Local Lagrangian (gauge) invariance compared to global symmetry

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In the last session we discussed a Lagrangian invariant under a global transformation:

$$u_i \to u_i' = u_i(x) + \delta u_i(x)$$

and

$$\delta u_i(x) = T^k_{ij}\epsilon_k u_j(x).$$

The transformations we discuss here are more general and the generators of the group are T^k_{ij} and ϵ_k are the coefficients. In the case of a global symmetry ϵ_k has no space-time dependence. We have derived the following expression, which can be written in this general case as:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)}\delta u_j = \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)} T^k_{ij}\epsilon_k u_j(x)$$

In the case of a local transformation we have to allow $\epsilon(x)$ to have a space-time dependence and hence.

$$\delta(\partial_\mu u_i(x)) = T^k_{ij}\epsilon_k(x)\partial_\mu u_j(x) + T^k_{ij}u_j(x)\partial_\mu \epsilon_k(x)$$

For the variation of the Lagrangian we get:

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial u_i}\delta u_i(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)}\delta(\partial_\mu u_i) =$$

$$= \frac{\partial \mathcal{L}}{\partial u_i} T^k_{ij}\epsilon_k(x)u_j(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)} T^k_{ij}\epsilon_k(x)\partial_\mu u_j(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)} T^k_{ij}u_j(x)\partial_\mu \epsilon_k(x)$$

The first two terms are zero, since our Lagrangian is invariant under global symmetry transformations and we are left with:

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)} T^k_{ij}u_j(x)\partial_\mu \epsilon_k(x) \neq 0$$

However, we want the Lagrangian to by invariant also under local transformations. The way to achieve this is the introduction of extra fields with such transformation properties that the above term is canceled. We call the fields A_l' and the Lagrangian is now $\mathcal{L}(u_i,\, \partial_\mu u_i,\, A_l')$. If one performs the variation of this Lagrange function and imposes the invariance of it under local transformations one gets the equations of motion and the transformation properties of the gauge field and observes that it can be absorbed in the derivative and hence "covariantise" it. The kinetic term of the gauge field has to be also added to the Lagrangian so that we have:

$$\mathcal{L} = \mathcal{L}(D_\mu u_i,\, u_i) + \mathcal{L}(F_{\mu\nu})$$

The explicit calculation gives also:

$$D_\mu u_i = \partial_\mu u_i - T^k_{ij}u_j A^k_\mu,$$

$$F^m_{\mu\nu} = \partial_\mu A^m_\nu - \partial_\nu A^m_\mu - \frac{1}{2} f_{klm} \left(A^k_\mu A^l_\nu - A^k_\nu A^l_\mu \right)$$

and

$$\delta A^k_\mu = f_{klm} A^m_\mu \epsilon_l(x) + \partial_\mu \epsilon_k(x)$$

with f_{klm} the structure coefficients of the group.

The invariance under local gauge transformations reads now as:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial A^k_\mu} \delta A^k + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^k_\mu)} \delta(\partial_\nu A^k_\mu) + \frac{\partial\mathcal{L}}{\partial u_i} \delta u_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu u_i)} \delta(\partial_\mu u_i)$$

Again as above we use the equations of motion to rewrite this:

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^k_\nu)} \delta A^k_\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu u_i)} \delta u_i \right) = 0$$

Inserting here the explicit form of the variations gives:

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(D_\mu u_i)} T^k_{ij} u_j + 2 \frac{\partial\mathcal{L}}{\partial F^m_{\mu\nu}} f_{klm} A^l_\nu \right) \epsilon_k +$$

$$+ \left[\frac{\partial\mathcal{L}}{\partial(D_\mu u_i)} T^k_{ij} u_j + 2 \frac{\partial\mathcal{L}}{\partial F^m_{\mu\nu}} f_{klm} A^l_\nu + 2\partial_\nu \left(\frac{\partial\mathcal{L}}{\partial F^k_{\mu\nu}} \right) \right] \partial_\mu \epsilon_k(x) +$$

$$+ \left(\frac{\partial\mathcal{L}}{\partial F^k_{\mu\nu}} + \frac{\partial\mathcal{L}}{\partial F^k_{\nu\mu}} \right) \partial_\mu \partial_\nu \epsilon_k(x) = 0$$

Since ϵ_k and $\partial_\mu \epsilon_k$ are arbitrary functions we have independent equations:

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(D_\mu u_i)} T^k_{ij} u_j + 2 \frac{\partial\mathcal{L}}{\partial F^m_{\mu\nu}} f_{klm} A^l_\nu \right) = 0 \tag{1}$$

and using:

$$\partial_\nu \left(\frac{\partial\mathcal{L}}{\partial F^k_{\mu\nu}} \right) = \frac{1}{2} \frac{\partial\mathcal{L}}{\partial A^k_\mu},$$

we get:

$$\frac{\partial\mathcal{L}}{\partial(D_\mu u_i)} T^k_{ij} u_j + 2 \frac{\partial\mathcal{L}}{\partial F^m_{\mu\nu}} f_{klm} A^l_\nu + \frac{\partial\mathcal{L}}{\partial A^k_\mu} = 0. \tag{2}$$

It is sensible to call $J^\mu_k = \frac{\partial\mathcal{L}}{\partial A^k_\mu}$ the current, since we expect the coupling $\mathcal{L} \sim ... + A^k_\mu J^\mu_k + ...$ in the Lagrangian. And so (2) gives us for J^μ_k:

$$J^\mu_k = -\frac{\partial\mathcal{L}}{\partial(D_\mu u_i)} T^k_{ij} u_j - 2 \frac{\partial\mathcal{L}}{\partial F^m_{\mu\nu}} f_{klm} A^l_\nu$$

Note: In the case of the global symmetry we had the first term only and also without the "covariant" derivative.

Equation (1) shows that this quantity we defined as current is conserved i.e.:

$$\partial_\mu J^\mu_k = 0 \;\forall k$$

We have proven that the invariance under a local symmetry gives rise to a more general form of the conserved current.