{EQ}_{N_1}) - \Gamma_W N_{B-L}\,, \tag{2.59}$$

where $x = \frac{M_1}{T}$. In chapter 4 we will derive very similar equations in the framework of a simple toy model. Here we can only try to understand them intuitively.

N_{N_1} and N_{B-L} are calculated for a co-moving volume element (see section 3.8).

The first equation describes the evolution of the heavy Majorana neutrino abundance. It can be solved independent of the second one.

Γ_D accounts for decays and inverse decays:

$$\Gamma_D = \frac{1}{8\pi}(h^\dagger h)_{11}M_1\frac{K_1(z)}{K_2(z)}\,,$$

[27] In literature these equations are usually refered to as the Boltzmann equations. We find that this is not a good terminology, as we will explain below.

where K_1 and K_2 are modified Bessel functions, see appendix B. The rate of inverse decays is given by $\Gamma_{ID} = \frac{n_{N_1}^{EQ}}{n_l}\Gamma_D$. Γ_S describes t- and s-channel scattering processes of N_1 with the Higgs field.

$$\Gamma_S = 2\Gamma_{\phi,t}^{N_1} + 4\Gamma_{\phi,s}^{N_1}$$

Finally there is the washout rate which reduces an existing $B-L$ asymmetry by $\Delta L = 1$ and $\Delta L = 2$ scattering processes

$$\Gamma_W = \left(\frac{1}{2}\Gamma_{ID} + 2\Gamma_{\phi,t}^{l} + \Gamma_{\phi,s}^{l}\frac{N_{N_1}}{N_{N_1}^{EQ}}\right) + 2\Gamma_N^l + 2\Gamma_{N,t}^l$$

From (2.58), we can see that both, Γ_D and Γ_S try to change the N_{N_1} abundance towards its equilibrium value $N_{N_1}^{EQ}$. The decay processes also act as a source term for the $B-L$ abundance, however only, if $N_{N_1} - N_{N_1}^{EQ} \neq 0$ i.e. if there is a deviation from equilibrium. Γ_W competes with the source term in a way that it tries to decrease any existing $B-L$ asymmetry.

Equations (2.58) and (2.59) can approximately be solved analytical in some limiting cases. In general this has to be done numerically. The result for typical parameters is depicted in figure 2.9.

Figure 2.9: Starting from zero initial abundance, $N_{N_1} = 0$ approaches its equilibrium value $N_{N_1}^{EQ}$. At the same time a nonvanishing $B-L$ asymmetry evolves, because this initial condition is an extreme non-equilibrium situation. When N_{N_1} reaches its equilibrium value the asymmetry is erased by washout processes. Finally, due to the rapid expansion the Majorana neutrinos decay out of equilibrium and the final asymmetry comes into existence. The parameters are: $M_1 = 10^{10}\,\text{GeV}, \epsilon_1 = 10^{-6}, \tilde{m}_1 = 10^{-3}\,\text{eV}$ and $\bar{m} = 0.05\,\text{eV}$ ($\tilde{m}_1$: effective neutrino mass; $\bar{m}^2 = m_1^2 + m_2^2 + m_3^2$: absolute neutrino mass scale). This picture has been taken from [28].

When only decays and inverse decays are taken into account, one can replace $\Gamma_D + \Gamma_S \to \Gamma_D$ and $\Gamma_W \to \Gamma_{ID}$ in equations (2.58) and (2.59). The second equation can then formally be integrated

to give

$$N_{B-L} = N^i_{B-L} e^{-\int_{z_i}^{z} dz' \Gamma_{ID}(z')} - \frac{3}{4}\epsilon_1 \kappa(z; \tilde{m}_1)\,, \tag{2.60}$$

with efficiency factor κ:

$$\kappa(z) = \frac{4}{3}\int_{z_i}^{z} dz' \Gamma_D \left(N_{N_1} - N^{EQ}_{N_1}\right) e^{-\int_{z'}^{z} dz'' \Gamma_{ID}(z'')}\,. \tag{2.61}$$

N^i_{B-L} is a possible initial asymmetry. The integrals in (2.61) can be evaluated approximately in the regime of strong washout (where the decay rate is much larger than the expansion rate of the universe $\Gamma_D \gg H$) and in the regime of weak washout (where $H \gg \Gamma_D$). Since κ does not depend on the CP asymmetry ϵ_1 we find, setting any initial asymmetry to zero, that the generated N_{B-L} is proportional to ϵ_1.

Relation Between B, L and B − L

In order to find out, how large the final B asymmetry produced by sphaleron processes out of the nonzero L asymmetry will be, one assumes that the involved leptons, quarks and Higgs particles are interacting rapidly enough via Yukawa, gauge and sphaleron processes to stay in thermal equilibrium. This induces relations between the chemical potentials of the various particle species. Using the relation (3.72) one finds, that (close to chemical equilibrium) lepton number density $n_L \sim \frac{LT^2}{3}$ and baryon number density $n_B \sim \frac{BT^2}{6}$ can be expressed in terms of the chemical potentials (see section 3.8). This gives on the other hand with the SM baryon and lepton numbers:

$$\begin{aligned} B &= 3 \times \frac{1}{3}\sum_i (2\mu_{qi} + 2\mu_{ui} + 2\mu_{di})\,, \\ L &= \sum_i (2\mu_{li} + 2\mu_{ei})\,. \end{aligned} \tag{2.62}$$

The effective sphaleron Lagrangian in (2.27) yields

$$\sum_i (3\mu_{qi} + \mu_{li}) = 0\,. \tag{2.63}$$

The condition that the total hypercharge must vanish results in the relation[28]

$$\begin{aligned} &\sum_i \left(3\frac{1}{3}2\mu_{qi} + 3\frac{4}{3}\mu_{ui} + 3(-\frac{2}{3})\mu_{di} + 2(-1)\mu_{li} + (-2)\mu_{ei} + \frac{1}{N}(1)\mu_\phi\right) \\ &= \sum_i \left(\mu_{qi} + 2\mu_{ui} - \mu_{di} - \mu_{li} - \mu_{ei} + \frac{2}{N}\mu_\phi\right) \\ &= 0\,, \end{aligned} \tag{2.64}$$

with $N = \frac{N_g}{n_H} = 3$.
The Yukawa interactions, when in equilibrium establish

$$\mu_{li} = \mu_l,\ \mu_{qi} = \mu_q\,, \tag{2.65}$$

[28]Here we use $Y_q = \frac{1}{3}$, $Y_u = \frac{4}{3}$, $Y_d = -\frac{2}{3}$, $Y_l = -1$, $Y_{e^-} = -2$, $Y_\phi = 1$.

and so on (for the different generations).

Employing all the above relations one can express μ_q, μ_u, μ_d, μ_e, μ_ϕ in terms of μ_l:

$$\begin{aligned}
\mu_q &= -\frac{1}{3}\mu_l \,, \\
\mu_u &= \frac{2N-1}{6N+3}\mu_l \,, \\
\mu_d &= -\frac{6N+1}{6N+3}\mu_l \,, \\
\mu_e &= \frac{2N+3}{6N+3}\mu_l \,, \\
\mu_\phi &= \frac{4N}{6N+3}\mu_l \,.
\end{aligned} \tag{2.66}$$

Inserting these results into (2.62) this gives after some calculation

$$\begin{aligned}
B &= -\frac{4N}{3}\mu_l \,, \\
L &= \frac{14N^2+9N}{6N+3}\mu_l \,, \\
B-L &= -\frac{22N^2+13N}{6N+3}\mu_l \,,
\end{aligned} \tag{2.67}$$

and finally

$$c_s \equiv \frac{B}{B-L} = \frac{8N+4}{22N+13} = \frac{28}{79} \,. \tag{2.68}$$

Note that this relation holds for $T \gg v$ only.

Thus we found by the analysis of the chemical potentials of all particle species in the high-temperature phase, that N_B and N_L are related by

$$N_B = c_s N_{B-L} = \frac{c_s}{c_s - 1} N_L \,, \tag{2.69}$$

with $c_s = 28/79$.

In this sense lepton number violation is necessary in order to explain the cosmological baryon asymmetry.

Putting the results of the previous paragraphs together we can formally write for the generated baryon asymmetry

$$\eta = \frac{n_B - n_{\bar{B}}}{n_\gamma} = d\epsilon_1\kappa_f \sim 10^{-9} \tag{2.70}$$

(ϵ_1: CP asymmetry; $\epsilon_1 \sim 10^{-6}$; $d \sim 0.01$: dilution factor that accounts for increase of the number of photons in a co-moving volume element between baryogenesis and today; $\kappa_f \sim 0.1$: efficiency factor, determination requires solution of Boltzmann equation).

The baryon asymmetry is generated at a temperature $T_B \sim M_1 \sim 10^{10}$ GeV corresponding to a time $t_B \sim 10^{-26}$ s.

Constraints on Neutrino Masses

By virtue of the seesaw mechanism, leptogenesis yields bounds for the light neutrino masses.

An upper bound for the CP asymmetry $\epsilon_1 \leq \epsilon_1^{max}(\tilde{m}_1, M_1, \bar{m})$ from theory implies a maximal baryon asymmetry:

$$\eta \leq \eta^{max} \simeq d\,\epsilon_1^{max}(\tilde{m}_1, M_1, \bar{m})\,\kappa(\tilde{m}_1, M_1, \bar{m}^2)\,. \tag{2.71}$$

Since both, CP asymmetry and κ can be expressed in terms of neutrino masses, requiring

$$\eta^{\text{max}} \geq \eta^{CMB}\,,$$

yields constraints for the neutrino masses. Buchmï¿½ller et al. find as upper (lower) bound for the light (heavy) neutrino mass [36]:

$$\begin{aligned} m_i &< 0.12\,\text{eV}\,, \\ M_2, M_3 &\gg M_1 > 4 \times 10^8\,\text{GeV}\,. \end{aligned} \tag{2.72}$$

We see that the light neutrino mass bounds are competitive to the mass bounds presented in section 2.4. However we should keep in mind that there exist numerous theories trying to explain the observed baryon asymmetry. Probably the mass bounds should rather be understand as a test condition for thermal leptogenesis.

Look-Out

Equations (2.58) and (2.59) are derived from a set of (full) Boltzmann equations by means of a certain approximation technique. This kind of approximation has first been applied to baryogenesis theories, where very similar Boltzmann equations are encountered [11].

During this derivation various assumptions are made concerning the momentum distributions of the interacting particles. Unfortunately, due to the complexity of the Boltzmann equations it is hardly possible to estimate the error imposed by these simplifications. Therefore it is desirable to solve the Boltzmann equations independently in order to find out how large the error bars are.

Another more fundamental problem is that the Boltzmann equations are in principle intended for the description of systems of dilute classical gasses. The early universe at the timescales of leptogenesis however consists of a dense strongly interacting plasma. In such extreme situations effects will occur which are not described by kinetic equations.

In order to find out how important these processes are a completely quantum field theoretic description of non-equilibrium process such as leptogenesis is needed. This task lies way beyond the scope of this thesis, but we will try to specify why the Boltzmann equations are imperfect with respect to the primordial plasma.

Chapter 3

Kinetic Theory

Conventional equilibrium thermodynamics deals with macroscopic quantities such as particle number, energy or pressure which are obtained as average values of a large ensemble of identical systems. In order to understand non-equilibrium phenomena however it is necessary to follow the evolution of a specific system in detail. Therefore one needs a microscopic description of many particle systems. Such a description is given by kinetic theory.

The basic object of this theory is the *single particle distribution function* $f(x,k)$. It is defined such that $f(x,k)\Delta^3\mathbf{x}\Delta^3\mathbf{k} = \Delta N$ is the average number of particles with momenta in the range $(\mathbf{k},\ \mathbf{k}+\Delta\mathbf{k})$ located in the volume $(\mathbf{x},\ \mathbf{x}+\Delta\mathbf{x})$ at a time t.[1]

Once this distribution function is known all macroscopic quantities can in principle be calculated from it.

In the context of relativistic kinetic theory it is important to note that f is a Lorenz scalar. This can be seen as follows. Consider a fixed number of (massive) particles ΔN. Say, viewed by an observer in a co-moving coordinate system, the size of a virtual box enclosing all ΔN particles is $\Delta\mathbf{x}' = \Delta x'\Delta y'\Delta z'$. A different observer will see this volume element Lorenz contracted (assuming parallel orientation of his coordinate system) with size $\Delta\mathbf{x} = \Delta x\Delta y\Delta z = \Delta x'\Delta y'\gamma^{-1}\Delta z'$. Since the energy of the particles seen by this observer is $E = \gamma m$ it is clear that $E\Delta\mathbf{x}$ is a Lorenz scalar. By similar argument one finds that $\Delta\mathbf{k}/E$ is also invariant. Therefore we can conclude that $f(x,k) = \Delta N/(\Delta\mathbf{x}\Delta\mathbf{k})$ is indeed a Lorenz invariant quantity. This reasoning can be generalized to massless particles.

3.1 The Boltzmann Equation

The central equation of kinetic theory was derived by Ludwig Boltzmann over hundred years ago as a description of dilute mono-atomic gasses. It is an integro-differential equation which governs the time evolution of the particle distribution function f.

Although originally derived by Boltzmann for the classical systems of mono-atomic gasses the Boltzmann equation has been applied to a broad range of physical problems. Among them are the diffusion of neutrons in nuclear reactors, the behaviour of electrons and quasi-particles in solids as well as various astrophysical problems.

[1] We use relativistic coordinates $x = x^\mu = (t,\mathbf{x})$. Note that k^0 is not an independent variable as the particle's 4-momentum is confined to a hyperboloid in momentum space $k^0 = \sqrt{\mathbf{k}^2+m^2} = E_k$. (i.e. f depends on seven independent variables only.)

In this chapter we will briefly introduce the Boltzmann equation in its relativistic and non-quantum mechanical form. Then we will successively generalize it to fit our needs of a general relativistic description of the primordial plasma. Some attention will be payed to the discussion of its limitations. Finally, we will rewrite the equations in a form suitable for an efficient numeric solution.

In its most abstract form the Boltzmann equation can be written as

$$L[f] = C[f]\,. \tag{3.1}$$

On the left hand side of the equation we have the Liouville operator

$$L[f](x,k) = k^\mu \partial_\mu f(x,k) = k^0 \left(\partial_t + \mathbf{u}\cdot\nabla\right) f(x,k)\,, \tag{3.2}$$

describing the time evolution of a phase space volume element. $\mathbf{u}$ is the three-velocity $\mathbf{u} = \mathbf{k}/k^0$.

The collision term or collision integral $C[f]$ on the right hand side describes the change of the distribution function due to binary interactions between the particles:[2]

$$C[f](x,k) = \int \left[f(x,q)f(x,r)W(q,r|k,p) - f(x,k)f(x,p)W(k,p|q,r)\right] d^3p\, d^3q\, d^3r\,. \tag{3.3}$$

The integrand is composed of two terms, of which the first one (called *gain term*) accounts for an increase of particle number in the phase-space element $\Delta^4 x \Delta^3 \mathbf{k}$ around x in Minkowski-space and around momentum $\mathbf{k}$ due to collisions of two particles with momenta q and r outside of this volume, which result in a final state inside this region.

The second one (called *loss term*) accounts for the loss of particles due to collisions inside this phase-space element, which result in states outside accordingly. Each of this terms is proportional to the product of the densities of the two incident particles. This prescription is known as *Stosszahlansatz*.

The proportionality factor $W(k,p|q,r)$ is the transition rate (i.e. the probability per unit time and volume that two particles with momenta k and p are scattered into the final states q and r). In a quantum field theory it can be expressed in terms of the invariant matrix elements as

$$W(k,p|q,r) = \frac{1}{(2\pi)^3 2p^0 (2\pi)^3 2q^0 (2\pi)^3 2r^0} (2\pi)^4 \delta^{(4)}(k+p-q-r)|\mathcal{M}(k,p\to q,r)|^2\,, \tag{3.4}$$

where the delta-function ensures energy and momentum conservation. $|\mathcal{M}|^2$ is the invariant matrix element squared. When the particles have spin $|\mathcal{M}|^2$ is replaced by $\overline{|\mathcal{M}|^2}$, the amplitude averaged over initial and final spin states.

Because of the scalar character of the distribution function and the invariance of the matrix amplitude as well as the phase space elements

$$\frac{d^3v}{(2\pi)^3 2v^0} \tag{3.5}$$

it is evident, that the collision integral (3.3) is Lorentz covariant.

[2] We suppress here and in the following any statistical factors for identical particles in the in and out-states and introduce them later when needed.

If the amplitude is invariant under interchange of initial and final momenta we can write[3]

$$W(k,p|q,r) = W(q,r|k,p)\,. \tag{3.6}$$

This is the so called *detailed balance* property. Then we can write eqn. (3.3) as

$$C[f](x,k) = \int W(q,r|k,p)\big[f(x,q)f(x,r) - f(x,k)f(x,p)\big]\, d^3p\, d^3q\, d^3r \tag{3.7}$$

For simplicity we will use this form frequently in the subsequent sections although the property of detailed balance is lost in case of CP violating processes such as in baryogenesis.

3.2 Macroscopic Variables

Now we list some macroscopic quantities that can be defined in terms of the distribution function f.

The first one is the particle four-flow $N^\mu(x)$ defined as

$$N^\mu(x) = \int f(x,p)\frac{p^\mu}{p^0}\frac{d^3p}{(2\pi)^3}\,. \tag{3.8}$$

In isotropic systems the only nonzero component is the number density[4]

$$n = N^0 = \frac{g}{2\pi^2}\int p^2 f(p)\, dp\,. \tag{3.9}$$

The spatial part gives the particle flow $\mathbf{j}(x,t)$ which is not important in the isotropic case.

Then there is the energy-momentum tensor $T^{\mu\nu}$, defined to be

$$T^{\mu\nu}(x) = \int f(x,p)\frac{p^\mu p^\nu}{p^0}\frac{d^3p}{(2\pi)^3}\,. \tag{3.10}$$

Its 00-component gives the energy density

$$\rho = \frac{g}{2\pi^2}\int p^2 p^0 f(p)\, dp\,. \tag{3.11}$$

The spatial components give the pressure tensor, which gives in the isotropic case

$$g^{ij}P = T^{ij} = g^{ij}\frac{g}{2\pi^2}\int \frac{p^4}{3p^0} f(p)\, dp\,. \tag{3.12}$$

This equation defines the pressure P.[5] The energy-momentum tensor takes only the rest energy and the kinetic energy of the particles into account. The interaction of the particles does not

[3]This is a consequence of combined symmetry under time reversal $W(k,p|q,r) = W(-q,-r|-k,-p)$ and spatial inversion i.e. $W(k,p|q,r) = W(-k,-p|-q,-r)$.

