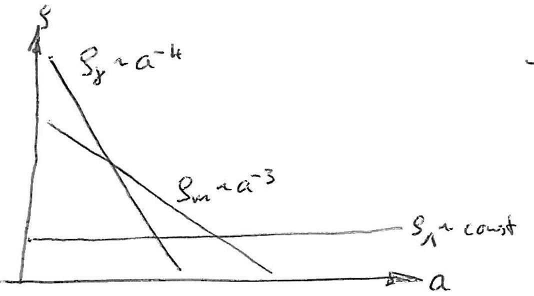
- standard Big Bang cosmology $\to$ flatness & horizon problems

$\Rightarrow$ solved by an early phase of inflation $\leftrightarrow$ exponential expansion $a \propto e^{Ht}$

- adding a cosmological constant would yield such an expansion, but:




$\to$ since the energy densities of radiation and matter decrease with rising scale factor, an early phase of $\Lambda$-dominated universe would remain $\Lambda$-dominated forever!

$\Rightarrow$ therefore we need a vacuum-like energy that is time-dependent:
- at early times $a \propto e^{Ht}$
- allows transition into radiation dominated phase

$\Rightarrow$ a scalar field is the simplest way to implement this behaviour.

## 1. Einstein equation & one scalar field

- we start with the action including a minimally coupled scalar field $\phi$

$$S = \int d^4x \sqrt{-g} \left( \frac{M_{Pl}^2}{16\pi} R + \mathcal{L}^\phi \right)$$

determinant of the metric; Ricci scalar; Lagrangian for $\phi$:

$$\mathcal{L}^\phi = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$$

$\to$ this action is called minimally coupled, because there are no terms included that mix $\phi$ and $R$

$\to$ many such mixing terms could be absorbed by field redefinitions

$\to$ possible generalizations are:
- replace $R$ with some function of $R$ ($\to$ e.g. Starobinski model)
- non-canonical kinematic terms $\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \mapsto F(\phi, g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi)$
  (such terms arise in string theories)
- multiple scalar fields

$\to$ adding a scalar field can be seen as an effective description in order to understand the mechanism of inflation (physically more meaningfull fields can be more complicated)

- assumptions:
  - isotropic & homogenious universe ( perturbations are treated seperately )
  - homogeneous scalar field : $\vec{\nabla}\phi = 0$

$\Rightarrow$ Friedmann - Robertson - Walker metric: $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -a^2(t) & & \\ & & -a^2(t) & \\ & & & -a^2(t) \end{pmatrix}$

$\Rightarrow$ the energy-momentum tensor, which can be derived from the Lagrangian for a scalar field as

$T^{\phi}_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\mathcal{L}^{\phi}$

takes then the simple form $T^{\phi\mu}{}_{\nu} = \begin{pmatrix} \rho(t) & & & \\ & -p(t) & & \\ & & -p(t) & \\ & & & -p(t) \end{pmatrix}$

with $\rho$ - energy density

$p$ - pressure

- the dynamics ~~are~~ now ~~be~~ determined by the equation of motions:

$\rightarrow \dfrac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Rightarrow \quad G_{\mu\nu} = \dfrac{8\pi}{M_{Pl}^2} T^{\phi}_{\mu\nu}$ (Einstein equation)

using the assumptions above

$$\boxed{\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3M_{Pl}^2}\rho}$$ ( Friedmann equation ~ FE )

$$\boxed{\frac{\ddot a}{a} = -\frac{4\pi}{3M_{Pl}^2}(\rho + 3p)}$$ ( Raychaudhurie equ. ~ RE )

[ here we have set the curvature $k$ to zero, because if inflation works, then it will drive the universe exponentially fast to a flat space ]

$\rightarrow \dfrac{\delta S}{\delta \phi} = 0 \quad \Rightarrow$

$$\boxed{\ddot\phi + 3H\dot\phi - \underbrace{\frac{\vec\nabla^2\phi}{a^2}}_{\to 0 \text{ homogenous field}} + V'(\phi) = 0}$$ ( EOM )

$\rightarrow$ the continuity equation can also be derived from the action, but it is not independent of (FE), (RE):

$\nabla_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \boxed{\dot\rho + 3\frac{\dot a}{a}(p+\rho) = 0}$ (CE)

$\rightarrow$ for the energy-momentum tensor we find:

$$\boxed{\begin{aligned} \rho &= \tfrac{1}{2}\dot\phi^2 + V(\phi) \\ p &= \tfrac{1}{2}\dot\phi^2 - V(\phi) \end{aligned}}$$ (EMT)

# 2. slow roll parameters

- recall:

→ the equation of state, defining the relation between $\rho$ and $p$ determines the evolution of the scale factor and depends of the kind of matter we add to our universe.

