Exercises for: „Introduction to the Standard model II“
Winter term 11/12

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Sheet 3

Exercise 7: Dirac bilinears

The Dirac bilinears are given by:

$$S = \bar{\psi}\psi, \qquad V^\mu = \bar{\psi}\gamma^\mu\psi, \qquad A^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi, \qquad P = i\,\bar{\psi}\gamma^5\psi, \qquad T^{\mu\nu} = i\bar{\psi}\gamma^{[\mu}\gamma^{\nu]}\psi$$

a) Check the transformation behavior of the bilinears under Lorentz transformations, using:

$$\psi(x) \to \Lambda\psi(\Lambda^{-1}x)$$

$$\bar{\psi}(x) \to \bar{\psi}(\Lambda^{-1}x)\Lambda^{-1}$$

$$\Lambda^{-1}\gamma^\mu\Lambda = \Lambda^\mu{}_\nu\gamma^\nu$$

b) Investigate the transformation properties of the bilinears under Parity transformation, using:

$$\mathcal{P}\psi\mathcal{P} = \pm\gamma^0\psi(\tilde{x}),$$

$$\mathcal{P}\bar{\psi}\mathcal{P} = \pm\bar{\psi}(\tilde{x})\,\gamma^0,$$

with: $\tilde{x} = (x_0\,, -\vec{x})$.

Exercise 8: Pion decay I

a) Consider the following reaction, mediated by the strong interaction:

$$\pi\, d \to n\, n.$$

Here π is the pion (spin 0), d the deuteron (core of the deuterium with spin 1) and n is the neutron (spin $\frac{1}{2}$). The pion is captured by the deuteron from a $1S$ state (implying that $l = 0$ in the initial state). Therefore the total angular momentum of the initial state is $j = l + s = 1$. Parity of the initial state and parity of the final state are given by:

Initial: $(-1)^l P_\pi P_d$,

Final: $(-1)^l P_n P_n$.

The neutrons in the final state are identical fermions and hence the allowed two particle states are :$^1S_0$, $^3P_{0,1,2}$, ,$^1D_2$, $^3F_{2,3,4}$, ... The only state with $j = 1$ is the $^3P_1$ state and since total angular momentum has to be conserved, this will be the produced state. Now assuming that strong interactions preserve parity and knowing that the parity of the deuteron $P_d = 1$, determine the parity of the pion P_π.

b) Consider now the pion decay: $\pi^+ \to \mu^+ + \nu_\mu$.

The effective Lagrangian describing this decay is the Fermi type interaction:

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}} \cos\theta_c \left[\bar{u}\gamma^\mu(1-\gamma^5)d\right] \left[\bar{\mu}\gamma_\mu(1-\gamma_5)\nu_\mu\right].$$

Show that:

$$\langle 0|\bar{u}\gamma_\mu d\,|\pi^+(p)\rangle = 0.$$

And:

$$\langle 0|\bar{u}\gamma_\mu\gamma_5 d\,|\pi^+(p)\rangle = i\, f_\pi p_\mu.$$

Where f_π is the pion decay constant.

Exercise 9: Vacuum energy

The energy momentum tensor of a massive, real, scalar field is given by (convention used $(+,-,-,-)$):

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\partial_\lambda\phi\partial^\lambda\phi\right) + \frac{1}{2}g_{\mu\nu}m^2\phi^2$$

The vacuum expectation value of this quantity is now of interest: $\langle T_{\mu\nu}\rangle = \langle 0|\, T_{\mu\nu}\,|0\rangle$ with the energy density $\rho = \langle T_{00}\rangle$ and the pressure $p = \langle T_{ii}\rangle$.

The second quantized field is given by:

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3 2E}\left\{\hat{a}(\vec{k})e^{-ik_\mu x^\mu} + \hat{a}^\dagger(\vec{k})e^{ik_\mu x^\mu}\right\}$$

a) Using second quantization and the fact that $\hat{a}(k)$ annihilates the vacuum state, calculate the energy density and the pressure of the vacuum. Introduce a momentum cut-off Λ as the upper limit of the momentum integral. Then organize the terms in the result in powers of Λ using the condition that $\Lambda^2 \gg m^2$.

b) In cosmology the local conservation equation of the energy momentum tensor reads as (with a being the scale factor of the universe):

$$\partial_\mu T^{\mu 0} = \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p)$$

Check order by order in Λ whether the energy momentum of the vacuum is locally conserved. Furthermore discuss the implication of your result for cosmology by inserting your result in the second Friedmann equation:

$$3\frac{\ddot{a}}{a} = -4\pi G_N (\rho + 3p)\,.$$

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