[4]We introduced here the number of internal degrees of freedom g which will be important in the chapter about cosmology. It is equal to one for scalar particles.

[5]The components T^{0i} and T^{j0} are called energy flow and momentum density respectively. They play no important role in this work.

contribute. In dilute systems the interaction energy can usually be neglected compared to the kinetic energy.

Finally we define the entropy four-flow as

$$S^\mu = -\int \frac{d^3p}{p^0} p^\mu f(x,p) \log\left(f(x,p)\right) . \tag{3.13}$$

In the next section we will formulate the H-Theorem on the basis of this quantity. This relation expresses the fact that the Boltzmann equation exhibits irreversible behaviour. The second law of thermodynamics tells us that entropy is always increasing. We will see that the Boltzmann equation constitutes a microscopic basis for this statement.

3.3 Boltzmann H-Theorem and Collisional Invariants

In order to see what *collisional invariants* (also summational invariants) are, we define the following functional operator:[6]

$$I[\psi] = \int \frac{d^3k}{k^0} \psi(x,k) C(x,k) . \tag{3.14}$$

Inserting the explicit expression (3.7) and performing the transformations

$$\begin{aligned} (k,p) &\to (q,r) , \\ (k,q) &\to (p,r) \end{aligned}$$

on the integration variables, we can infer the following property of I:

$$I[\psi_k] = \frac{1}{4} I[\psi_k + \psi_p - \psi_q - \psi_r] , \tag{3.15}$$

where we used the index to label the particles (i.e. ψ_i is a function belonging to particle i). A function ψ is called collisional invariant if the microscopic conservation law

$$\psi_k + \psi_p = \psi_q + \psi_r \tag{3.16}$$

is satisfied, i.e. if ψ is a property conserved in every single collision. Since this is necessarily the case for the all components of the four-momentum k and because of the linearity of the functional I, collisional invariants will in general be a linear combination of these:

$$\psi_k(x,k) = a_k(x) + b_\mu(x) k^\mu , \tag{3.17}$$

where $a_k(x)$ and $b_\mu(x)$ are arbitrary functions except for the fact that $a_k(x)$ must be additively conserved in a binary collision.

When we know such an invariant we can use equations (3.1), (3.2) in order to write

$$\partial_\mu \int \frac{d^3k}{k^0} k^\mu \psi_k(x,k) f(x,k) = \int \frac{d^3k}{k^0} \psi_k(x,k) C(x,k) = I[\psi_k] = 0 . \tag{3.18}$$

[6] The concept of collisional invariants can be generalized to systems consisting of several particle species [37].

Armed with the concept of collisional invariants it is easy to derive conservation equations for the macroscopic quantities defined in the previous section. For example if we have particle conservation in the microscopic interaction we can choose $\psi = 1$. Then eqn. (3.18) immediately gives the conservation law for particle four-flow:[7]

$$\partial_\mu N^\mu(x) = 0\,. \tag{3.19}$$

In the same way the conservation law for the energy-momentum tensor can be derived:

$$\partial_\nu T^{\mu\nu}(x) = 0\,. \tag{3.20}$$

Using eqn. (3.13) we can define the local entropy production as

$$\sigma(x) = \partial_\mu S^\mu\,. \tag{3.21}$$

This can be calculated to be

$$\partial_\mu S^\mu = -\int \frac{d^3k}{k^0} k^\mu \left[\partial_\mu f(x,k)\log\left(f(x,k)\right) + \partial_\mu f(x,k)\right]. \tag{3.22}$$

The last term vanishes according to eqn. (3.19). We can therefore write, using (3.7):[8]

$$\begin{aligned}\partial_\mu S^\mu &= -\int \frac{d^3k}{k^0} k^\mu \partial_\mu f_k \log\left(f_k\right)\\ &= -I[\log\left(f_k\right)] = -\frac{1}{4} I[\log \frac{f_k f_p}{f_q f_r}]\\ &= -\frac{1}{4}\int \frac{d^3k}{k^0}\, d^3p\, d^3q\, d^3r k^\mu W(k,p|q,r) \log\left(\frac{f_k f_p}{f_q f_r}\right)\left[f_q f_r - f_k f_p\right].\end{aligned} \tag{3.23}$$

Since $(x-y)\ln\frac{x}{y} > 0$ for all $x,\, y > 0$ and equal to zero for (and only for) $x = y$ it follows that the entropy is always increasing. This property is known as the *H-theorem*. It is in agreement with the second law of thermodynamics.[9]

According to the above derivation $\ln\left(f_k\right)$ is a summational invariant. This means we can write

$$\ln f(x,k) = a(x) + b_\mu(x) k^\mu\,. \tag{3.24}$$

This in turn means, that f, which we identify as equilibrium distribution function can be written

$$f^{\text{eq}}(x,k) = \exp\left(a(x) + b_\mu(x) k^\mu\right). \tag{3.25}$$

For f^{eq} being a true equilibrium distribution it is necessary that the left hand side of the Boltzmann equation vanishes too:

$$L[f^{\text{eq}}] = 0\,.$$

[7] In case of homogeneous systems this gives the conservation of the total particle number $\partial_t N = 0$

[8] Note, that we employ the detailed balance property here. However this is not necessary. According to [37] it is sufficient to use the weaker condition of so-called bilateral normalization.

[9] One could ask, how this definite arrow of time occured in an equation derived from microscopic laws which are invariant under time reversal. The answer is that the assumption of molecular chaos makes all the difference. Under this assumption a collision has a definite direction in time, because two colliding particles must be uncorrelated before they collide. After the collision however they are correlated.

It can be shown, that this yields the following general form for the distribution function:

$$f^{\text{eq}}(k) = \exp\left(\frac{\mu - k^\mu U_\mu}{T}\right), \tag{3.26}$$

where U is a constant four-vector obeying $U^\mu U_\mu = 1$.

In the classical limit this gives the distribution function known as Maxwell-Boltzmann distribution (or Maxwellian), well known from statistical mechanics:

$$f(p) = e^{-(E_p - \mu)/T}\,. \tag{3.27}$$

For E_k being the relativistic energy this is also the correct solution for relativistic systems in the global rest frame, where $U = (1,\, 0,\, 0,\, 0)$.

We conclude that any initial distribution function inserted in the Boltzmann equation is evolved, with steadily increasing entropy until it reaches the equilibrium distribution given by (3.27).

This process of thermalization is shown in figure 3.1 for a gas of massless bosons. Figure 3.2 shows the time dependence of the number density, energy density and entropy density. The plots in this section show numerical results of the computer simulations performed within this work. See section 4.5 for details.

Figure 3.1: Thermalization of a homogeneous system of massless bosons. Successive time slices of the distribution function $f(t_i, |\mathbf{k}|)$. At start time t_0 the system is in an extreme non-equilibrium situation (tsunami distribution). In agreement with the H-theorem it is evolved into a Bose-Einstein distribution function (see below).

Figure 3.2: Time dependence of the macroscopic quantities number density, energy density and entropy density normalized to their initial value. As advertised, particle number and energy are conserved. The entropy density converges against its maximum value. This indicates that the system approaches thermal equilibrium.

We finish this general part on the Boltzmann equation with the remark that the major results of the preceding sections can be generalized to the modifications below, namely the quantum modifications and the generalization to mixtures.[10]

3.4 Limitations

Despite of its wide range of application the Boltzmann equation is suitable for the description of a specific problem only if certain conditions are satisfied. These are in particular the assumptions made during its derivation. Since there exist various derivations this conditions vary slightly in literature.[11] We discuss this subject at great length because it is usually neglected in treatments of kinetic theory in early universe cosmology.

The key assumptions brought into play in the derivation of the Boltzmann equation are as follows:

- A basic assumption of kinetic theory which justifies the definition of the distribution function is that f must be homogeneous and slowly varying in the range of the interaction.[12] This means that there may be no rapid density fluctuations. This can safely be assumed for most of the history of the early universe as we will see later.

[10] Another necessary, but far from obvious feature of the Boltzmann equation is the conservation of positivity of the distribution function. If $f(x, k) \geq 0$ at the initial time then this must be true at later times too, since f is a probability distribution. A simple argument for this property can be found in [38].

[11] See [39] for the derivation of the classical Boltzmann equation. A review on the derivation of the quantum Boltzmann equation has been given in [40]. A field-theoretic derivation can be found in [37]. Finally the Boltzmann equations can be derived from the Kadanoff-Baym equations, making several approximations. Amongst others a first-order gradient expansion and a quasi-particle approximation.

[12] Also the density distribution is assumed to vary slowly on the time scale where collisions happen.

- The mean free path is assumed to be much grater than the range of interaction $l \gg r_0$. This can equivalently be stated as $\tau \gg \tau_0$, i.e. the mean time between two distinct collision must be much greater than the time which a single interaction takes. In field theory the interactions will be of short range if the particles mediating the reaction are heavy enough ($l \sim 1/M$). If the above assumption is violated then the mediating particle could undergo several interactions while it propagates.

- Particle trajectories are rectilinear before and after collisions and the interactions are distinct and instantaneous. This assumption justifies the Stosszahlansatz. Between two interactions the involved particles are assumed to travel freely (on mass shell). In quantum field theory at finite temperatures, however off-shell processes can play an important role.

- Furthermore, there is the assumption of so called *molecular chaos*. This means that two particles about to collide must be uncorrelated i.e. they may not have interacted before. In systems of dilute gasses this is ensured by the fact that two particles, before they collide will undergo many interactions with different particles and therefore loose any correlation they may have had before.[13] In dense system this condition might be violated since successive collisions between two particles can occur frequently.

This short list makes clear that the Boltzmann equation is not intended for the application to very dense systems at high energies.

Another effect that could be important under such conditions are particle creation processes which are predicted by quantum field theories. These kind of processes are also not described by Boltzmann equations.

How could the Boltzmann equation be generalized, such as to account for dense systems like the early universe plasma?

In order to account for reactions involving more than two particles one could introduce higher order terms that represent interactions between three or more particles[14]. This could formally be written

$$\begin{aligned} L[f](k_1) = \int & dk_2 \cdots dk_n \, dp_1 \cdots dp_m \\ & \times \big[W(p_1 \cdots p_m | k_1 \cdots k_n)(1 - \xi_1 f(k_1)) \cdots (1 - \xi_n f(k_n)) f(p_1) \cdots f(p_m) - \\ & - W(k_1 \cdots k_n | p_1 \cdots p_m) f(k_1) \cdots f(k_n)(1 - \xi_1 f(p_1)) \cdots (1 - \xi_m f(p_m)) \big] \, , \end{aligned} \tag{3.28}$$

where W are rate factors (equal to the transition rate defined above in the case of $n = m = 2$).

However this will rapidly become impractical and the mathematical behaviour of such Boltzmann equations is quite unknown. Consequently the higher order terms are used by current authors for illustrational purposes only.

[13] This assumption must be made when deriving the Boltzmann equation from the Liouville equation. In this derivation one gets equations of motions for the multi-particle distribution functions, the celebrated BBGKY-hierarchy (Bogoliubov, Born, Green, Kirkwood, Yvon). In truncating these equations, keeping only the terms with single particle distribution functions, one necessarily makes the assumption of molecular chaos.

[14] Some authors do so, e.g. [15], [41]

Another correction would be to implement off-shell effects where the exchanged particle interacts with particles of the background plasma in an effective way by introducing modified propagators for the mediating particles with an effective mass of order the inverse mean free path [13].

Finally in order to account for spontaneous particle creation processes in the framework of a transport equation source terms can be introduced. An approach to this subject can be found in [42]. The author asserts the possibility of temporary violations of the H-theorem.

Although, today several different possible derivations of the Boltzmann equation are known, its success must in the end be attributed to the ingenuity of Ludwig Boltzmann, who constructed it by simple heuristic arguments. Later it has proved to be the correct approach in many physical situations. The decision whether the Boltzmann equation can be an appropriate description of the early universe plasma is therefore also ultimately left to experiment.

Currently, the only physical system believed to exhibit similar extreme conditions as the early universe, and that can be probed by experiments is the *quark gluon plasma* (QGP). The issue whether it can be described by ordinary hydrodynamics is closely related to the question if the early universe can be described by kinetic equations.

Recent experiments at the Relativistic Heavy Ion Collider (RHIC) suggest that the hydrodynamical description is valid for certain regions of momentum space and a small time interval (about $4-7$ fm/c according to hydrodynamics) between thermalization and hadronization [43], [44]. In this experiments high energetic Au ions collide and form a fireball consisting of nuclear matter at temperatures between 350 MeV and 1000 MeV. However it is not yet clear, whether the QCD phase transition, predicted by lattice calculations did in fact take place. Further experiments are necessary in order to clarify this issue.

A full quantum mechanical approach to out-off equilibrium phenomena in the early universe as well as the quark gluon plasma is non-equilibrium quantum field theory.[15] A comparison of the thermal equilibration described by Kadanoff-Baym equations, arising in thermal field theory and the Boltzmann equation in the framework of a scalar Φ^4 theory can be found in [47].

In this thesis we employ the Boltzmann equation for the calculation of the baryon production in the early universe. When we do so we should always keep in mind that it is presumably only a first order approximation.

3.5 Modifications and Generalizations

In this section we discuss some modifications, that we have to make in order to customize the Boltzmann equation for our specific problem, the calculation of the baryon asymmetry evolution in the early universe.

Spatial Homogeneous and Isotropic Case

In the following, we restrict ourselves to systems, spatially homogeneous and isotropic in momentum space. This simplification will be in agreement with the cosmological principle, when we turn to the discussion of the general relativistic equations. In particular it will simplify the numerical evaluation tremendously.

[15] An introduction to the subject of non-equilibrium quantum field theory can be found in [45]. For equilibrium thermal quantum field theory see [46].

Apart from the modifications, which we already discussed above, a major change in such systems is that we can drop the gradient terms in the Liouville Operator (3.2) and write simply

$$L[f](k) = \partial_t f(t,k)\,. \tag{3.29}$$

Here we have simultaneously rewritten the distribution function in terms of the remaining independent variables. We will often write

$$f(t,k) = f_k\,. \tag{3.30}$$

Quantum Modifications

In order to incorporate quantum statistics into the Boltzmann equation (3.3) so called *blocking* (for fermions) or *stimulated emission* factors $(1-\xi f)$ (for bosons) are introduced in the collision integral.[16]

For fermions these terms ensure that the Pauli exclusion principle is respected by the outgoing particles, because the blocking terms tend to zero when the distribution function becomes 1.

The collision integral modified in this way reads

$$C[f](k) = \int W(k,p|q,r)\big[(1-\xi f_k)(1-\xi f_p)f_q f_r - f_k f_p(1-\xi f_q)(1-\xi f_r)\big]\, d^3p\, d^3q\, d^3r\,. \tag{3.31}$$

The general statements made above can easily be adapted to this form of the Boltzmann equation.

The entropy functional is defined by[17]

$$H\,[f] = -\int \left(\xi^{-1}(1-\xi f)\ln(1-\xi f) + f\ln f\right)\, d^3p\,. \tag{3.32}$$

The condition for statistical balance is then given by

$$\left(\frac{f_k}{1-\xi f_k}\right)\left(\frac{f_p}{1-\xi f_p}\right) = \left(\frac{f_q}{1-\xi f_q}\right)\left(\frac{f_r}{1-\xi f_r}\right)\,. \tag{3.33}$$

This relation yields the correct quantum mechanical equilibrium distribution functions (Bose-Einstein and Fermi-Dirac distribution respectively):

$$f(p) = \frac{1}{e^{(E(p)-\mu)/T} + \xi}\,. \tag{3.34}$$

[16] Throughout we will use $\xi = +1$ for fermions, $\xi = -1$ for bosons and $\xi = 0$ for the classical case respectively.

[17] This is the time component of the entropy four flow only. The classical case is included in this formula as limit $\xi \to 0$.

The Boltzmann Equation in Curved Spacetime

In order to generalize the equations to general relativistic systems we have to impose the corresponding Liouville operator.[18] If we assume that the particle paths are geodesics (i.e. the particles are acted upon only by gravitation between their collisions) then the general relativistic form of the Liouville operator is[19]

$$L[f] = k^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma^\alpha{}_{\beta\gamma} k^\beta k^\gamma \frac{\partial f}{\partial k^\alpha} \,. \tag{3.35}$$

With help of the Christoffel symbols given in A Robertson-Walker metric this becomes (see [51])

$$L[f] = \frac{\partial f}{\partial t} - \frac{\partial R}{\partial t}\frac{1}{R} k \frac{\partial f}{\partial k} \,. \tag{3.36}$$

Because of the additional second term which represents the change of the distribution function due to the expansion of the universe, compared to (3.29) it is obvious, that (3.27) and (3.34) in the classical and quantum case respectively cannot be equilibrium distributions in Robertson-Walker space-time.[20]

The relativistic Liouville operator (3.36) can be written in a different form applying the following transformations to the variables t and k:

$$\begin{aligned} x &= MR(t)\,, \\ \tilde{k} &= kR(t)\,. \end{aligned} \tag{3.37}$$

The Liouville operator in this new coordinates then reads

$$L[f] = Hx \frac{\partial f(x, \tilde{k})}{\partial x} \,. \tag{3.38}$$

Mixtures

In physical systems consisting of N different components one obtains a network of coupled Boltzmann equations. Each equation describes the temporal evolution of the distribution function $f_k^n = f^n(k,t)$ corresponding to species n ($n = 1 \ldots N$) which incorporates the interactions with all other particle species. This system of equations can straight forward be written as

$$L\left[f^n\right](k) = \sum_{l=1}^{N} C_{nl}(k)\,, \tag{3.39}$$

where C_{nl} is the collision integral for interactions between species n and l:

[18] The collision term depends only on the phase space density f at particular space-time coordinates and is therefore independent of the large-scale properties of space-time. Thus the form of the collision integral remains unchanged.

