Exercises for: „Introduction to the Standard model II“
Winter term 11/12

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Sheet 2

Exercise 4: Complex scalar field

Let $\Phi = (\phi_1 + i\phi_2)\sqrt{2}$ denote a complex scalar field with a Lagrangian

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi a - a\phi^* \phi - b(\phi^* \phi)^2 - c. \tag{1}$$

The theory is invariant under global $U(1)$ transformations.

Derive the energy functional $E(\phi, \partial\phi) = \int d^3x\, H(x)$ corresponding to (1) and show that the ground state is space-time uniform.

Note: Here $H(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_0 \phi - \mathcal{L}$ is the Hamilton density derived via Legendre transform from the Lagrangian.

For $a < 0$, discuss the potential $V(\phi * \phi)$ and sketch it. Show that it has a continuous set of minima. Choose one of the minima as the ground state ($a < 0$) and consider perturbations around it. Derive the Lagrangian up to second order in the perturbations. Show that one of the scalar degrees of freedom remains massless in this ground state. Discuss this fact.

Exercise 5: $SU(2)$ Doublet

Consider a field theory with gauge group $SU(2)$ and a scalar field doublet in the fundamental representation $\Phi = (\phi_1, \phi_2)^T$, where $\phi_{1,2}$ are complex scalar fields. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} - (D_\mu \Phi)^\dagger (D_\mu \Phi) - a\Phi^\dagger \Phi - b(\Phi^\dagger \Phi)^2, \tag{2}$$

in an obvious notation.

Argue in a similar way to Exercise 4 that the vacuum of (2) should be also space-time uniform. From the family of gauge invariant vacua, choose a convenient one. Find the spectrum of small perturbations around this ground state. For convenience fix a unitary gauge. Show that the spectrum in this ground state consists of three massive vector fields with the same mass and one massive scalar field. Show that the model has ten degrees of freedom, in both the broken and unbroken phase.

Exercise 6: Scalar electrodynamics

Consider scalar electrodynamics equipped with a Higgs mechanism:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)^* D^\mu\phi - \lambda^2(\phi^*\phi)^2 + \mu^2(\phi^*\phi), \tag{3}$$

with the gauge covariant derivative $D_\mu\phi := \partial_\mu\phi - ieA_\mu\phi$.

Consider the case where $\partial^0\phi = \partial^0\vec{A} = 0$ and $A_0 = 0$. Derive the equation of motion for $\vec{A}$ and determine the current density $\vec{J}$ sourcing the gauge potential.

Show that in the spontaneously broken phase the current density $\vec{J} = e^2v^2\vec{A}$, where v denotes the Higgs vacuum expectation value (short vev), and thus $\Delta\vec{B} = e^2v^2\vec{B}$. Discuss the physical significance of this result.

The resistivity R for the system is defined by $\vec{E} = R\vec{J}$. Show that in the broken phase $R = 0$, i.e. the ground state is super conducting.

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