→ in general it is defined by

$$p = \omega \rho \xrightarrow{(CE),\,(FE)} \left(\frac{\dot a}{a}\right)^2 \propto t^{-3(1+\omega)}$$

→ de-Sitter universe ($\Lambda$-dominated):

$$p = -\rho \longrightarrow \left(\frac{\dot a}{a}\right)^2 = \text{const} \Rightarrow a \propto e^{Ht}$$

→ since $\omega = -1$ gives the required behaviour for inflation to happen, we demand this equation of state also for the inflaton (for early times):

$$(EMT) \Rightarrow p \triangleq -\rho \quad \text{if: } \underline{\underline{V(\phi) \gg \dot\phi^2}} \longleftrightarrow \text{"slow roll"} \longleftrightarrow a \propto e^{\int H(t)\,dt}$$

- more quantitatively:

$$\left.\begin{matrix}(FE)\\(RE)\end{matrix}\right\} \xrightarrow[\text{expressions for } \rho,\, p]{\text{insert}} \left\{\begin{aligned} H^2 &= \frac{8\pi}{3M_{pe}^2}\left(\frac{1}{2}\dot\phi^2 + V(\phi)\right) \\ \frac{\ddot a}{a} &= \frac{8\pi}{3M_{pe}^2}\left(\frac{1}{2}\dot\phi^2 + V(\phi)\right)(1-\varepsilon) = H^2(1-\varepsilon)\end{aligned}\right.$$

with the definition

$$\boxed{\varepsilon := \frac{3}{2}\left(\frac{p}{\rho}+1\right) = \frac{4\pi}{M_{pe}^2}\left(\frac{\dot\phi}{H}\right)^2}$$

→ this parameter is the first slow roll parameter. it is useful, because:

$\varepsilon > 1 \longleftrightarrow \ddot a < 0 \longleftrightarrow$ decelerated expansion

$\varepsilon = 1 \longleftrightarrow \ddot a = 0 \longleftrightarrow$ constant expansion

$\varepsilon < 1 \longleftrightarrow \ddot a > 0 \longleftrightarrow$ accelerated expansion

the de-Sitter limit ($p = -\rho$) is achieved for $\varepsilon \to 0$, therefore we require for inflation:

$$\varepsilon \ll 1$$ (inflation would start at very small $\varepsilon$ and ends at $\varepsilon = 1$)

→ the second slow roll parameter should also be small, ~~in order to~~ such that inflation holds on long enough (and $\varepsilon$ does not reach 1 too fast):

$$\boxed{\eta := -\frac{\ddot\phi}{H\dot\phi} \ll 1}$$

→ the number of e-folds, we need for infation to be an explanation for the Horizon & flatness problem:

$$\boxed{N = -\int H(t)\,dt \gtrsim 60} \longleftrightarrow a \propto e^{\int H dt} = e^{-N}$$

- it is convenient to express the conditions we impose on the ~~pot~~ scalar field (in order to explain flatness & horizon problem) in terms of the potential $V(\phi)$. then we could in principle take any potential satisfying the conditions and we receive sufficient inflation.

$$1 \gg \varepsilon = \frac{4\pi}{M_{pl}^2}\left(\frac{\dot\phi}{H}\right)^2 = \frac{M_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2$$

$$1 \gg \eta = -\frac{\ddot\phi}{H\dot\phi} = \frac{M_{pl}^2}{8\pi}\left[\frac{V''}{V} - \frac{1}{2}\left(\frac{V'}{V}\right)^2\right]$$

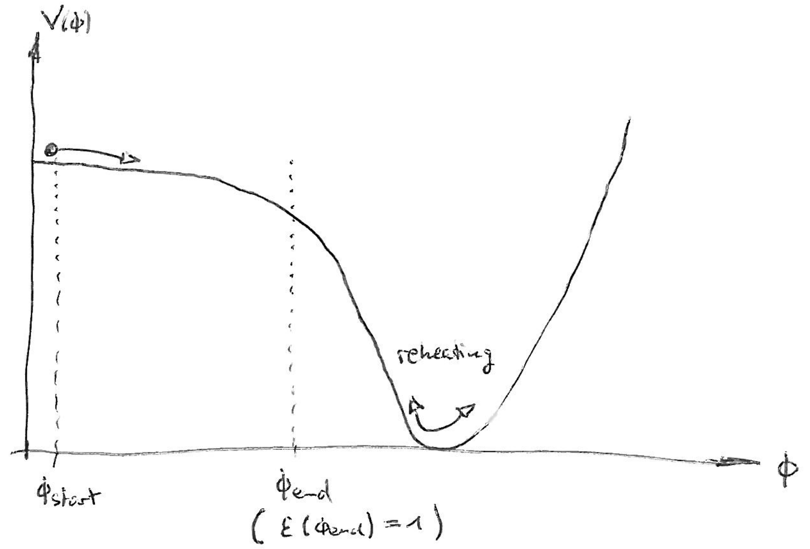
$$60 \lesssim N = -\int H\,dt = -\int_{\phi_{start}}^{\phi_{end}} \frac{H}{\dot\phi}\,d\phi = -\frac{8\pi}{M_{pl}^2}\int_{\phi_{start}}^{\phi_{end}} \frac{V(\phi)}{V'(\phi)}\,d\phi \quad \text{with } \phi_{end} \text{ from } \underline{\varepsilon(\phi_{end}) = 1}$$