[19] See [48], [49] and [50].

[20] In fact as pointed out above we have to make sure that $L[f] = 0$. Using the general solution (3.25), we can infer the equation $\frac{\dot{a}}{b} = E(k) - \frac{\dot{R}}{R}\frac{b}{\dot{b}}\frac{k^2}{E(k)}$. It has no general solution. However for massless particles such as photons one finds a solution with $a = 0$ and $b = 1/T \propto R$. This is why the CMB stays in perfect Planckian (i.e. equilibrium) shape although it has decoupled a long time ago.

$$C_{nl} = S_{kl} \int \big[(1-\xi_n f_k^n)(1-\xi_l f_p^l) f_q^n f_r^l W_{nl}(q,r|k,p) - \\ - f_k^n f_q^l (1-\xi_n f_q^n)(1-\xi_l f_r^l) W_{nl}(k,p|q,r)\big]\, d^3p\, d^3q\, d^3r \tag{3.40}$$

Here $W_{nl}(q,r|k,p)$ is the transition rate for the process $n+l \to n+l$. The symmetrization factor S_{kl} is $\frac{1}{2}$ in the case of identical particles and 1 for different particle species.

Inelastic Collisions and Particle Number Violating Processes

The transport equations encountered above include elastic collisions only. In relativistic theory, however also inelastic and even particle number violating collisions occur. The generalization follows immediately from (3.40), if we allow for general transitions like $A \to B$ and use the corresponding transition rates $W_{A\to B}$.

3.6 Reduction of the Collision Integral

In general no exact, closed form solutions to the nonlinear Boltzmann equation are known.[21] In most cases it has to be solved numerically.

In this section we show how the nine dimensional collision integral can be reduced to a two dimensional one. This simplification will speedup the numerical calculations significantly. We begin with the collision integral in the form (we use (3.4) for the transition rate)

$$C[f] = \frac{1}{2E_k} \int (2\pi)^4 \delta(E_k + E_p - E_q - E_r) \delta^3(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r})\, |\mathcal{M}|^2\, F(f) \prod_{\mathbf{v}=\mathbf{p},\mathbf{q},\mathbf{r}} \frac{d^3v}{(2\pi)^3 2E_v}\,, \tag{3.41}$$

where we have written symbolically

$$F(f) = (1-\xi f_k)(1-\xi f_p) f_q f_r - f_k f_p (1-\xi f_q)(1-\xi f_r)\,.$$

We can write the 3-dimensional delta-function as the Fourier transform of unity and switch to spherical coordinates:

$$\delta^3(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r}) = \int e^{i\lambda(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r})} \frac{d^3\lambda}{(2\pi)^3}\,,$$

$$d^3p = p^2\, dp\, d\Omega_p\,.$$

The collision term then becomes

$$C[f] = \frac{1}{64\pi^3 E_k} \int \delta(E_k + E_p - E_q - E_r) F(f) D(k,p,q,r) \frac{p\,dp}{E_p} \frac{q\,dq}{E_q} \frac{r\,dr}{E_r}\,. \tag{3.42}$$

[21] Except for a certain class of exact (similarity) solution for so called Maxwell molecules (BKW-modes) in the case of the classical, non-relativistic and homogeneous Boltzmann equation and the trivial equilibrium distribution functions of course.

Here we have defined D as

$$D(k,p,q,r) = \frac{pqr}{64\pi^5} \int \lambda^2 \, d\lambda \int e^{i\lambda \mathbf{k}} \, d\Omega_\lambda \int e^{i\lambda \mathbf{p}} \, d\Omega_p \int e^{-i\lambda \mathbf{q}} \, d\Omega_q \int e^{-i\lambda \mathbf{r}} \, d\Omega_r \, |\mathcal{M}|^2 \ . \tag{3.43}$$

For certain matrix elements $|\mathcal{M}|^2$ the integrals in D can be evaluated analytically[22]. As an example we perform this calculation for the most simple case of $|\mathcal{M}|^2 = const$. The calculation for more complicated amplitudes can be found in appendix E.

We can in general evaluate all the solid angle integrals, which $|\mathcal{M}|^2$ does not depend on in (3.43) as

$$\int e^{\pm i\lambda \mathbf{p}} \, d\Omega_p = \frac{4\pi}{\lambda p} \sin(\lambda p) \, . \tag{3.44}$$

Thus for $|\mathcal{M}|^2 = 1$ the function D simplifies to

$$D_1(k,p,q,r) = \frac{4}{k\pi} \int_0^\infty \sin(\lambda k) \sin(\lambda p) \sin(\lambda q) \sin(\lambda r) \frac{d\lambda}{\lambda^2} \, . \tag{3.45}$$

The integration gives

$$\begin{aligned} D_1 = \frac{1}{4k}((&-1 - s_1 + s_2 + s_3 + s_4 + s_5 - s_6 - s_7)k \\ &+(-1 - s_1 - s_2 + s_3 + s_4 - s_5 + s_6 + s_7)p \\ &+(-1 + s_1 + s_2 + s_3 - s_4 - s_5 - s_6 + s_7)q \\ &+(-1 + s_1 + s_2 - s_3 + s_4 - s_5 - s_7 + s_6)r) \, , \end{aligned} \tag{3.46}$$

with

$$\begin{aligned} s_1 &= \text{sgn}(k + p - q - r) \\ s_2 &= \text{sgn}(k - p + q + r) \\ s_3 &= \text{sgn}(k + p + q - r) \\ s_4 &= \text{sgn}(k + p - q + r) \\ s_5 &= \text{sgn}(k - p - q - r) \\ s_6 &= \text{sgn}(k - p + q - r) \\ s_7 &= \text{sgn}(k - p - q + r) \, . \end{aligned} \tag{3.47}$$

In case of massless particles this can be simplified further (using the Energy conservation) to

$$\frac{1}{2k}(q + r - |q - k| - |r - k|) \, .$$

Because of the term in front of the integral of (3.45) it looks like the formula does not hold for $k = 0$. However it turns out that we can treat this case as limit $k \to 0$. D_1 then becomes ($\sin(\lambda k)/k \to \lambda$)

$$D_1'(p,q,r) = \frac{4}{\pi} \int_0^\infty \sin(\lambda p) \sin(\lambda q) \sin(\lambda r) \frac{d\lambda}{\lambda} \, , \tag{3.48}$$

[22] For example for products of two four momenta such as $k_\mu p^\mu$ for which the calculation is straight forward (see [52]). A different approach has been developed in [53].

which can again be written in terms of signum functions:

$$D_1' = \frac{1}{2}\left(\text{sgn}\,(p-q+r) - \text{sgn}\,(p-q-r) + \text{sgn}\,(p+q-r) - 1\right)$$

After calculating D we can proceed with the integration of the remaining energy delta-function in eqn. (3.42). This is easily done, using $\frac{p\,dp}{E_p} = dE_p$ and changing the lower limit to m_p accordingly. Integrating over dE_p then fixes p.

The Boltzmann equation with the reduced two-dimensional integral finally reads

$$L\,[f] = \frac{1}{64\pi^3 E_k}\int\int \Theta(E_p - m_p)F(f)\sum_i D_i(k,p,q,r)\frac{q\,dq}{E_q}\frac{r\,dr}{E_r}\,, \tag{3.49}$$

where $p = \sqrt{E_p^2 - m_p^2}$ and $E_p = E_q + E_r - E_k$. The Heaviside-function prevents us from integrating over combinations of q and r which are not allowed kinematically.

In appendix E we generalize these steps to collision integrals for decays and inverse decays of the form

$$\begin{aligned} C^{1\leftrightarrow 2}(k) = \frac{1}{2E_k}\int &(2\pi)^4\delta^{(4)}(k+q-r)\,|\mathcal{M}|^2\,[(1-\xi f_k)f_q f_r - f_k(1-\xi f_q)(1-\xi f_r)] \\ &\times \frac{d^3q}{(2\pi)^3 2E_q}\frac{d^3r}{(2\pi)^3 2E_r}\,. \end{aligned} \tag{3.50}$$

Furthermore we perform the integration for the various amplitudes $|\mathcal{M}|^2$ encountered in the subsequent.

3.7 The Discrete Boltzmann Equation

Since the solution of integro-differential equations is not a standard task of numerical analysis we will briefly discuss how this can be carried out in the case of the Boltzmann equation.[23]

We will perform this calculation for the most simple form of the Liouville operator $L[f] = \partial f/\partial t$. It is straight forward to generalize the results to the form (3.38), which we actually employ later on.

In order to solve the Boltzmann equation on a computer we need to establish a discrete velocity model for it.[24] For this purpose we choose a grid of discrete momenta $\mathbf{k}_i$, and divide the physical relevant momentum space (i.e. we consider only momenta up to a maximum of k_{max}) $V \subset \mathbb{R}^3$ in a set of M disjoint but arbitrary domains ΔV_i with $\mathbf{k}_i \in V_i$.

Then we make the approximation

$$\int_{\Delta V_i} f(|\mathbf{k}|\,,t)\,d^3k \simeq f(|\mathbf{k}_i|\,,t)\Delta k_i \equiv f_i \Delta k_i\,,$$

[23] For numerical standard tasks we refer to [54]

[24] Due to the numerical complexity of the Boltzmann equation numerous different numerical techniques have been developed to solve it. One of these methods are discrete velocity models. In kinetic theory the velocity is commonly used instead of the momentum as the independent variable.

where the size of the domain is given by $\Delta k_i = \int_{\Delta V_i} d^3k$. Because of the isotropy in momentum space we can choose the domains as concentric spheric shells about the origin with volume $\Delta k_i = 4\pi k_i^2 D_k$ with some constant small number D_k and an arbitrary vector $\mathbf{k}_i$ contained in this volume, because our distribution functions depend only on the magnitude of this vectors. By integrating eqn. (3.49) over the domain ΔV_l we obtain for the left hand side

$$\int_{\Delta V_l} d^3k \frac{\partial f}{\partial t} \simeq \frac{\partial f_l}{\partial t} \Delta k_l \,, \tag{3.51}$$

with $f_l = f(k_l)$, and for the right hand side

$$\frac{1}{64\pi^3 E_{k_l}} \Delta k_l \sum_{\substack{i,j \\ E_p \geq m_p, p \leq M D_k}}^{M} F(f_l, f_p, f_i, f_j) D(k_l, p, k_i, k_j) \frac{k_i D_k}{E_{k_i}} \frac{k_j D_k}{E_{k_j}} \,, \tag{3.52}$$

where $p = \sqrt{(E_{k_i} + E_{k_j} - E_{k_l})^2 - m_p^2}$.

With the help of the discrete velocity model we turned our continuous Boltzmann equation into a set of M partial differential equations for the discrete functions f_l:

$$\frac{\partial f_l}{\partial t} = \frac{D_k^2}{64\pi^3 E_{k_l}} \sum_{\substack{i,j \\ E_p \geq m_p, p \leq M D_k}}^{M} F(f_l, f_p, f_i, f_j) D(k_l, p, k_i, k_j) \frac{k_i}{E_{k_i}} \frac{k_j}{E_{k_j}}, \ (l = 1 \ldots M) \,. \tag{3.53}$$

3.8 Statistical Mechanics and Standard Cosmology

The basis of cosmology is formed by the *cosmological principle*.[25] It summarizes the observations that *the universe is homogeneous and isotropic on large scales.*[26].

A direct kinematic consequence of this requirements together with the theory of general relativity is, that spacetime is, on large scales, described by the Robertson-Walker metric The line element of Robertson-Walker metric is given by

$$ds^2 = dt^2 - R^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right) \,, \tag{3.54}$$

where $R(t)$ is the cosmic scale factor and θ, ϕ and r are polar coordinates. The spatial separation between co-moving points increases with $R(t)$. k is the curvature constant which determines whether the spatial section of space time is positively curved ($k = +1$), flat ($k = 0$) or negatively curved ($k = -1$). Current astronomical data (in particular the WMAP data) suggests that the universe is flat. We will therefore assume $k = 0$ in the following.

The dynamical evolution of the universe is governed by the Einstein Equation (without cosmological constant):

$$R_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} = 8\pi G T_{\mu\nu} \,, \tag{3.55}$$

[25] See e.g. [4], [5], [55], [56].

[26] The experimental evidence supporting the cosmological principle comes mainly from galaxy surveys and the measurement of the cosmic microwave background. The latter exhibits that the universe has been homogeneous except for variations with magnitude 10^{-4} at the time when the radiation decoupled (at a temperature of $3000\,\mathrm{K}$, which we see red shifted today to a temperature of $2.725\,\mathrm{K}$)

with the energy-momentum-tensor T as source term determining the curvature expressed by the Ricci-Tensor $R_{\mu\nu}$. G denotes Newton's constant. In our units:

$$G{m_{\text{Pl}}}^2 = 1\,, \tag{3.56}$$

$$m_{\text{Pl}} = 1.2 \cdot 10^{19}\,\text{GeV} = 1.39 \cdot 10^{32}\,\text{K}\,. \tag{3.57}$$

This equation, combined with the cosmological principle (i.e. the corresponding form of the energy momentum tensor) yields the Friedmann Equation

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{R^2}\,, \tag{3.58}$$

where we defined the Hubble parameter H. ρ_i are the different energy density components. The Friedmann equation relates the rate of increase of the scale factor to the total energy density of all matter in the universe. We define the critical energy density as

$$\rho_c = \frac{3H^2}{8\pi G}\,, \tag{3.59}$$

for which the universe would be precisely flat. With this notation at hand we can write the energy densities in terms of density parameters

$$\Omega_i = \frac{\rho_i}{\rho_c}\,. \tag{3.60}$$

Current observational data suggests, that $\Omega_{\text{total}} \simeq 1$. This means that the universe is flat.[27].
The Hubble parameter relates the speed v of a receding galaxy to its distance d via Hubble's law:

$$v \simeq Hd\,. \tag{3.61}$$

The present value of the Hubble parameter H_0 is often parametrized

$$H_0 = 100h\frac{\text{km}}{\text{s Mpc}}\,,$$

with reduced Hubble parameter

$$h \simeq 0.71\,. \tag{3.62}$$

A second equation obtained in this way is the acceleration equation:

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\sum_i (\rho_i + 3p_i)\,. \tag{3.63}$$

For $k = 0$ the Friedmann equation yields for the scale factor

$$R(t) = R_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}\,, \tag{3.64}$$

[27] According to WMAP the total density parameter is partitioned in $\Omega_{\text{Dark Energy}} = 0.73$, $\Omega_{\text{Matter}} = 0.27$ and $\Omega_{\text{B}} = 0.044$. Wee see that the universe today is dominated by dark energy. Dark energy has a negative pressure. It drives the accelerated expansion of the universe observed today (equation of state parameter $w = -1$)

where the equation of state parameter w has been defined by

$$p = w\rho\,. \tag{3.65}$$

With $w = 0$ for the matter dominated universe (Einstein-de Sitter universe) and $w = 1/3$ for the radiation dominated universe (Tolman universe) respectively.

So that we have in particular for the radiation dominated universe

$$R \propto \sqrt{t}\,, \tag{3.66}$$

or

$$H = \frac{1}{2t}\,. \tag{3.67}$$

In cosmology the expansion of the universe is assumed to be adiabatic and quasi-static. This means that it passes smoothly from one equilibrium state to another and that the entropy is conserved.

In order to calculate quantities such as the number density and the energy density of a particular particle species i, we can therefore presume that its equilibrium distribution function is

$$f_i(p) = \frac{1}{e^{(E_i(p)-\mu_i)/T} + \xi}\,, \tag{3.68}$$

where is $E_i(p) = \sqrt{p_i^2 + m_i^2}$ for massive particles and $E(p) = p$ for massless particles respectively and μ_i is the chemical potential of species i.

According to (3.9), (3.11) and (3.12) number density, energy density and pressure are given by

$$n_i = \frac{g_i}{2\pi^2}\int p^2 f_i(p)\,dp\,, \tag{3.69}$$

$$\rho_i = \frac{g_i}{2\pi^2}\int p^2 E(p) f_i(p)\,dp\,, \tag{3.70}$$

$$p_i = \frac{g_i}{2\pi^2}\int \frac{p^4}{3E(p)} f_i(p)\,dp\,. \tag{3.71}$$

These integrals can be performed explicitly in the ultrarelativistic ($T \gg m$) and non-relativistic ($T \ll m$) limits. The results for zero chemical potential can be found in the following table (ζ denotes the Riemann zeta function).

	rel. bosons	rel. fermions	non-rel. (both)
n_i	$\frac{\zeta(3)}{\pi^2} g_i T^3$	$\frac{3\zeta(3)}{4\pi^2} g_i T^3$	$g_i \left(\frac{m_i T}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m_i}{T}}$
ρ_i	$\frac{\pi^2}{30} g_i T^4$	$\frac{7}{8}\frac{\pi^2}{30} g_i T^4$	$m_i n_i^{\text{non-rel.}}$
p_i	$\frac{\rho_i}{3}$	$\frac{\rho_i}{3}$	$n_i T$

Figure 3.3: Macroscopic quantities number density, energy density and pressure in thermal equilibrium. In the non-relativistic case the given formulas can be applied to bosons and fermions.

For particles close to chemical equilibrium with their antiparticles i.e. $\frac{\mu}{T} \ll 1$ and $\mu_i = \bar{\mu}_i$ one finds using (3.68):

$$\begin{aligned} n_i - n_{\bar{i}} &\simeq \frac{g_i T^3}{3}\left(\frac{\mu_i}{T}\right) \quad \text{(bosons)}\,, \\ n_i - n_{\bar{i}} &\simeq \frac{g_i T^3}{6}\left(\frac{\mu_i}{T}\right) \quad \text{(fermions)} \end{aligned} \tag{3.72}$$

We met these relations before, in section 2.5.

