Exercises for: „Introduction to the Standard model II“
Winter term 11/12

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Sheet 9

Exercise 18: Matrix stories (clarification of last session's discussion)

a) Show that any complex $n \times n$ matrix $M \in GL(n, \mathbb{C})$ can be diagonalized by a biunitary transformation.

$$V^{\dagger} M U = \mathrm{diag}(m_1, m_2, ...)$$

b) Show that any symmetric complex matrix M can be diagonalized in the following way:

$$U^T M \, U = \mathrm{diag}(m_1, m_2, ...).$$

Exercise 19: Neutrino oscillations in the two flavour case

Time evolution of the wave function in the mass eigenbasis can be described by:

$$|\nu(t)\rangle = U^*_{a\,j} \, e^{-iE_j t} \, |\nu_j\rangle.$$

The mass and the flavour basis do not coincide and hence the probabilities for oscillations between various flavour states are given by the square of the amplitude:

$$A(\nu_a \to \nu_b;\, t) = \langle \nu_b | \nu(t) \rangle.$$

a) Compute the general expression for the oscillation probability

$$P(\nu_a \to \nu_b;\, t\,).$$

b) Suppose now the neutrinos are relativistic and take $E_i \cong p + \frac{m_i^2}{2E}$ as an approximation for the energy of the neutrinos, note that it is assumed that the different mass states have the same momentum. On the other hand you could assume that the different mas states have the same energy, write

$p_i \cong E - \frac{m_i^2}{2E}$ and describe the spatial propagation of each mass eigenstate with the phase factor $e^{ip_i x}$. Compare the impact of the different assumptions on your formula for the probability. Are the assumptions justified?

c) Consider the two flavour case, where (with $c = \cos(\theta_0)$ and $s = \sin(\theta_0)$) the mixing is given by:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.$$

Take here the approximation for the neutrino energy $E_i \cong p + \frac{m_i^2}{2E}$ and compute the transition probability $P(\nu_e \to \nu_\mu;\, t)$.

You should get:

$$P(\nu_e \to \nu_\mu;\, t) = \sin^2(2\theta_0) \sin^2\left(\frac{\Delta m^2}{4E}\, t\right) \qquad \text{with} \quad \Delta m^2 = m_2^2 - m_1^2.$$

Exercise 20: Neutrino oscillations in the three flavour case and CP-violation

Consider now the three flavour case. The neutrino flavour eigenstate and mass eigenstate fields are related via:

$$\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}.$$

The oscillation has a CP odd part denoted as $\Delta P_{ab} := P(\nu_a \to \nu_b;\, t) - P(\bar{\nu}_a \to \bar{\nu}_b;\, t)$.

a) Use the general expression for the transition probability (Exercise 19 a) and show that:

$$\Delta P_{ab} = 2 \sum_{j>k} Im\left\{U_{aj}^* U_{bj} U_{ak} U_{bk}^*\right\} \sin(\Delta_{jk} t) \text{ with } \Delta_{jk} = \frac{m_j^2 - m_k^2}{2E}$$

b) Use unitarity of U to prove that $U_{aj}^* U_{bj} U_{ak} U_{bk}^*$ is invariant for different choices of a and b ($a \neq b$).

Note: The same invariant can be constructed from the CKM matrix and measures the strength of CP violation in the meson sector.

c) Using the following parametrization for U:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

with $c_{ij} = \cos(\theta_{ij})$ and $s_{ij} = \sin(\theta_{ij})$. Convince yourself that:

$$\Delta P_{e\mu} = \Delta P_{\mu\tau} = \Delta P_{\tau e} =$$

$$= 4\, s_{12}c_{12}s_{13}c_{13}^2 s_{23}c_{23}\ \sin(\delta) \times \left[\sin\left(\frac{\Delta m_{12}^2}{2E}\, t\right) + \sin\left(\frac{\Delta m_{23}^2}{2E}\, t\right) + \sin\left(\frac{\Delta m_{31}^2}{2E}\, t\right) \right].$$

d) Discuss your result and the impact of the mixing parameters on CP violation.

Exercise 21: Prepare for X-mas

a) Have a good X-mas vacation.

b) In case you get bored, read: „The quantum state can not be interpreted statistically“ (http://arxiv.org/abs/1111.3328) and for a better background understanding this explanatory article (http://mattleifer.info/2011/11/20/can-the-quantum-state-be-interpreted-statistically/). We will discuss this paper in the new year.

Once again a merry X-mas and a happy new year.

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