A further useful result is the equilibrium number density of a species obeying Maxwell-Boltzmann statistics. It can be expressed in terms of modified Bessel functions (see appendix B):

$$n_{MB} = \frac{T^3}{2\pi^2}\frac{m}{T}K_2(\frac{m}{T})\,. \tag{3.73}$$

With help of these relations and the first law of thermodynamics we can derive the entropy of the photon gas:

$$\begin{aligned} dS &= \frac{1}{T}\big(\,d(\rho_\gamma R^3) + \frac{1}{3}\rho_\gamma\, d(R^3)\big) = \frac{1}{T}\big(\frac{4}{3}\rho_\gamma\, d(R^3) + R^3\, d\rho_\gamma\big) \\ &= \frac{4\pi^2}{45}\big(T^3\, d(R^3) + 3T^2 R^3\, dT\big) = \frac{4\pi^2}{45}\, d(T^3 R^3)\,. \end{aligned}$$

It follows that the co-moving entropy is given by

$$s = \frac{S}{R^3} = \frac{4\pi^2}{45}T^3\,. \tag{3.74}$$

One can employ this result and the formulas for the energy density of relativistic particles to define an effective number of relativistic degrees of freedom for the energy density:

$$g_* = \sum_{\text{bosons}} g_i\left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{\text{fermions}} g_i\left(\frac{T_i}{T}\right)^4\,, \tag{3.75}$$

and for the entropy density

$$g_{*S} = \sum_{\text{bosons}} g_i\left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{\text{fermions}} g_i\left(\frac{T_i}{T}\right)^3\,. \tag{3.76}$$

The numeric values for g_* and g_{*S} lie typically very close to each other, however both depend on the epoch the universe is in. For the standard model the approximate results are[28]

$$g_* \simeq g_{*S} \simeq \begin{cases} 100 & \text{if } T > 300\,\text{MeV}, \text{ above the QCD phase transition} \\ 10 & \text{if } 300\,\text{MeV} > T > 1\,\text{MeV} \\ 3 & \text{if } T < 1\,\text{MeV}, \text{ below } e^+e^-\text{-annihilation} \end{cases} \tag{3.77}$$

A more accurate value for g_* at $T > 300\,\text{MeV}$ is $g_* = 106.75$.

[28] Strictly speaking g_* changes whenever $T \approx$ mass of a species. As long as g_* is constant T decreases as R^{-1}

With help of these expressions we can write the total energy and entropy density as

$$\rho = \frac{\pi^2}{30} g_* T^4 \,, \tag{3.78}$$

and

$$s = \frac{2\pi^2}{45} g_{*S} T^3 \,. \tag{3.79}$$

By plugging (3.78) into the Friedmann equation we obtain a direct relation between H and the temperature for the radiation dominated universe:

$$H = 1.66\sqrt{g^*}\frac{T^2}{m_{\rm Pl}} \,. \tag{3.80}$$

3.9 Non-Equilibrium Phenomena

As we have seen standard cosmology is based upon the assumption of a quasi-stationary expansion where most of the particle content stays in thermal equilibrium. However many important processes in the thermal history of the universe are non-equilibrium processes. Examples are inflation, Big Bang nucleosynethesis, decoupling of neutrinos, decoupling of the CMB radiation, and of course baryogenesis or leptogenesis.

Despite of the fundamental problems outlined above, Boltzmann equations are *the* tool for the investigation of non-equilibrium phenomena in early universe cosmology.

In order to derive the form of the Boltzmann equation commonly used in cosmology we integrate (3.36) over the remaining three-momentum:[29]

$$\frac{g}{(2\pi)^3}\int L[f]\, d^3k = \partial_t \int \frac{g}{(2\pi)^3} f_k \, d^3k - \frac{\dot{R}}{R}\int \frac{g}{(2\pi)^3} k \frac{\partial}{\partial k} f_k \, d^3k \,.$$

The first integral in this expression is simply the particle number density and the second one can be evaluated via integration by parts:

$$\int \frac{g}{(2\pi)^3} k \frac{\partial}{\partial k} f_k \, d^3k = \frac{g}{2\pi^2}\int_0^\infty k^3 \partial_k f_k \, dk = -3\frac{g}{2\pi^2}\int_0^\infty k^2 f_k \, dk = -3 n_k \,.$$

We get

$$\frac{g}{(2\pi)^3}\int L[f] = \partial_t n_k + 3H n_k \,. \tag{3.81}$$

This result tells us that every kind of particles even without interaction is diluted by the expansion, at a rate

$$\partial_t n_k = -3H n_k \,.$$

A popular transformation which exploits the conservation of entropy in the universe $d(sR^3) = 0$ is to introduce[30]

$$Y = \frac{n}{s} \,. \tag{3.82}$$

[29] This equation is commonly refered to as Boltzmann equation in literature about cosmology. However, due to the integration of the left hand side the microscopic information contained in the distribution function is lost. The resulting equations cannot be solved without further approximations, whereas the Boltzmann equation in principle can be.

[30] Since the photon number density is proportional to the entropy density $n_\gamma \sim s$ it is equivalent to define $Y = \frac{n}{n_\gamma}$

This gives us

$$s\dot{Y} = \dot{n} + 3Hn\,. \tag{3.83}$$

Usually after some successive simplifications the collision integral becomes a function of temperature rather than time. Therefore it makes sense to transform to the new dimensionless variable

$$x = \frac{m}{T}\,, \tag{3.84}$$

where m is an arbitrary mass scale e.g. the mass of the particle species under investigation. In order to see how this transformation acts on (3.83) we combine (3.80) and (3.67) to

$$t = 0.301\frac{1}{\sqrt{g_*}}\frac{m_{\rm Pl}}{m^2}x^2\,,$$

which leads to

$$s\frac{dY}{dt} = \frac{H(m)}{x}\frac{dY}{dx}\,, \tag{3.85}$$

with $H(m) = 1.66\sqrt{g_*}\frac{m^2}{m_{\rm Pl}}$.

On the other hand, we have to evaluate the collision integral (right hand side of (3.41); we write $g_i\frac{dk^3}{(2\pi)^3 2E_k^i} = d\Pi_k^i$ where we omit the index i when only one particle species is involved)

$$\begin{aligned}\frac{g}{(2\pi)^3}\int C[f]\,d^3k = \int &(2\pi)^4\delta^{(4)}(k+p-q-r)\,|\mathcal{M}|^2 \\ &\times\left[(1-\xi f_k)(1-\xi f_p)f_q f_r - f_k f_p(1-\xi f_q)(1-\xi f_r)\right]d\Pi_k\,d\Pi_p\,d\Pi_q\,d\Pi_r\,.\end{aligned} \tag{3.86}$$

Performing all these transformations, we can write the integrated Boltzmann equation as

$$\begin{aligned}Y' = \frac{x}{H(m)s}\int &(2\pi)^4\delta^{(4)}(k+p-q-r)\,|\mathcal{M}|^2 \\ &\times\left[(1-\xi f_k)(1-\xi f_p)f_q f_r - f_k f_p(1-\xi f_q)(1-\xi f_r)\right]d\Pi_k\,d\Pi_p\,d\Pi_q\,d\Pi_r\,,\end{aligned} \tag{3.87}$$

where Y' denotes the derivative of Y with respect to x.

Strictly speaking the equilibrium distribution functions of statistical mechanics cannot be solutions of the transport equation in Robertson-Walker space-time. However it turns out that these distributions are approximate solutions as long as the total rate of all interactions of a particle species is much larger than the expansion of the universe [51] $\Gamma \gg H$.

In cosmology one distinguishes two cases

- $\Gamma \gtrsim H$: The particles interact rapidly. They are approximately in equilibrium and the number density obeys the relations stated above.
- $\Gamma \lesssim H$: The interaction rate fails to keep up with the expansion. The species is said to be frozen out. The number density is governed by the expansion.

The interesting physics however happens for $\Gamma \sim H$.

Freeze-Out

The abundance of a particle species staying in thermal equilibrium decreases exponentially (see table 3.3). Today its abundance would hardly be measurable. When a particle species is to survive till today it must decouple before its equilibrium abundance has become to small.[31]

The freeze-out of a particle species is usually analyzed by means of a simple model. Let us assume that the reaction rate of a (stable) particle species is dominated by the annihilation reaction

$$\psi\bar{\psi} \longleftrightarrow X\bar{X}\,. \tag{3.88}$$

When the $X's$ are subject to some different particle number conserving reactions (typically that will be electromagnetic interactions with the background plasma), which occur rapidly enough to keep the species in equilibrium the resulting Boltzmann equation can be solved approximately by analytic methods.

For simplicity one assumes that the particles obey Maxwell-Boltzmann statistics. Then one writes $f_{X,\bar{X}} = \exp\left(-E_{X,\bar{X}}/T\right)$ i.e. the chemical potentials are set to zero. Furthermore the quantum terms in the Boltzmann equations are neglected.

Exploiting the energy conservation enforced by the delta function, tells us that

$$f_q^X f_r^{\bar{X}} = \exp\left(-(E_q^X + E_r^{\bar{X}})/T\right) = \exp\left(-(E_k^\psi + E_p^\psi)/T\right) = f_k^{\psi^{EQ}} f_p^{\psi^{EQ}}\,.$$

We use $f_k^{i^{EQ}} = \exp\left(-E_i/T\right)$ to distinguish it from any other non-equilibrium distribution only when this is not obvious. Then we can write eqn. (3.87)

$$Y' = \frac{x}{H(m)s}\int (2\pi)^4\delta^{(4)}(k+p-q-r)\,|\mathcal{M}|^2\left[f_q^{\psi^{EQ}} f_r^{\bar{\psi}^{EQ}} - f_k^\psi f_p^{\bar{\psi}}\right] d\Pi_k\, d\Pi_p\, d\Pi_q\, d\Pi_r\,. \tag{3.89}$$

Defining the thermally-averaged cross section times relative velocity by[32]

$$\left\langle \sigma_{\psi\psi\to X\bar{X}}\,|v|\right\rangle = \frac{1}{(n_\psi^{EQ})^2}\int (2\pi)^4\delta^{(4)}(k+p-q-r)\,|\mathcal{M}|^2\, f_q^{\psi^{EQ}} f_r^{\bar{\psi}^{EQ}}\, d\Pi_k\, d\Pi_p\, d\Pi_q\, d\Pi_r\,, \tag{3.90}$$

we may write

$$Y' = \frac{x\left\langle \sigma_{\psi\psi\to X\bar{X}}\,|v|\right\rangle s}{H(m)}\left[Y_{EQ}^2 - Y^2\right]. \tag{3.91}$$

Since $H(m) = x^2 H(T)$ we can interpret $\frac{\langle\sigma_{\psi\psi\to X\bar{X}}|v|\rangle s}{H(m)} \sim \frac{\langle\sigma_{\psi\psi\to X\bar{X}}|v|\rangle n_\gamma}{H} \sim \frac{\Gamma}{H}$ as efficiency factor for the annihilation processes. That means, that if $\Gamma/H \lesssim 1$ the annihilation rate will decrease and the $\psi's$ will freeze out. The time (or temperature) of decoupling can therefore be infered from the details of the interactions of a given particle species. Equation (3.91) must in general be solved numerically.

[31] In this description we follow Kolb and Turner [15].

[32] The interaction rate for a particle with crosssection σ is typically of the form $\Gamma = n\,\langle\sigma v\rangle$.

After freeze out $Y_{EQ}(x_{\text{freeze-out}}) = \frac{n_\psi}{s}\big|_{x_{\text{freeze-out}}}$ stays constant. (Just insert the appropriate expressions from table 3.3.) This observation implies bounds for the masses of relic particles (such as neutrinos), because the following relation for the density parameter must hold:

$$\Omega_\psi h^2 = \frac{\rho_\psi}{\rho_c} h^2 < \Omega_{\text{total}} h^2 \lesssim 1 \tag{3.92}$$

A discussion of the non-equilibrium effects for the phenomenon of particle freeze-out can be found in [57]. The authors of this paper find that these effects are small for particles decoupling when strongly non-relativistic. The quantum terms are neglected in these considerations.

Out-of-Equilibrium Scenario

The standard scenario commonly employed to satisfy the third Sakharov condition in baryogenesis theories is the out of equilibrium decay scenario.[33] It is based on the observation that a heavy and relatively long-lived particle-species can deviate significantly from its equilibrium distribution before the particles begin to decay. By this means the back reaction, which would erase any baryon asymmetry in thermal equilibrium, is suppressed.

If the particle decays are governed by a small coupling constant α, then the decays (rate $\Gamma_D \propto \alpha M$; M is the particle mass) are the dominant processes governing the particle number density, since the annihilation rate will be of order α^2. As long as $T \gtrsim M$ the decays will be fast enough to maintain thermal equilibrium. When the universe has become as old as the particles lifetime (and the temperature has become low enough so that back reactions are suppressed, because their typical energy $\sim T \lesssim M$) they begin to decay freely.

Taking only decays and inverse decays into account the decoupling of a massive particle species can be modeled by the following Boltzmann equation:

$$\begin{aligned} L[f^X](k) =& \frac{1}{2E_k^X} \int (2\pi)^4 \delta^{(4)}(k-q-r)\, |\mathcal{M}|^2 \\ & \times \left[(1+f_k^X) f_q^{EQ} f_r^{EQ} - f_k^X (1+f_q^{EQ})(1+f_r^{EQ})\right] \frac{d^3q}{(2\pi)^3 2E_q} \frac{d^3r}{(2\pi)^3 2E_r}\,, \end{aligned} \tag{3.93}$$

where we assumed that the particles, which X decays into stay in perfect equilibrium and obey Bose-Einstein statistics (say these are photons of the background plasma). The numerical solution of this equation is shown in figure 3.4.

[33] The former popular scenario of electroweak baryogenesis, in which the non equilibrium condition was created by supposed first-order phase-transition is essentially ruled-out.

Figure 3.4: Out-of-equilibrium decay of a massive particle species (bosons) interacting with the background plasma, plotted for different coupling strengths. The particles first become non-relativistic and afterwards decouple when $\Gamma \sim H$. For smaller couplings (upper lines) the particles depart significantly from thermal equilibrium (solid line). As soon as the energy of the colliding particles becomes lower than the mass of the decaying ones, the number density drops rapidly.

Figures 3.5 and 3.6 show the rates of decay and inverse decay compared to the Hubble rate. We see that the decoupling does indeed take place, when $\Gamma \sim H$.

Figure 3.5: Rates of decay and inverse decay compared to the Hubble rate for a massive particle species close to equilibrium.

Figure 3.6: Rates of decay and inverse decay compared to the Hubble rate for a massive particle species far from equilibrium.

Equilibrium Abundance of Baryons

As advertised in the introduction, we will now calculate the equilibrium abundance of baryons (protons and neutrons).

For $T \leq 1\,\text{GeV} \simeq m_N$ the equilibrium abundance of protons and neutrons ($n_B = n_{\bar{B}}$) is according to table 3.3 given by ($g_i = 8$ for nucleons)

$$\begin{aligned}\eta &= \frac{n_B}{n_\gamma} = \frac{n_{\bar{B}}}{n_\gamma} \\ &= \frac{\sqrt{2\pi} g_i}{8\zeta(3)} \left(\frac{m_N}{T}\right)^{\frac{3}{2}} e^{-\frac{m_N}{T}} \,. \end{aligned} \tag{3.94}$$

The freeze-out temperature is given by

$$n_B \langle \sigma |v| \rangle \simeq H = 1.66\sqrt{g^*}\frac{T_f^2}{m_{\text{Pl}}} \,. \tag{3.95}$$

In order to get an order of magnitude approximation we may take $\langle \sigma |v| \rangle \sim \frac{c}{m_\pi^2} = \frac{1}{m_\pi^2}$ ($m_\pi \simeq 135\,\text{GeV}$ is the pion mass) and obtain

$$\frac{g_i}{m_\pi^2}\left(\frac{m_N T_f}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m_N}{T_f}} = H \,. \tag{3.96}$$

Rearranging this and plugging in the numbers yields

$$\left(\frac{m_N}{T_f}\right)^{-1/2} e^{\frac{m_N}{T_f}} \simeq 5 \times 10^{19} \,. \tag{3.97}$$

The solution is $T_f \simeq 20\,\mathrm{MeV}$, which in turn results in

$$\eta = \frac{n_B}{n_\gamma} = \frac{n_{\bar{B}}}{n_\gamma} \simeq 2 \times 10^{-18}\,. \tag{3.98}$$

We see that η would indeed be way to small if the nucleons would stay in thermal and chemical equilibrium until they freeze out.

In the next section we are going to discuss a simple model, in which the baryon asymmetry is generated by the out of equilibrium decay of a heavy particle species.

Chapter 4

Baryogenesis in a Toy Model

We will now discuss a simple scalar toy model which possesses all the features needed for the dynamical production of a net baryon asymmetry. This means in particular, that it satisfies all the Sakharov conditions. We will provide the set of Boltzmann equations governing the time evolution of all the different particle species contained in this model. Then we will solve them by two distinct methods. First we follow the classic simplification scheme developed in [11]. Secondly we will solve it exact in the numerical sense applying the methods presented in the preceding chapters.[1]

4.1 The Model

In order to introduce the model we postulate the following free Lagrangian $\mathscr{L}_0$ and interaction Lagrangian: $\mathscr{L}_I$.[2]

$$\mathscr{L}_0 = \frac{1}{2}\partial_\mu X \partial^\mu X - \frac{1}{2}M^2 X^2 + \partial_\mu \bar{b}\partial^\mu b\,, \tag{4.1}$$

$$\mathscr{L}_I = \frac{\lambda}{4!}b^4 + \frac{\lambda}{4!}\bar{b}^4 + \frac{\lambda}{2!2!}(\bar{b}b)^2 + \frac{g}{2!}Xbb + \frac{g^*}{2!}X\bar{b}\bar{b}\,, \tag{4.2}$$

$$\mathscr{L} = \mathscr{L}_0 - \mathscr{L}_I\,. \tag{4.3}$$

The model contains a real scalar field X (i.e $X = \bar{X}$) and a complex and massless scalar field b coupled to X by a scalar Yukawa interaction.

We restricted ourselves to a specific choice of the coupling constants. We take λ to be real but $\mathrm{Im}(g) \neq 0$. The imaginary part of g accounts for the CP asymmetry as we will see. For $\lambda > 0$ the Lagrangian (4.3) with its Yukawa coupling terms is stable.[3] Furthermore we demand λ to be of the same order as g^2 ($\mathcal{O}(\lambda) = \mathcal{O}(g^2)$). We note that λ is dimensionless and g has mass dimension 1.

Since we like to explain the baryon number generation on the basis of this model, we have to assign charges to the particles (which we will call baryon or lepton number). Since the interaction Lagrangian cannot be charge neutral whatever charges we choose, we have to define this quantum number by means of the free Lagrangian $\mathscr{L}_0$ and treat the terms in $\mathscr{L}_I$ as perturbation.

[1] Parts of this chapter can also be found in [11], [15], [14] and [58]

[2] For consistency we denote the complex conjugate of the fields by an overbar.

[3] This is not obvious, but can easily be shown, inserting $g = g_1 + ig_2$ and $b = b_1 + ib_2$.

The free part of the action $S_0 = \int d^4x\mathcal{L}_0$ is obviously invariant under $U(1)$ symmetry transformations of the b-fields. By Norther's theorem this yields a conserved current

$$j_B{}^\mu = i\left(\bar{b}\partial^\mu b - b\partial^\mu \bar{b}\right) , \tag{4.4}$$

and the conserved charge

$$B = \int d^3x j_B^0 \, . \tag{4.5}$$

We assign the charges $B = 1/2$ and $-1/2$ to b and $\bar{b}$ respectively. Since X is its own antiparticle it carries zero baryon number. With this assignment it is obvious that $\mathscr{L}_I$ violates baryon number conservation. In particular the X decays associated with the Yukawa terms in the Lagrangian will be baryon number violating.

Since scalar fields are invariant under parity transformations (P)

$$P: \ \phi \to \phi \, ,$$

parity is a symmetry of the free as well as the interaction Lagrangian.

Charge conjugation transforms a scalar field in its complex conjugate:[4]

$$C : \phi \to \bar{\phi} \, ,$$

i.e. $b \to \bar{b}, \ \bar{b} \to b$, whereas X is a real field and transforms into itself: $X \to X$.

$\mathscr{L}_0$ is left invariant by this transformation, but due to $g \neq g*$ the interaction Lagrangian is not.

Because $\mathscr{L}_I$ is invariant under P, but not under C, the combined CP symmetry is also violated.

We summarize that our model satisfies the first two Sakharov conditions (B, C and CP violation). In order to investigate the third one in the framework of kinetic theory we need to know the transition amplitudes. We are going to calculate them in the next section.

4.2 Transition Amplitudes

Figure (4.2) depicts the Feynman rules arising from (4.1) and (4.2). We are interested in b violating reactions mainly, because they affect the evolution of the baryon asymmetry directly in contrast to the b conserving ones which are important for the thermalization process only.

[4]We suppress arbitrary phase factors here, because they vanish anyway when physical quantities are calculated.

Figure 4.1: Feynman rules of the toy model

Apart from the tree level graphs it is necessary to take loop corrections into account, because the amplitudes $\mathcal{M}^2_{X,\,bb}$ and $\mathcal{M}^2_{X,\,\bar{b}\bar{b}}$ are just equal at tree level. [5]

We demand λ to be of the same order as g^2 and consider only terms up to $\mathcal{O}(g^5)$ in perturbation theory. Furthermore the propagator for the X-field must be renormalized.[6] We postpone this calculations to appendix C.

The amplitudes for X decay can be parametrised as (tree level and one loop):[7]

$$\mathcal{M}^2_{X,\,bb} = \frac{1}{2}(1+\epsilon)\,|\mathcal{M}_X|^2 \ , \tag{4.6}$$

$$\mathcal{M}^2_{X,\,\bar{b}\bar{b}} = \frac{1}{2}(1-\epsilon)\,|\mathcal{M}_X|^2 \ , \tag{4.7}$$

where (μ is an arbitrary energy scale)

$$|\mathcal{M}_X|^2 = \mathcal{M}^2_{X,\,bb} + \mathcal{M}^2_{X,\,\bar{b}\bar{b}} = 2\,|g|^2 \left(1 - \frac{|g|^2}{96 m_X^2} + \frac{\lambda}{8\pi^2\,|g|^2}(\log\frac{m_X^2}{\mu^2} - 2)\mathrm{Re}(g^2)\right) \ . \tag{4.8}$$

[5] This is similar to GUT baryogenesis and leptogenesis where the CP -violating terms also stem from the interference of the tree level and one-loop diagrams.

[6] In case of the b-particles we assume that the renormalized mass is negligible. Then the renormalized propagator equals the free one.

[7] We use $\mathcal{M}^2_{a,\,b}$ as abbreviation for $\overline{|\mathcal{M}(a\to b)|}^2$.

This means that if we consider a number of X-particles in a box decaying freely, the resulting mean net baryon number would be ϵ. This is in agreement with the definition of the CP asymetry:

$$\epsilon = \frac{\Gamma(X \to bb) - \Gamma(X \to \bar{b}\bar{b})}{\Gamma(X \to bb) + \Gamma(X \to \bar{b}\bar{b})} = \frac{\lambda}{8\pi\,|g|^2}\mathrm{Im}(g^2) = \frac{\lambda}{4\pi\,|g|^2}\mathrm{Re}(g)\mathrm{Im}(g)\,. \tag{4.9}$$

Since the amplitudes (4.6) and (4.7) are constants, the decay rates for these processes are given simply by (appendix E):

$$\Gamma(X \to bb) = \frac{\mathcal{M}^2_{X,\,bb}}{16\pi m_X}\,, \tag{4.10}$$

$$\Gamma(X \to \bar{b}\bar{b}) = \frac{\mathcal{M}^2_{X,\,\bar{b}\bar{b}}}{16\pi m_X}\,. \tag{4.11}$$

The amplitudes for the inverse decays can immediately be deduced from CPT invariance:

$$\mathcal{M}^2_{bb,\,X} = \frac{1}{2}(1-\epsilon)\,|\mathcal{M}_X|^2\,, \tag{4.12}$$

$$\mathcal{M}^2_{\bar{b}\bar{b},\,X} = \frac{1}{2}(1+\epsilon)\,|\mathcal{M}_X|^2\,. \tag{4.13}$$

Note that because of preferred decay of $X \to bb$ according to (4.6) and the suppressed inverse decay of $bb \to X$ according to (4.12), taking only decays and inverse decays into account would inevitably generate baryon number, even in thermal equilibrium.[8] In order to construct an instructive model we need to include further processes which tend to diminish baryon number in thermal equilibrium.

Up to $\mathcal{O}(g^5)$ only the vertices in figure 4.2 and the s, t and u-channel diagrams depicted in figure 4.2 contribute to the 2-2 scattering amplitude:[9]

$$\begin{aligned}
\mathcal{M}^2_{bb,\,\bar{b}\bar{b}} = &\left|\lambda + (g^*)^2\left(\frac{1}{s-m_X^2+im_X\Gamma_X} + \frac{1}{t-m_X^2} + \frac{1}{u-m_X^2}\right)\right|^2 = \\
&\lambda^2 + |g|^4\left(\frac{1}{(s-m_X^2)^2+(m_X\Gamma)^2} + \left(\frac{1}{t-m_X^2} + \frac{1}{u-m_X^2}\right)^2\right. \\
&\left. + \frac{2(s-m_X^2)}{(s-m_X^2)^2+m_X^2\Gamma_X^2}\left(\frac{1}{t-m_X^2} + \frac{1}{u-m_X^2}\right)\right) \\
&+ 2\lambda\mathrm{Re}(g^2)\left(\frac{1}{t-m_X^2} + \frac{1}{u-m_X^2} + \frac{s-m_X^2}{(s-m_X^2)^2+(m_X\Gamma)^2}\right) \\
&- 2\lambda\mathrm{Im}(g^2)\frac{m_X\Gamma_X}{(s-m_X^2)^2+(m_X\Gamma_X)^2}\,.
\end{aligned} \tag{4.14}$$

[8] This is not to be considered as a violation of Sakharov's third condition. In this case our model would just be incomplete.

[9] Here we keep the full X-propagator in the s-channel. TODO: why exactly do I have to do this?

Figure 4.2: s-,t- and u-channel.

Since $\Gamma_X = \frac{1}{16\pi m_X} |\mathcal{M}_X|^2 = \mathcal{O}(g^2)$ and $\mathcal{O}(\lambda) = \mathcal{O}(g^2)$ the last term containing the imaginary part will be of order $\mathcal{O}(g^6)$ and we drop it:

$$\begin{aligned}\mathcal{M}^2_{bb,\,\bar{b}\bar{b}} =& \lambda^2 + |g|^4 \left(\frac{1}{(s-m_X^2)^2 + (m_X\Gamma)^2} + \left(\frac{1}{t-m_X^2} + \frac{1}{u-m_X^2} \right)^2 \right. \\ &\left. + \frac{2(s-m_X^2)}{(s-m_X^2)^2 + m_X^2\Gamma_X^2} \left(\frac{1}{t-m_X^2} + \frac{1}{u-m_X^2} \right) \right) \\ &+ 2\lambda \mathrm{Re}(g^2) \left(\frac{1}{t-m_X^2} + \frac{1}{u-m_X^2} + \frac{s-m_X^2}{(s-m_X^2)^2 + (m_X\Gamma)^2} \right) .\end{aligned}$$

The amplitude for $\mathcal{M}^2_{bb,\,\bar{b}\bar{b}}$ emerges by substitutions $b \leftrightarrow \bar{b}$ and $g \leftrightarrow g^*$. Thus we have to this order of perturbation theory: [10]

$$\mathcal{M}^2_{bb,\,\bar{b}\bar{b}} = \mathcal{M}^2_{\bar{b}\bar{b},\,bb} \,. \tag{4.15}$$

For the Boltzmann equation we need the amplitudes with the real intermediate state s-channel contributions subtracted out, because these are already included in the Boltzmann equation as successive inverse decay and decay $bb \to X \to \bar{b}\bar{b}$ [14]. The t and u-channel diagrams do not receive contributions from physical intermediate states at lowest order:

$$\mathcal{M}^{\mathrm{RIS}\,2}_{\bar{b}\bar{b},\,bb} = \frac{\mathcal{M}^4_{X,\,bb}}{(s-m_X^2)^2 + (m_X\Gamma_X)^2} \,, \tag{4.16}$$

$$\mathcal{M}^{\mathrm{RIS}\,2}_{bb,\,\bar{b}\bar{b}} = \frac{\mathcal{M}^4_{X,\,\bar{b}\bar{b}}}{(s-m_X^2)^2 + (m_X\Gamma_X)^2} \,. \tag{4.17}$$

In the narrow width approximation

$$\lim_{h\to 0} \frac{h}{x^2+h^2} = \pi\delta(x) \,,$$

this can be written as:

[10] If we would consider only the two amplitudes $\mathcal{M}^2_{bb,\,\bar{b}\bar{b}}$ and $\mathcal{M}^2_{bb,\,X}$ and the corresponding results for $\bar{b}\bar{b}$ this would clearly violate (2.10), obtained from CPT -invariance and unitarity.

$$\mathcal{M}^{\text{RIS}\,2}_{\bar{b}\bar{b},\,bb} = \mathcal{M}^4_{X,\,bb}\frac{\pi}{m_X\Gamma_X}\delta(s-m_X^2)\,, \tag{4.18}$$

$$\mathcal{M}^{\text{RIS}\,2}_{bb,\,\bar{b}\bar{b}} = \mathcal{M}^4_{X,\,\bar{b}\bar{b}}\frac{\pi}{m_X\Gamma_X}\delta(s-m_X^2)\,. \tag{4.19}$$

The amplitudes for 2-2 scattering are then, without the RIS contribution:

$$\mathcal{M}'^2_{bb,\,\bar{b}\bar{b}} = \mathcal{M}^2_{bb,\,\bar{b}\bar{b}} - \mathcal{M}^{\text{RIS}\,2}_{bb,\,\bar{b}\bar{b}}\,, \tag{4.20}$$

$$\mathcal{M}'^2_{\bar{b}\bar{b},\,bb} = \mathcal{M}^2_{\bar{b}\bar{b},\,bb} - \mathcal{M}^{\text{RIS}\,2}_{\bar{b}\bar{b},\,bb}\,. \tag{4.21}$$

We note that these contributions do violate CP whereas the full amplitudes for $2-2$ scattering do not (4.15).

4.3 Boltzmann Equations

Now, that we have calculated all the necessary transition amplitudes, we can easily set up the system of Boltzmann equations which describes the time evolution of the X, b and $\bar{b}$ density distributions, according to the guidelines elaborated in chapter 3.

For the X-particle evolution (X decays and inverse decays) we get:

$$\begin{aligned} C_k^X = \frac{1}{2E_k^X}\int (2\pi)^4\delta^{(4)}(k-q-r)[&(1+f_k^X)f_q^b f_r^b\mathcal{M}^2_{bb,\,X} - f_k^X(1+f_q^b)(1+f_r^b)\mathcal{M}^2_{X,\,bb} \\ &+(1+f_k^X)f_q^{\bar{b}}f_r^{\bar{b}}\mathcal{M}^2_{\bar{b}\bar{b},\,X} - f_k^X(1+f_q^{\bar{b}})(1+f_r^{\bar{b}})\mathcal{M}^2_{X,\,\bar{b}\bar{b}}]\,d\Pi_q^b\,d\Pi_r^b\,. \end{aligned} \tag{4.22}$$

The b-particle evolution (X decays and inverse decays and 2-2 scattering) is described by

$$\begin{aligned} C_k^b = &\frac{1}{E_k^b}\int (2\pi)^4\delta^{(4)}(k+q-r)[(1+f_k^b)(1+f_q^b)f_r^X\mathcal{M}^2_{X,\,bb} - f_k^b f_q^b(1+f_r^X)\mathcal{M}^2_{bb,\,X}]\,d\Pi_q^b\,d\Pi_r^X \\ &+\frac{1}{2E_k^b}\int (2\pi)^4\delta^{(4)}(k+p-q-r)[(1+f_k^b)(1+f_p^b)f_q^{\bar{b}}f_r^{\bar{b}}\mathcal{M}'^2_{\bar{b}\bar{b},\,bb} - f_k^b f_p^b(1+f_q^{\bar{b}})(1+f_r^{\bar{b}})\mathcal{M}'^2_{bb,\,\bar{b}\bar{b}}]\times \\ &\times\ d\Pi_p^b\,d\Pi_q^b\,d\Pi_r^b\,. \end{aligned} \tag{4.23}$$

The Boltzmann equation for the $\bar{b}$-particle evolution can be obtained from the above one, interchanging b and $\bar{b}$ (X decays and inverse decays and 2-2 scattering):

$$\begin{aligned} C_k^{\bar{b}} = &\frac{1}{E_k^{\bar{b}}}\int (2\pi)^4\delta^{(4)}(k+q-r)[(1+f_k^{\bar{b}})(1+f_q^{\bar{b}})f_r^X\mathcal{M}^2_{X,\,\bar{b}\bar{b}} - f_k^{\bar{b}} f_q^{\bar{b}}(1+f_r^X)\mathcal{M}^2_{\bar{b}\bar{b},\,X}]\,d\Pi_q^{\bar{b}}\,d\Pi_r^X \\ &+\frac{1}{2E_k^{\bar{b}}}\int (2\pi)^4\delta^{(4)}(k+p-q-r)[(1+f_k^{\bar{b}})(1+f_p^{\bar{b}})f_q^b f_r^b\mathcal{M}'^2_{bb,\,\bar{b}\bar{b}} - f_k^{\bar{b}} f_p^{\bar{b}}(1+f_q^b)(1+f_r^b)\mathcal{M}'^2_{\bar{b}\bar{b},\,bb}]\times \\ &\times\ d\Pi_p^{\bar{b}}\,d\Pi_q^{\bar{b}}\,d\Pi_r^{\bar{b}}\,. \end{aligned} \tag{4.24}$$

Note, that we added an extra factor 2 in front of the integrals for decays and inverse decays in (4.23) and (4.24) relative to (4.22). This factor accounts for the particle-number violating character of these integrals. A single collision in (4.22) leads to an increase (or decrease) of the total X-particle number by 1, whereas a single collision in (4.23) increases or decreases the total b number density by 2.[11] And we exploited the fact that $d\Pi^{\bar{b}} = d\Pi^{b}$.

4.4 Simplified Boltzmann Equations

Solving the Boltzmann equations [4.22],[4.23] and [4.24] directly is numerically expensive (but possible when making the appropriate analytic simplifications). Usually some standard approximations are made which turn the above equations into a system of ordinary, linear differential equations.

- Our first approximation is to assume that the density distributions are very close to thermal equilibrium. This is usually a good one, if the reaction rate is greater than the expansion rate $\Gamma \gg H$. In the early universe the particles will undergo other thermalizing reactions such as $\gamma + b \rightarrow \gamma + b$ and so on (typically at higher rates) which maintain thermal equilibrium. These processes are of course not included in our toy model.

- Furthermore we assume that the chemical potentials are small (i.e. that the system is close to chemical equilibrium). The chemical potentials of b and $\bar{b}$ are equal but opposite in sign. This follows from the existence of reactions like $b+\bar{b} \longleftrightarrow \gamma+\gamma$ which gives in equilibrium $\mu_b + \mu_{\bar{b}} = 2\mu_\gamma = 0$.

- The third approximation will be to use Maxwell-Boltzmann distribution functions instead of Bose-Einstein or Fermi-Dirac distributions respectively. This also means that we will completely neglect the quantum mechanical blocking and stimulated emission terms introduced in chapter 3. Thus the particles are treated fully as classical particles. [12]The Maxwell-Boltzmann distributions can then be expanded in $\frac{\mu}{T}$:

$$f(p) = e^{-(E-\mu)/T} \simeq e^{-\frac{E}{T}}(1 + \frac{\mu}{T}) = f_k{}^{EQ}(1 + \frac{\mu}{T})\,. \tag{4.25}$$

- Finally we will assume that the entropy is conserved (i.e. $sR^3 = const$). This common cosmological approximation is believed to be satisfied for the bulk of the thermal history of the universe. However it is in general violated when non-equilibrium processes (such as baryogenesis and leptogenesis itself) occur. Such processes produce additional entropy.[13]

Unfortunately because of the complexity of the Boltzmann equation it is hard to estimate the error introduced by these approximations. And it will remain unclear until we solve the equations in a completely different way in order to compare the results of both approaches.

[11]This seems to be neglected elsewhere.

[12]In literature this approximation is sometimes justified by the argument that the particle densities at high temperatures are very similar for bosons and fermions respectively. This is indeed true for particle number densities (cf. table 3.3). However the distribution functions can have a quite different shape. (e.g. by looking at equation (3.68) it becomes clear that the distribution function for fermions can never exceed 1 whereas the that for bosons can get very large for $\mu = 0$ and small momenta and even infinite for $m = 0$).

[13]We will not consider these effects in our ab initio numerical solution within this work. However they can be incorporated in principle. One then has to solve coupled system of Boltzmann and Einstein equations.

Let us perform the approximations step by step. First of all we integrate the equations over the remaining phase space element

$$\frac{dk}{(2\pi)^3 E_k}\,.$$

The Liouville operator will then be given by (3.81).

The collision integral for decays and inverse decays will look like (only the first part of (4.22), neglecting the quantum terms)

$$\int (2\pi)^4 \delta^{(4)}(k-q-r)[f_q^b f_r^b \mathcal{M}^2_{bb,\,X} - f_k^X \mathcal{M}^2_{X,\,bb}]\, d\Pi_k^X\, d\Pi_q^b\, d\Pi_r^b \quad . \tag{4.26}$$

Performing the above approximations for the distribution functions gives

$$\int (2\pi)^4 \delta^{(4)}(k-q-r)[f_k^{X\,EQ}(1+\frac{2\mu_b}{T})\mathcal{M}^2_{bb,\,X} - f_k^X \mathcal{M}^2_{X,\,bb}]\, d\Pi_k^X\, d\Pi_q^b\, d\Pi_r^b\,. \tag{4.27}$$

Where we exploited the fact that due to energy conservation

$$f_k^{X\,EQ} = f_q^{b\,EQ} f_r^{b\,EQ} e^{+2\mu/T} \simeq f_q^{b\,EQ} f_r^{b\,EQ}(1+2\mu_b/T)\,.$$

The second part of (4.22) follows, substituting $b \to \bar{b}$ (this implies $\mu_b \to \mu_{\bar{b}} = -\mu_b$). Combining the two parts and plugging in the Amplitudes (4.6), (4.12) yields

$$\dot{X} \simeq \int (2\pi)^4 \delta^{(4)}(k-q-r)[f_k^{X\,EQ}(1-\epsilon\frac{2\mu_b}{T}) - f_k^X]\,|\mathcal{M}_X|^2\, d\Pi_k^X\, d\Pi_q^b\, d\Pi_r^b\,. \tag{4.28}$$

Since μ_b and ϵ are small we can neglect the $\epsilon\frac{2\mu_b}{T}$ term.

The phase space integral over $f_X(k)\,|\mathcal{M}_X|^2$ gives the thermally averaged X-decay width times the number density of X, so that we can write it

$$\frac{1}{2}\langle\Gamma_X\rangle\left[n_X^{EQ} - n_X\right]\,. \tag{4.29}$$

We will determine the temperature dependence of $\langle\Gamma_X\rangle$ later.

The integrated collision integral (4.23) can be simplified in the same way to

$$\begin{aligned} &2\int (2\pi)^4\delta^{(4)}(k+q-r)\left[f_r^X\frac{1}{2}(1+\epsilon)\,|\mathcal{M}_X|^2 - f_r^{X\,EQ}(1-\frac{2\mu}{T})\frac{1}{2}(1-\epsilon)\,|\mathcal{M}_X|^2\right] d\Pi_q^b\, d\Pi_r^b\, d\Pi_r^X \\ &+\int (2\pi)^4\delta^{(4)}(k+p-q-r) f_k^{b\,EQ} f_p^{b\,EQ}\left[(1-\frac{2\mu}{T})\mathcal{M}'^2_{\bar{b}\bar{b},\,bb} - (1+\frac{2\mu}{T})\mathcal{M}'^2_{bb,\,\bar{b}\bar{b}}\right] d\Pi_k^b\, d\Pi_p^b\, d\Pi_q^b\, d\Pi_r^b\,. \end{aligned} \tag{4.30}$$

The corresponding integral (4.24) can again be obtained from this result by simple substitutions. In order to get rid of the full amplitudes $\mathcal{M}^2_{bb,\,\bar{b}\bar{b}}$ and $\mathcal{M}^2_{\bar{b}\bar{b},\,bb}$, we can subtract these equations and get a single one governing the evolution of the net baryon number $n_B = \frac{1}{2}(n_b - n_{\bar{b}})$

Performing the integrals over the real intermediate state delta function at the same time (see appendix C), and using

$$\mathcal{M}'^2_{\bar{b}\bar{b},\,bb} - \mathcal{M}'^2_{bb,\,\bar{b}\bar{b}} = -16\pi^2\epsilon\,|\mathcal{M}_X|^2\,\delta(s-m_X^2)\,,$$

we find (neglecting higher order terms):

$$\frac{1}{2}\langle\Gamma_X\rangle\left[n_X^{EQ}-n_X\right]\epsilon-\frac{12\zeta(3)}{\pi^2}n_B\left(\langle\Gamma_X\rangle\frac{n_X^{EQ}}{n_\gamma}+n_\gamma\left\langle\sigma'\left|v\right|\right\rangle\right), \tag{4.31}$$

where we defined the averaged cross section times velocity (with real intermediate state subtracted out) analogous to (3.90) by

$$\begin{aligned}\left\langle\sigma'\left|v\right|\right\rangle =&\frac{1}{n_\gamma^2}\int(2\pi)^4\delta^{(4)}(k+p-q-r)e^{-(E_k+E_p)/T}\\ &\times\left[2\mathcal{M}^2_{bb,\overline{b}\overline{b}}-\frac{\pi}{2m_X\Gamma_X}\left|\mathcal{M}\right|^4\delta(s-m_X^2)\right]d\Pi_k\,d\Pi_p\,d\Pi_q\,d\Pi_r\,.\end{aligned} \tag{4.32}$$

This expression has again a temperature dependence to be determined later.

Now, writing all equations in terms of the variables $X=\frac{n_X}{s}$ and $B=\frac{n_B}{s}$ introduced in section 3.9, we get the advertised system of ordinary differential equations describing the evolution of the X-number density and the baryon asymmetry:

$$\begin{aligned}\tilde{X}' &= -xK\gamma_X(\tilde{X}-\tilde{X}_{EQ})\,,\\ \tilde{B}' &= xK\gamma_X(\tilde{X}-\tilde{X}_{EQ})-xK\gamma_B\tilde{B}\,,\end{aligned} \tag{4.33}$$

where we made the definitions

$$\tilde{X} = g_*X\,, \tag{4.34}$$

$$\tilde{B} = g_*\epsilon^{-1}B\,. \tag{4.35}$$

Note the similarity to the Boltzmann equations encountered in section 2.5 in the framework of thermal leptogenesis.

γ_X and γ_B are dimensionless (These quantities are called reaction densities.) and for $T\sim M$ they equal the rates of B violating reactions (i.e. decays and $2-2$ scattering processes):

$$\gamma_X = \frac{\langle\Gamma_X\rangle}{\langle\Gamma_X\rangle_{x=1}}=\frac{K_2(1)}{K_1(1)}\frac{K_1(x)}{K_2(x)}\,, \tag{4.36}$$

$$\gamma_B = \frac{24\zeta(3)}{\pi^2}\left(1.80\gamma_X\tilde{X}_{EQ}+n_\gamma\frac{\langle\sigma'\left|v\right|\rangle}{\langle\Gamma_X\rangle_{x=1}}\right), \tag{4.37}$$

with efficiency parameter K:

$$K=\frac{\langle\Gamma_X\rangle_{x=1}}{H(m_X)}\,.$$

For the temperature dependence of $\langle\sigma'\left|v\right|\rangle$, $\langle\Gamma_X\rangle$ and $\tilde{X}_{EQ}$ one finds (see appendix D):

$$\left\langle\sigma'\left|v\right|\right\rangle\simeq\frac{\lambda^2}{T^2}\frac{4Cm_X^2+T^2}{T^2(m_X^2+\frac{T^2}{C})}\,,$$

$$\langle\Gamma_X\rangle=\frac{K_1(\frac{m_X}{T})}{K_2(\frac{m_X}{T})}\Gamma_X\,,$$

with

$$C = \frac{1}{512\pi\zeta(3)^2}$$

and

$$\tilde{X}_{EQ}(T) = \frac{1}{4\,1.80\zeta(3)}\frac{m_X^2}{T^2}K_2(\frac{m_X}{T})\,.$$

Equations (4.33) can be motivated as follows. The departure from equilibrium with respect to the X number density $\tilde{X} - \tilde{X}_{EQ}$ drives the generation of the baryon asymmetry. γ_B, which describes the inverse decays and $2-2$ scattering governs the decrease of baryon number. $K = \langle\Gamma_X\rangle_{x=1}/H(m_X)$ i.e. the reaction rate divided by the Hubble rate can be interpreted as a measure for the effectiveness of the decays.

These equations can in general not be solved analytically. Therefore we do some numerical calculations in the next section.

4.5 Numerical Results

For the calculations we choose $m_X = 10^{10}$ GeV (the mass scale of the lightest heavy Majorana neutrino) and $\lambda = \frac{g^2}{m_X^2}$.

Solution of the Simplified Equations

We solve the system of the simplified equations (4.33) first. Figures 4.3, 4.4 and 4.5 show the results for several values of the efficiency parameter K. All graphs are plotted over the dimensionless parameter $x = m_X/T$.

Figure 4.3: Evolution of the relative X abundance.

Figure 4.4: The baryon asymmetry $\tilde{B} = g_* \epsilon^{-1} B$, generated by the toy model for various K. For decreasing K the final asymmetry approaches a maximum value which belongs to the completely free decay of the X particles without any back-reactions. For large values of K the baryon number generation is highly suppressed, because the system cannot deviate sufficiently far from equilibrium.

Figure 4.5: Deviation from equilibrium of the relative abundances $\tilde{X} - \tilde{X}_{EQ}$. For smaller values of K the deviation from thermal equilibrium increases. Since $\tilde{X} - \tilde{X}_{EQ}$ acts as a source term driving the baryon number generation $\tilde{B}$ also increases with decreasing K.

Full Numerical Solution

In this section the results obtained for the full numerical solution of the unaltered Boltzmann equation are presented.[14]

The coupled set of Boltzmann equations (4.22), (4.23) and (4.24) for the toy model has been solved using the analytical reduction formalism outlined in section 3.6. For this purpose the D integrals have been calculated in appendix E. The reduced equations are still integro-differential equations with two or one integrals in case of $2-2$ scattering and decays respectively. These equations are discretized using the discrete velocity model described in section 3.7.

A fourth-order Runge-Kutta Integrator with adaptive step size control was used in order to solve the discrete system of differential equations.

There are various subtleties with respect to the discretization grid.[15] For the calculations presented in this thesis a non-equidistant grid has been chosen in order to account for the different behavior of the equilibrium distributions for small or large momenta (i.e. large variations for small momenta and the long smooth Boltzmann tail for large momenta).

Furthermore there is the fundamental problem that the momentum grid is effectively shrinking due to the expansion (i.e. the momenta are red shifted). This implies that the b and $\bar{b}$ particles produced in the decay of the X will eventually fall of the upper bound of the grid, when the upper bound becomes equal to half of the X mass. This means that the upper bound has to be chosen high enough, so that during the relevant time interval for which the equations are to be solved this problem cannot occur. This means however that the density of the grid points is lessened when the number of points, i.e. the available computing power stays constant. The non-equidistant grid fits these needs by far better than an equidistant one. The infinity of the Bose-Einstein equilibrium distribution function at $k=0$ for zero particle masses also requires a lower threshold greater than zero.

Good values for the upper bound and lower bounds can be found by demanding the numerical value for the number density integrated on the respective grid to be close to the exact value.

In principle it is necessary to show, that successive refinement of the grid (this implies decreasing the grid intervals as well as increasing the upper bound). The quality of the calculated momentum and time distribution can roughly be estimated, testing the conservation laws encountered in the model. In our case this means that the relation $2n_X + n_b + n_{\bar{b}} = const$ must hold.

For a first approximation it is sufficient to consider only decays and inverse decays as well as the real intermediate state contribution to $2-2$ scattering.[16] These are the only CP violating processes, because as we have seen that $\mathcal{M}_{bb,\,\bar{b}\bar{b}} = \mathcal{M}_{\bar{b}\bar{b},\,bb}$. in this case the only free parameters are $|\mathcal{M}_X|^2$ and ϵ. The later does not enter the simplified Boltzmann equations, because B depends only linear on ϵ. With the relation

$$K = \frac{m_{\text{Pl}}\,|\mathcal{M}_X|^2}{4.5\,16\pi\sqrt{g_*}m_X^3}\,, \tag{4.38}$$

one can directly compare the results of both approaches.

[14] Note, that the numerical calculation with respect to the unaltered Boltzmann equations turned out to be problematic. The algorithm has not always been stable. There has not been enough time for extensive testing, so that the results presented in this must be considered preliminary.

[15] Due to the homogeneity and isotropy of the equations we are of course referring to an one-dimensional grid here.

[16] The real intermediate state $2-2$ scattering allays needs to be taken into account in order to preserve the combined CPT -unitarity condition in (2.10).

Considering only decays and inverse decays would always generate a baryon asymmetry in thermal equilibrium.

We used the parameter set

$$m_X = 10^{10}\,\text{GeV},\ \epsilon = 1 \times 10^{-3} \text{ and } K = 5.22 \times 10^{-4}\,. \tag{4.39}$$

Note the small value of K compared to the values in figures 4.3, 4.4 and 4.5. These special values were enforced partly by the necessity of numerical stability of the software routines. The one to one comparison between the full and the simplified equations for these parameters is shown in figure 4.6. We learn that the errors induced by the simplification of the Boltzmann equations are small in the regime $K \ll 1$. This is not surprising because the full Boltzmann equations as well as the simplified ones are then governed by the expansion rate, rather than by the interactions of the particles. In order to clarify how important the corrections to the linearized Boltzmann equations are it will be necessary also to examine the regime of strong coupling $K \gg 1$.

Figure 4.6: The solutions of the simplified equations compared to the full numeric approach for $K = 5.22 \times 10^{-4}$. Note that this value corresponds to a far from equilibrium situation with maximum baryon production (cf. figure 4.4). One can expect a larger difference for bigger values of K, because the interaction will than become more important. The upper case letters belong to the simplified equations.

Figure 4.7: Total particle number ($2n_X + n_b + n_{\bar{b}}$) and total entropy normalized by their initial value. We see that there is significant entropy production. This observation contradicts the assumption of conserved entropy made in the derivation of the simplified Boltzmann equations.

Chapter 5

Conclusions and Outlook

We have seen that, in the framework of the seesaw mechanism, leptogenesis can in a natural way explain the observed baryon asymmetry through the decay of heavy, speculative Majorana neutrino states. The importance of these theories increases with decreasing likeliness of baryogenesis theories such as electroweak baryogenesis.

If leptogenesis is indeed the cause for the baryon asymmetry in our universe, this would have the delightful aspect, that the lightest and most unimpressive known particles, the neutrinos would in the end be responsible for our own existence.

Although, in view of the numerous different baryogenesis and leptogenesis scenarios the uncertainties in the underlying physics will dominate the error in the state-of-the-art calculations, it is desirable to achieve a reliable numerical simulation of the Boltzmann network for a given baryogenesis or leptogenesis theory.

We have seen that the entropy production, caused by the CP violating out of equilibrium decay of the heavy particle species causing baryon or lepton number generation, might produce entropy to a great extent. The assumption of conserved total entropy might therefore be inappropriate, and the relation between Hubble rate and temperature might be altered. This well known phenomenon of entropy production already led to corrections to the expected neutrino spectrum. Which effect these and other corrections such as the employment of the correct quantum statistics (Fermi or Bose-Einstein statistics) have must be examined on the basis of a particular and realistic baryogenesis model.

Because of the observed similarity of the Boltzmann equations for the toy model on the one hand and thermal leptogenesis on the other, the latter would be suited well for such investigations.

Furthermore, the numerical algorithms developed in line with this work can in principle be adjusted to all kinds of non-equilibrium phenomena in the expanding universe, limited mainly by computer power and the theoretical uncertainties in the underlying physics. It has been shown that the numerical solution of the full Boltzmann equations for such leptogenesis models is feasible. The incorporation of entropy generating effects (i.e. the solution of the coupled Boltzmann-Einstein system) implies only minor changes in the software routines, so that this issue can be investigated easily.

However the most urgent issue associated with the current treatment of leptogenesis with respect to the out of equilibrium decay, is still whether the kinetic approach itself is an appropriate description of the early universe plasma and the respective non-equilibrium phenomena. We outlined the problems encountered in connection with this approach. The Boltzmann equation has been de-

veloped for systems of classical dilute gases. Many effects expected to be important in a dense quantum plasma are not incorporated within this description. Sporadic attempts to supplement the equations accordingly will in general be arbitrary, if not motivated by superior techniques.

In order to clarify this question a full quantum field theoretic treatment needs to be carried out. In order to reject or to ratify the validity of the Boltzmann equations it will be necessary to compare the numerical solutions of the unaltered Boltzmann equations to the results of a full quantum field theoretic calculation.

The toy model described in the second part of this thesis might proof useful in this context. According to [59] the assembly and numerical solution of the Kadanoff-Baym equations for this toy model is feasible. Thus a direct comparison of these two different approaches is within reach.

Appendix A

Robertson Walker Metric

In polar coordinates the line element of Robertson-Walker metric is given by

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\right) . \tag{A.1}$$

The only non-vanishing Christoffel symbols are the following:

$$\begin{aligned}
\Gamma^0{}_{11} &= \frac{R\dot{R}}{1-kr^2} \\
\Gamma^0{}_{22} &= r^2 R\dot{R} \\
\Gamma^0{}_{33} &= r^2\sin^2\theta R\dot{R} \\
\Gamma^1{}_{11} &= \frac{kr}{1-kr^2} \\
\Gamma^1{}_{22} &= -r(1-kr^2) \\
\Gamma^0{}_{33} &= -r(1-kr^2)\sin^2\theta \\
\Gamma^2{}_{12} &= \Gamma^3{}_{13} = \frac{1}{r} \\
\Gamma^2{}_{33} &= \sin\theta\cos\theta \\
\Gamma^3{}_{23} &= \cot\theta \\
\Gamma^1{}_{01} &= \Gamma^2{}_{02} = \Gamma^3{}_{03} = \frac{\dot{R}}{R}
\end{aligned} \tag{A.2}$$

In the spatially flat case ($k = 0$) and when we write the line elements in terms of cartesian coordinates:

$$ds^2 = dt^2 - R^2(t)\left(dx_1{}^2 + x_2{}^2 + x_3{}^2\right) . \tag{A.3}$$

we get for the remaining Christoffel symbols

$$\begin{aligned}
\Gamma^0{}_{ij} &= R\dot{R}\delta_{ij} , \\
\Gamma^i{}_{0j} &= \frac{\dot{R}}{R}\delta^i{}_j , \\
\Gamma^i{}_{jk} &= 0 .
\end{aligned} \tag{A.4}$$

Appendix B

Modified Bessel Functions

The modified Bessel functions of the second kind are defined as

$$K_\nu(z) = \frac{\sqrt{\pi}\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_1^\infty e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}\,dt\,, \tag{B.1}$$

where Γ is the Gamma function

$$\Gamma(n+\frac{1}{2}) \underset{n=0,1,2,\ldots}{=} \frac{(2n)!\sqrt{\pi}}{n!2^{2n}}\,. \tag{B.2}$$

Thus the first and second Bessel function are

$$K_1(z) = z\int_1^\infty e^{-zt}\sqrt{t^2-1}\,dt = \int_z^\infty e^{-t}\sqrt{t^2-z^2}\,dt \tag{B.3}$$

and

$$K_2(z) = \frac{z^2}{3}\int_1^\infty e^{-zt}(t^2-1)^{\frac{3}{2}}\,dt = \frac{1}{z^2}\int_z^\infty te^{-t}\sqrt{t^2-z^2}\,dt\,. \tag{B.4}$$

The second integral can be found via integration by parts from the first one in either case. This relations can be used to express the equilibrium number density in case of Maxwell Boltzmann statistics by

$$n_{MB} = \frac{1}{2\pi^2}\int e^{-\frac{\sqrt{p^2+m^2}}{T}}\,dp = \frac{T^3}{2\pi^2}\int_{\frac{m}{T}}^\infty te^{-t}\sqrt{t^2-\left(\frac{m}{T}\right)^2} = \frac{T^3}{2\pi^2}\left(\frac{m}{T}\right)^2 K_2(\frac{m}{T})\,. \tag{B.5}$$

Appendix C

Matrix Elements

Unitarity condition

Unitarity of the S-matrix requires

$$\sum_j |\mathcal{M}(i \to j)|^2 = \sum_j |\mathcal{M}(j \to i)|^2 \ . \tag{C.1}$$

This can be seen as follows. From the unitarity of the S-matrix $S^\dagger S = SS^\dagger = 1$ and the definition of the T-matrix $S = 1 + iT$ we get $TT^\dagger = -i(T - T^\dagger) = T^\dagger T$ and thus $\langle i| TT^\dagger |i\rangle = \langle i| T^\dagger T |i\rangle$.[1] We insert $\sum_j |j\rangle \langle j| = 1$ and get

$$\sum_j \langle i| T |j\rangle \langle j| T^\dagger |i\rangle = \sum_j \langle i| T^\dagger |j\rangle \langle j| T |i\rangle \ ,$$

or using $\langle a| U^\dagger |b\rangle = \langle b| U |a\rangle^*$

$$\sum_j |\langle i| T |j\rangle|^2 = \sum_j |\langle j| T |i\rangle|^2 \ .$$

Recalling the definition of the invariant matrix element $\mathcal{M}$

$$\langle i| T |j\rangle = (2\pi)^4 \delta^{(4)}(p_i - p_j) \mathcal{M}(i \ \to j) \, ,$$

we can immediately deduce equation (2.8).

Renormalization of the toy model amplitudes

Figure C.1 shows one loop contributions to the self energy $M(p^2)$ for the X and b fields. As pointed out in chapter 4 we renormalize only the X-field propagator and assume that the b propagator is equal to the free one (due to the much smaller mass of the b particles).

This is the sum of all one-particle-irreducible insertions (1PI)

$$M(p^2) = \frac{|g|^2}{8\pi^2} \left(\ln \frac{|p^2|}{\mu^2} - 2 - i\pi\theta(p^2) \right) - C_1 p^2 + C_2 m_X^2 \, , \tag{C.2}$$

[1] We use the abbreviation $|i\rangle = |\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle$ as an abbreviation for a complete state of incoming or outgoing particles.

where μ is an arbitrary mass scale. The full two point function is then given by

$$\frac{i}{p^2 - m_X^2 - M^2(p^2)} \,. \tag{C.3}$$

The physical X mass is defined by

$$\mathrm{Re}(i(G^X)^{-1}(\tilde{M}^2)) = 0 \,, \tag{C.4}$$

which gives us

$$\tilde{M}^2 = m_X^2 Z_m \,, \tag{C.5}$$

with

$$Z_m = 1 + C_1 + C_2 + \frac{|g|^2}{8\pi^2 m_X^2}\left(\ln\frac{m_X^2}{\mu^2} - 2\right) \,. \tag{C.6}$$

And the decay width Γ_X is

$$\mathrm{Im}(i(G^X)^{-1}(\tilde{M}^2)) = -\tilde{M}\Gamma \,. \tag{C.7}$$

This gives

$$\Gamma_X = \frac{|g|^2}{8\pi m_X} \,. \tag{C.8}$$

The residue is defined by

$$\frac{d}{dp^2} i(G^X(p^2))^{-1}\big|_{p^2=m_X^2} = 1 - \frac{|g|^2}{8\pi^2 m_X^2} + C_1 = Z^{-1} \,. \tag{C.9}$$

For $Z = Z_m = 1$ we have to choose

$$\begin{aligned} C_1 &= \frac{|g|^2}{8\pi^2 m_X^2} \,, \\ C_2 &= \frac{|g|^2}{8\pi^2 m_X^2}\big(\ln\frac{m_X^2}{\mu^2} - 3\big) \,. \end{aligned} \tag{C.10}$$

Furthermore, we need to compute one loop corrections for the vertices. The corresponding diagrams for Xbb can be found in figure C. The corrections to $X\bar{b}\bar{b}$ can be obtained by substituting $g \leftrightarrow g^*$.

Figure C.1: X and b self energy

Figure C.2: One loop contributions to the vertices Xbb in $\overline{\text{MS}}$-prescription calculated by means of the Feynman rules in chapter 4. The vertices for $X\bar{b}\bar{b}$ are analogous (substitute $g \leftrightarrow g^*$).

The amplitude $\mathcal{M}^2_{X,\,bb}$ is found by summing up the vertex in figure 4.2 and all the loop corrections, to be (up to $\mathcal{O}(g^5)$)

$$\mathcal{M}^2_{X,\,bb} = |g|^2 - \frac{|g|^4}{96 m_X^2} + \frac{\lambda}{8\pi^2}\left((\ln(\frac{m_X^2}{\mu^2}) - 2)(\text{Re}(g^2) + |g|^2) + \pi \text{Im}(g^2)\right) . \qquad \text{(C.11)}$$

Of course the amplitude $\mathcal{M}^2_{X,\,\bar{b}\bar{b}}$ can again be found interchanging g and $g*$ ($\mathrm{Im}(g^2) = -\mathrm{Im}(g^{*2})$):

$$\mathcal{M}^2_{X,\,\bar{b}\bar{b}} = |g|^2 - \frac{|g|^4}{96 m_X^2} + \frac{\lambda}{8\pi^2}\left((\ln(\frac{m_X^2}{\mu^2}) - 2)(\mathrm{Re}(g^2) + |g|^2) - \pi \mathrm{Im}(g^2)\right) . \tag{C.12}$$

We define

$$|\mathcal{M}_X|^2 = \mathcal{M}^2_{X,\,bb} + \mathcal{M}^2_{X,\,\bar{b}\bar{b}} = 2\,|g|^2\left(1 - \frac{|g|^2}{96 m_X^2} + \frac{\lambda}{8\pi^2\,|g|^2}(\ln\frac{m_X^2}{\mu^2} - 2)\mathrm{Re}(g^2)\right) \tag{C.13}$$

and

$$\epsilon = \frac{\lambda}{8\pi\,|g|^2}\mathrm{Im}(g^2)\,. \tag{C.14}$$

Then the amplitudes for X decay can be parametrised as

$$\mathcal{M}^2_{X,\,bb} = \frac{1}{2}(1+\epsilon)\,|\mathcal{M}_X|^2\,,\;\; \mathcal{M}^2_{X,\,\bar{b}\bar{b}} = \frac{1}{2}(1-\epsilon)\,|\mathcal{M}_X|^2\;. \tag{C.15}$$

This is the form we used in chapter 4. Up to $\mathcal{O}(g^5)$ loop corrections do not contribute to the amplitudes $\mathcal{M}^2_{bb,\,\bar{b}\bar{b}}$ and $\mathcal{M}^2_{\bar{b}\bar{b},\,bb}$.

Appendix D

Temperature Dependence of the Scattering Parameters

In section 4.4 we encounter the quantities $\tilde{X}_{EQ}$ and $\langle \sigma \, |v| \rangle$, of which we need to know the temperature dependence in order to solve the simplified Boltzmann equations.

Let us first solve the following integral encountered in the derivation of the simplified Boltzmann equations:

$$I = \int d\Pi_k \int d\Pi_p e^{\frac{-E_k}{T}} e^{\frac{-E_p}{T}} \delta(s - m_X^2) \,, \tag{D.1}$$

where $s = (k+p)^2$ is a Mandelstamm variable. Performing the angle integrals leaves

$$\begin{aligned} I =& \frac{\pi^2}{(2\pi)^6} \int dE_k \int dE_p e^{\frac{-E_k}{T}} e^{\frac{-E_p}{T}} \theta(4E_k E_p - m_X^2) \\ =& \frac{\pi^2}{(2\pi)^6} T m_X K_1 \left(\frac{m_X}{T}\right) = \frac{\langle \Gamma_X \rangle}{\Gamma_X} \frac{2\pi^4 n_X^{EQ}}{(2\pi)^6 m_X} \,. \end{aligned} \tag{D.2}$$

Now let us turn to the temperature dependence of $\tilde{X}_{EQ}$ and $\langle \sigma \, |v| \rangle$. With the results of appendix C we find

$$\tilde{X}_{EQ}(T) = \frac{1}{4\,1.80\zeta(3)} \frac{M^2}{T^2} K_2(\frac{M}{T}) \,.$$

In order to evaluate

$$\begin{aligned} \left\langle \sigma' \, |v| \right\rangle =& \frac{1}{n_\gamma^2} \int (2\pi)^4 \delta^{(4)}(k+p-q-r) e^{-(E_k+E_p)/T} \\ & \times \left[2\mathcal{M}_{bb,\,\overline{b}\overline{b}}^2 - \frac{\pi}{2m_X \Gamma_X} \left|\mathcal{M}\right|^4 \delta(s - m_X^2) \right] d\Pi_k \, d\Pi_p \, d\Pi_q \, d\Pi_r \,, \end{aligned} \tag{D.3}$$

with

$$\begin{aligned}\mathcal{M}^2_{bb,\overline{b}\overline{b}} =&\lambda^2 + |g|^4 \left(\frac{1}{(s-m_X^2)^2 + (m_X\Gamma)^2} + \left(\frac{1}{t-m_X^2} + \frac{1}{u-m_X^2} \right)^2 \right.\\ &\left. + \frac{2(s-m_X^2)}{(s-m_X^2)^2 + m_X^2\Gamma_X^2} \left(\frac{1}{t-m_X^2} + \frac{1}{u-m_X^2} \right) \right)\\ &+ 2\lambda \mathrm{Re}(g^2) \left(\frac{1}{t-m_X^2} + \frac{1}{u-m_X^2} + \frac{s-m_X^2}{(s-m_X^2)^2 + (m_X\Gamma)^2} \right) .\end{aligned}$$

We consider two limiting cases: $T \gg m_X$ and $T \ll m_X$. Since the average momenta are of the same order as the temperature we can in the first case make the zeroth order approximation $\mathcal{M}_{bb,\overline{b}\overline{b}} \simeq \lambda^2$. Ignoring the real intermediate state subtraction we find

$$\langle \sigma |v| \rangle \simeq \frac{\lambda^2}{T^2} C , \tag{D.4}$$

where C is a constant factor given by (performing the transformation $k' = \frac{k}{T}$, $d\Pi_k^{'} = \frac{d\Pi_k}{T^2}$)

$$C = \frac{\pi^4}{4\zeta(3)^2} \int (2\pi)^4 \delta^{(4)}(k' + p' - q' - r') e^{-(E_k' + E_p')/T} \, d\Pi_k^{'} \, d\Pi_p^{'} \, d\Pi_q^{'} \, d\Pi_r^{'} . \tag{D.5}$$

C can be evaluated analytically to give

$$C = \frac{1}{512\pi\zeta(3)^2} . \tag{D.6}$$

In the low energy limit $T \ll M$, the momenta can be put to zero for a first approximation, which yields

$$\langle \sigma |v| \rangle \simeq \frac{1}{T^2} C \left(\lambda^2 + 9\frac{|g|^2}{m_X^4} - 6\frac{\lambda \mathrm{Re}(g^2)}{m_X^2} \right) . \tag{D.7}$$

For the real intermediate contribution

$$\begin{aligned}\langle \sigma |v| \rangle - \langle \sigma' |v| \rangle =& \frac{1}{n_\gamma^2} \int (2\pi)^4 \delta^{(4)}(k + p - q - r) e^{-(E_k + E_p)/T}\\ &\times \left[\frac{\pi}{2m_X\Gamma_X} |\mathcal{M}|^4 \, \delta(s - m_X^2) \right] d\Pi_k \, d\Pi_p \, d\Pi_q \, d\Pi_r ,\end{aligned} \tag{D.8}$$

we find with the help of (D.2)

$$\langle \sigma |v| \rangle - \langle \sigma' |v| \rangle = \frac{\pi^2}{8\zeta(3)} m_X^2 T^{-5} K_1 \left(\frac{m_X}{T} \right) . \tag{D.9}$$

This part is essentially zero in the regime $T \ll m_X$ and negligible in the case $T \gg m_X$ compared to the contribution by (D.4).

The total $2-2$ velocity averaged cross section is

$$\langle \sigma' |v| \rangle \simeq \begin{cases} C\frac{\lambda^2}{T^2} & (T \gg m_X) \\ C\frac{\lambda^2}{T^2} & (T \ll m_X) \end{cases} \tag{D.10}$$

These results can be plugged together to give the smooth interpolating function

$$\left\langle \sigma' \left| v \right| \right\rangle \simeq \frac{\lambda^2}{T^2} \frac{4Cm_X^2 + T^2}{T^2(m_X^2 + \frac{T^2}{C})} \,, \tag{D.11}$$

which is used for the numerical evaluation.

Appendix E

Reduction of the Collision Integrals

Reduction of $\mathbf{C^{1\leftrightarrow 2}}$ like Collision Integrals

In the upcoming calculations we will assume ($k, p, q, r > 0$) for simplicity. The other cases must be treated separately.

The generalization of the collision integrals for decays and inverse decays is

$$C^{1\leftrightarrow 2}[f] = \frac{1}{2E_k}\int (2\pi)^4\delta^{(4)}(k-q-r)\,|\mathcal{M}|^2\,[(1-\xi f_k)f_q f_r - f_k(1-\xi f_q)(1-\xi f_r)] \\ \times \frac{d^3q}{(2\pi)^3 2E_q}\frac{d^3r}{(2\pi)^3 2E_r}\,. \tag{E.1}$$

Performing the same steps as those, which led us from eqn. (3.41) to eqn. (3.42), we derive

$$C^{1\leftrightarrow 2}[f] = \frac{1}{32\pi E_k}\int \Theta(E_q - m_q)F(f)\sum_i D_i(k,q,r)\frac{r\,dr}{E_r}\,, \tag{E.2}$$

where $E_q = E_k - E_r$, $q = \sqrt{E_q^2 - m_q^2}$ and we have defined the function D as

$$D(k,q,r) = \frac{qr}{8\pi^4}\int \lambda^2\,d\lambda \int e^{i\lambda\mathbf{k}}\,d\Omega_\lambda \int e^{-i\lambda\mathbf{q}}\,d\Omega_q \int e^{-i\lambda\mathbf{r}}\,d\Omega_r\,|\mathcal{M}|^2\,.$$

The simplification of the collision integral described above relies on the analytic computation of the integrals D for our particular matrix elements. In this section we will calculate these integrals for the matrix amplitudes encountered above.

Mandelstamm variables

The three Mandelstamm variables are defined as

$$\begin{aligned} s &= (k+p)^2 = (q+r)^2\,, \\ t &= (q-k)^2 = (r-p)^2\,, \\ u &= (r-k)^2 = (q-p)^2\,. \end{aligned} \tag{E.3}$$

In the limit of massless particles we get

$$\begin{aligned} s &= 2kp = 2\,|\mathbf{k}|\,|\mathbf{p}|\,(1-\cos\theta_{kp})\,, \\ t &= -2kq = -2\,|\mathbf{k}|\,|\mathbf{q}|\,(1-\cos\theta_{kq})\,, \\ u &= -2kr = -2\,|\mathbf{k}|\,|\mathbf{r}|\,(1-\cos\theta_{kr})\,. \end{aligned} \tag{E.4}$$

The Mandelstamm variables are not independent. Rather than that they satisfy the relation

$$s+t+u=\sum_{i=1}^{4} m_i^2\,. \tag{E.5}$$

Calculation of D for some matrix elements

In the main part we have seen, that reducing the collision integral means that we have to calculate

$$D(k,p,q,r)=\frac{pqr}{64\pi^5}\int\lambda^2\,d\lambda\int e^{i\lambda\mathbf{k}}\,d\Omega_\lambda\int e^{i\lambda\mathbf{p}}\,d\Omega_p\int e^{-i\lambda\mathbf{q}}\,d\Omega_q\int e^{-i\lambda\mathbf{r}}\,d\Omega_r\,|\mathcal{M}|^2$$

(denoting the surface element with $d\Omega$ and the inner product by $\lambda\mathbf{k}=(\lambda,k)$ and the absolute value of a vector by $k=|k|$)
for the respective matrix element $|\mathcal{M}|^2$. $\mathcal{M}$ will in general exhibit a dependence on the angles between the various (momentum) vectors k,p,q,r of the form $\mathbf{kp}$, $\mathbf{kq}$ and $\mathbf{kr}$.

Cases where the dependencies on $\mathbf{kp}$, $\mathbf{kq}$ and $\mathbf{kr}$ separate and thus each of the integrations over $d\Omega_q$, $d\Omega_r$ and $d\Omega_p$ can be done on its own, and where $\mathcal{M}$ is a homogeneous polynomial (in $\mathbb{R}^3$), can be treated by the following formula found in [60]:

$$\int e^{i\lambda\hat{\eta}}P^l(\hat{\eta})\,d\Omega_\eta=4\pi\left(\frac{i}{2}\right)^l\sum_{k=0}^{[l/2]}\frac{(-)^k}{k!}\left(\frac{2}{\lambda}\right)^{l-k}j_{l-k}(\lambda)(\Delta^k P^l)(\lambda)\,.$$

Here P^l is a homogeneous polynomial of degree l, Δ is the Laplace operator j_ν denotes the spherical Bessel functions of the first kind, $\Gamma(z)$ is the Gamma function and $\hat{\eta}$ is a unit vector. Because of the relation

$$j_n(z)=(-1)^n z^n\left(\frac{d}{z\,dz}\right)^n\frac{\sin z}{z}$$

these can always be expressed in terms of trigonometric functions and powers. For the simple case of $\mathcal{M}=1$ ($l=0$) the formula gives the correct results immediately (cf. eqn. (3.45)):

$$D_1(k,p,q,r)=\frac{4}{k\pi}\int_0^\infty\sin(\lambda k)\sin(\lambda p)\sin(\lambda q)\sin(\lambda r)\frac{d\lambda}{\lambda^2}\,.$$

This can then be written in terms of signum functions as seen above.

In particular we encountered the matrix element

$$\begin{aligned} \mathcal{M}^2_{bb,\overline{bb}} =& \lambda^2+|g|^4\left(\frac{1}{(s-m_X^2)^2+(m_X\Gamma)^2}+\left(\frac{1}{t-m_X^2}+\frac{1}{u-m_X^2}\right)^2\right. \\ &\left.+\frac{2(s-m_X^2)}{(s-m_X^2)^2+m_X^2\Gamma_X^2}\left(\frac{1}{t-m_X^2}+\frac{1}{u-m_X^2}\right)\right) \\ &+2\lambda\mathrm{Re}(g^2)\left(\frac{1}{t-m_X^2}+\frac{1}{u-m_X^2}+\frac{s-m_X^2}{(s-m_X^2)^2+(m_X\Gamma)^2}\right)\,. \end{aligned}$$

We can treat the different terms separately.
First, we calculate D for matrix elements of the form

$$|\mathcal{M}|^2 = \frac{1}{t - m_X^2}\,,$$

$$t = -2kq(1 - \cos\theta_{kq})\,.$$

All the angle integrals which $|\mathcal{M}|^2$ does not depend on can be integrated out using relation (3.44). Thus we have to compute the integral

$$D = -\frac{1}{4\pi^3 k}\int \sin(\lambda p)\sin(\lambda r) I\, d\lambda\,, \tag{E.6}$$

where we have defined

$$I = \int e^{i\lambda\mathbf{k}} \int e^{-i\lambda\mathbf{q}} \frac{1}{1 + \frac{m_X^2}{2kq} - \cos\theta_{kq}}\, d\Omega_q\, d\Omega_\lambda\,.$$

For the integration we can choose to orientate the z-axis of the coordinate system in the direction of $\mathbf{k}$ and the x-axis in the direction of the projection of $\mathbf{q}$ on the xy-plane. Then I becomes ($a = 1 + \frac{m_X^2}{2kq}$)

$$I = \int e^{i\lambda\mathbf{k}} \int e^{-i\lambda q(\cos\theta_\lambda\cos\theta_q + \sin\theta_\lambda\sin\theta_q\cos\phi_\lambda)} \frac{1}{a - \cos\theta_q}\, d\Omega_q\, d\Omega_\lambda\,.$$

The integration over ϕ_q and ϕ_λ gives

$$\int\int e^{\pm i\lambda q\sin\theta_\lambda\sin\theta_q\cos\phi_\lambda}\, d\phi_q\, d\phi_\lambda = (2\pi)^2 J_0(\lambda q\sin\theta_\lambda\sin\theta_q)\,,$$

where J_0 is a Bessel function of the first kind. Thus I is

$$\begin{aligned} I =& (2\pi)^2 \int\int e^{i\lambda\mathbf{k}} e^{-i\lambda q\cos\theta_\lambda\cos\theta_q} \frac{J_0(\lambda q\sin\theta_\lambda\sin\theta_q)}{a - \cos\theta_q}\, d\cos\theta_\lambda\, d\cos\theta_q \\ =& (2\pi)^2 \int \frac{d\cos\theta_q}{a - \cos\theta_q} \int e^{i\lambda(k - q\cos\theta_q)\cos\theta_\lambda} J_0(\lambda q\sin\theta_\lambda\sin\theta_q)\, d\cos\theta_\lambda \\ =& (2\pi)^2 \int \frac{d\cos\theta_q}{a - \cos\theta_q} \int_0^\pi \sin\theta_\lambda\cos(\lambda(k - q\cos\theta_q)\cos\theta_\lambda) J_0(\lambda q\sin\theta_q\sin\theta_\lambda)\, d\theta_\lambda\,. \end{aligned} \tag{E.7}$$

The integral

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin x\cos(\beta\cos x) J_0(\alpha\sin x)\, dx &= \sqrt{\frac{\pi}{2}}(\alpha^2 + \beta^2)^{-\frac{1}{4}} J_{\frac{1}{2}}(\sqrt{\alpha^2 + \beta^2}) \\ &= \frac{1}{\sqrt{\alpha^2 + \beta^2}} \sin\left(\sqrt{\alpha^2 + \beta^2}\right) \end{aligned} \tag{E.8}$$

can be found in the literature [61]. With $\alpha = \lambda q\sin\theta_q$ and $\beta = \lambda(k - q\cos\theta_q)$ this yields

$$I = 8\pi^2 \int_{-1}^{1} \frac{\sin(\lambda\sqrt{q^2 + k^2 - 2kqx})}{\lambda\sqrt{q^2 + k^2 - 2kqx}(a - x)}\, dx\,.$$

Substituting $\sqrt{q^2+k^2-2kqx}=y$ we find ($f(y)=\frac{1}{2akq-q^2-k^2+y^2}$)

$$I=16\pi^2\int_{|q-k|}^{q+k} f(y)\frac{\sin(\lambda y)}{\lambda}\,dy\,.$$

Interchanging the integration over dy and $d\lambda$ the total expression for D becomes

$$D=-\frac{4}{\pi k}\int_{|q-k|}^{q+k} f(y)\int\frac{d\lambda}{\lambda}\sin(\lambda p)\sin(\lambda r)\sin(\lambda y)\,dy\,,$$

where the inner integral is known from (3.48):

$$\begin{aligned}D_1'=&\frac{4}{\pi}\int\frac{d\lambda}{\lambda}\sin(\lambda p)\sin(\lambda r)\sin(\lambda y)\,d\lambda\\ =&\frac{1}{2}\left(\text{sgn}\left(y-(r-p)\right)+\text{sgn}\left(y-(p-r)\right)-\text{sgn}\left(y-(p+r)\right)-1\right)\,.\end{aligned}$$

Now we can evaluate the outer integral, using the relation

$$\int_a^b f(y)\text{sgn}(y-h)\,dy=\text{sgn}(b-h)\left(F(b)-F(h)\right)+\text{sgn}(h-a)\left(F(a)-F(h)\right)\,,$$

where F is the anti-derivative of f. The result is

$$D=-\frac{1}{2k}\int_{|q-k|}^{q+k} f(y)\left(\text{sgn}\left(y-(r-p)\right)+\text{sgn}\left(y-(p-r)\right)-\text{sgn}\left(y-(p+r)\right)-1\right)\,.$$

This can also be written as

$$\begin{aligned}D=-\frac{1}{2k}\Big\{&\text{sgn}\big(q+k-(r-p)\big)\big(F(q+k)-F(r-p)\big)\\ &+\text{sgn}\big(r-p-|q-k|\big)\big(F(|q-k|)-F(r-p)\big)\\ &+\text{sgn}\big(q+k-(p-r)\big)\big(F(q+k)-F(p-r)\big)\\ &+\text{sgn}\big(p-r-|q-k|\big)\big(F(|q-k|)-F(p-r)\big)\\ &-\text{sgn}\big(q+k-(p+r)\big)\big(F(q+k)-F(p+r)\big)\\ &-\text{sgn}\big(p+r-|q-k|\big)\big(F(|q-k|)-F(p+r)\big)\\ &-\big(F(q+k)-F(|q-k|)\big)\Big\}\,.\end{aligned}$$

In order to calculate

$$F(y)=\int f(y)\,dy=\int\frac{1}{2akq-q^2-k^2+y^2}\,dy$$

we have to distinguish two cases: $2akq-q^2-k^2=m_X^2-(q-k)^2<0$ and >0. In the first case we have

$$F(y)=\int\frac{1}{y^2-c^2}\,dy=\frac{1}{2c}\ln\left(\frac{y-c}{y+c}\right),$$

with $c^2=-(m_X^2-(q-k)^2)$ and in the second case

$$F(y)=\int\frac{1}{y^2+c^2}\,dy=\frac{1}{c}\text{arctan}\left(\frac{y}{c}\right)\,,$$

with $c^2 = m_X^2 - (q-k)^2$.
The integrals involving

$$|\mathcal{M}|^2 = \frac{1}{u - m_X^2} \ ,$$

$$|\mathcal{M}|^2 = \frac{1}{(t - m_X^2)^2} \ ,$$

and

$$|\mathcal{M}|^2 = \frac{1}{(u - m_X^2)^2}$$

can be evaluated in the same way.
Now we calculate D for

$$|\mathcal{M}|^2 = \delta(s - m_X^2) \, ,$$
$$s = 2kp(1 - \cos\theta_{kp}) \, .$$

We can start from (cf. (E.6))

$$D = \frac{p}{4\pi^3} \int \sin(\lambda q) \sin(\lambda r) I \, d\lambda \, , \tag{E.9}$$

where this time

$$I = \int e^{i\lambda \mathbf{k}} \, d\Omega_\lambda \int e^{i\lambda \mathbf{p}} \delta(2kp(1 - \cos\theta_{kp}) - m_X^2) \, d\Omega_p \, .$$

Repeating the steps above we can write the second integral as

$$2\pi \int d\cos\theta_p e^{i\lambda p \cos\theta_\lambda \cos\theta_p} J_0(\lambda p \sin\theta_\lambda \sin\theta_p) \delta(2kp(1 - \cos\theta_p) - m_X^2) \, .$$

The delta function can be rewritten as

$$\delta(2kp(1 - \cos\theta_p) - m_X^2) = \frac{1}{2kp} \delta(\cos\theta_p - (1 - \frac{m_X^2}{2kp})) \, .$$

Then the integration can easily be performed to give

$$\frac{\pi}{kp} e^{i\lambda p \cos\theta_\lambda \cos\theta_p} J_0(\lambda p \sin\theta_\lambda \sin\theta_p) \theta(4kp - m_X^2) \, ,$$

where the theta function makes sure that the peak of the delta function is located within the region of integration and θ_p now is fixed such that $\cos\theta_p = 1 - \frac{m_X^2}{2kp}$.
Plugging this into D yields

$$D = \frac{1}{4k\pi^2} \int d\lambda \sin(\lambda q) \sin(\lambda r) \int d\Omega_\lambda e^{i\lambda(k + p\cos\theta_p)\cos\theta_\lambda} J_0(\lambda p \sin\theta_\lambda \sin\theta_p) \theta(4kp - m_X^2) \, .$$

Now we can again apply (E.7):

$$D = \frac{1}{2k\pi^2}\theta(4kp - m_X^2)\int \frac{d\lambda}{\lambda}\sin(\lambda q)\sin(\lambda r)\frac{\sin(\lambda\sqrt{k^2+p^2-2kp\cos\theta_p})}{\lambda\sqrt{k^2+p^2-2kp\cos\theta_p}}\,.$$

The last integral can now be evaluated just like the above one to

$$D = \frac{1}{16ck\pi}\theta(4kp - m_X^2)\left(\text{sgn}\left(c-(r-q)\right) + \text{sgn}\left(c-(q-r)\right) - \text{sgn}\left(c-(q+r)\right) - 1\right)\,,$$

where we defined

$$c = \sqrt{k^2+p^2-2kp\cos\theta_p} = \sqrt{(k+p)^2 - m_X^2}\,.$$

Because of $(k+p)^2 - m_X^2 = (k-p)^2 + 4kp - m_X^2 \geq 4kp - m_X^2$ the radical of c is always positive, when the argument of the Heaviside function is, such that D is well defined.

For the term

$$\left(\frac{1}{t-m_X^2} + \frac{1}{u-m_X^2}\right)^2 = \frac{1}{m_X^4}\frac{\left((a-1)(b-y)+(b-1)(a-x)\right)^2}{(a-x)^2(b-y)^2} = f(x,y)\,,$$

with $x = \cos\theta_q$, $y = \cos\theta_r$ and

$$a = 1 + \frac{m_X^2}{2kq}\,,$$

$$b = 1 + \frac{m_X^2}{2kr}\,,$$

we try to find an approximating function $g(x,y)$ because the corresponding integrals are hard to solve. We can make the ansatz

$$\begin{aligned} g(x,y) =& \frac{1}{m_X^4}\frac{\alpha\,(a-1)^2(b-y)^2 + \beta\,(b-1)^2(a-x)^2 + \left(\gamma\,(x-y)^2+\delta\right)(a-x)^2(b-y)^2}{(a-x)^2(b-y)^2} \\ =& \frac{1}{m_X^4}\left(\alpha\frac{(a-1)^2}{(b-y)^2} + \beta\frac{(b-1)^2}{(a-x)^2} + \gamma\,(x-y)^2 + \delta\right)\,. \end{aligned} \tag{E.10}$$

The integration over the different terms can then be performed according to the results above. Demanding that $g(x,y) = f(x,y)$ in the points $(x,y) \in \{(0,0),\,(1,0),\,(0,1)\,(1,1)\}$, we find for the parameters α, β, γ and δ

$$\left\{\alpha = \frac{4ab-a-b}{b\,(2\,a-1)}, \beta = \frac{4ab-a-b}{a\,(2\,b-1)}, \gamma = -\frac{1}{ab}, \delta = -\frac{(a-b)^2}{(4\,ab-2\,a-2\,b+1)\,ab}\right\}$$

It is important to notice, that $g(x,y)$ is still symmetric under exchange of q and r. The quality of this approximation depends on the parameters q and r, but for $y = x$ it is always exact.

Appendix F

Integrals

In this chapter we compute some integrals encountered in the main text.

Two particle decay rate

The differential 2-particle decay rate $d\Gamma(A \to BC)$ in the rest frame of the decaying particle is given by [20]

$$d\Gamma = \frac{1}{2m_A} \frac{d^3p_B}{(2\pi)^3 2E_B} \frac{d^3p_C}{(2\pi)^3 2E_C} (2\pi)^4 \delta^{(4)}(p_A - p_B - p_C) \left|M(A \to BC)\right|^2 \ . \tag{F.1}$$

For a two particle decay process we have $\left|M(A \to BC)\right|^2 = const.$[1] In this case both integrations can be carried out and one [62]

$$d\Gamma(A \to BC) = \frac{\left|M(A \to BC)\right|^2}{8\pi m_A^2} \left|p_B\right| \ , \tag{F.2}$$

with

$$\left|p_B\right| = \left|p_C\right| = \frac{\sqrt{(m_A^2 - (m_B + m_C)^2)(m_A^2 - (m_B - m_C)^2)}}{2m_A} \ . \tag{F.3}$$

For radiative decays $m_B = m_C = 0$ one finds

$$\Gamma(A \to BC) = \frac{\left|M(A \to BC)\right|^2}{16\pi m_A} \ . \tag{F.4}$$

[1] One can infer this general property by expressing all possible Lorentz invariant combinations of momenta in terms of the masses. Since the amplitude is invariant it can only be made up by these combinations and thus it must be constant.

Acknowledgements

I want to take the opportunity in order to thank all those who helped, in one way or another, making this work become a success.

First of all I thank my supervisor Prof. Dr. Manfred Lindner for enabling me to work on this interesting subject, and for taking care of all the annoying, but necessary organisational stuff. Furthermore, for giving me the opportunity to attend the summer school in Maria Laach which gave me a broader view over all the aspects of particle physics.

Special thanks go to Markus Michael Müller, who has been my tutor, and who was always ready to answer my questions. He did proof read this thesis in very short time, although he has been busy with his own work. He has also been very helpful in questions concerning programming and other computer related stuff.

Claudia Hagedorn for sharing her office and time with me, for her useful hints regarding the symmetries of the toy model and for all her critical remarks.

My buddies Joachim Kopp, Alexander Merle and Florian Plentinger, who were doing their thesis at the same time, as well as Andreas Weinberger and Michael Wick from the neighbour group for driving to Maria Laach and making the visit there more entertaining.

Furthermore, all the other regular group members Florian Bauer, Mathias Garny, Dr. Werner Rodejohann, Mark Rolinec, Michael Schmidt who have always been helpful when asked.

Our guest scientists Prof. Srubabati Goswami, Prof. Naoyuki Haba, Toshifumi Jittoh, Dr. Toshihiko Ota, Prof. Joe Sato and Yoshio Sato for the international flair in the group and especially Prof. Evgeny Akhmedov for his funny and interesting stories during the coffee breaks.

Finally, our secretary Karin Ramm for providing me with all the materials one needs for a thesis in theoretical physics, and for her help with all the other bureaucratic things.

In particular I like to thank my parents Hans and Hermine Hohenegger for their absolute support during the time of my studies and ever before.

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