[here, we use (FE) and (EOM) in the de-Sitter limit (i.e. $\varepsilon \to 0$) and $\eta \to 0$ }:

$$(FE) \longrightarrow H^2 = \frac{8\pi}{3M_{pl}^2} V$$

$$(EOM) \longrightarrow \dot\phi = -\frac{V'}{3H}$$
]

- this leads us to the general picture:




→ as time evolves, the field $\phi$ roles down the potential; as long as "slow roll" ($\varepsilon \ll 1$) is fulfilled, we have an quasi-exponential expansion

→ when $\phi$ reaches the minimum, it oscillates & decays into SM particles during this process the energy originally stored in the large potential is transformed into SM particles, including radiation

$\Rightarrow$ at this point the radiation dominated phase starts.

## 3. example: $V(\phi) = \lambda \phi^4$

- for such a potential we have

$$(\text{EOM}) \underset{\substack{\text{in}\\ \text{slow roll}\\ \text{limit}}}{\Longrightarrow} \quad \dot{\phi} = -\frac{V'}{3H} \overset{(\text{FE})}{\propto} \frac{V'}{\sqrt{V}} \propto \frac{\phi^3}{\phi^2} = \phi$$

$\Rightarrow$ even though the potential does not behave like in the schematic picture above, for very large initial values of $\phi$ the slow roll condition is satisfied:

$$V(\propto \phi^4) \overset{!}{\gg} \dot{\phi}^2 (\propto \phi^2)$$

- more quantitatively:

$$\varepsilon = \frac{M_{Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 = \frac{M_{Pl}^2}{\pi \phi^2} \longrightarrow 1 = \varepsilon(\phi_{end}) \Rightarrow \underline{\underline{\phi_{end} = \frac{M_{Pl}}{\sqrt{\pi}}}}$$

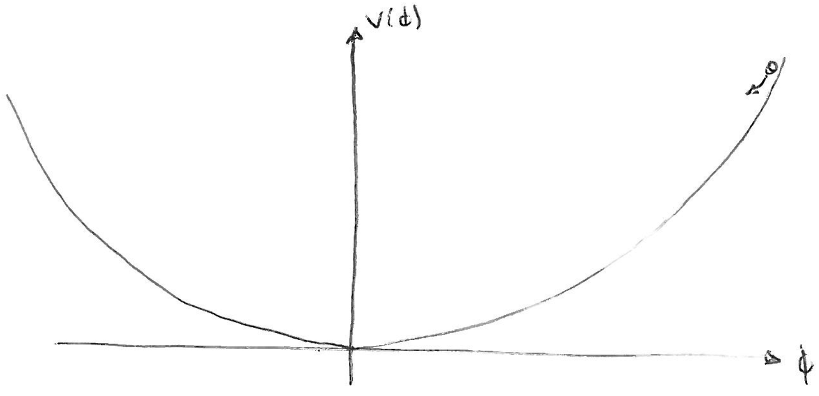
$$N = -\frac{8\pi}{M_{Pl}^2} \int_{\phi_N}^{\phi_{end}} \frac{V(\phi)}{V'(\phi)}\, d\phi = -\frac{2\pi}{M_{Pl}^2} \int_{\phi_N}^{\phi_{end}} \phi \, d\phi = \pi \frac{\phi_N^2}{M_{Pl}^2} - 1 \Rightarrow \underline{\underline{\phi_N = M_{Pl}\sqrt{\frac{N+1}{\pi}}}}$$

$\phi_N$ refers to the initial value of the field in order to have N e-folds

In order to have enough e-folds to explain flatness & horizon problems we set

$$60 \gtrsim N \longrightarrow \phi_{60} \gtrsim 4{,}4\, M_{Pl}$$

$\Rightarrow$ in this model we have inflation from $\phi_{start} \sim 4{,}4\, M_{Pl}$ to $\phi_{end} \sim 0{,}6\, M_{Pl}$, which is at very large $\phi$.
In order to avoid transplanckian values for the physical relevant quantity $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \sim \lambda \phi$ one can set the coupling $\lambda$ smaller than 1.

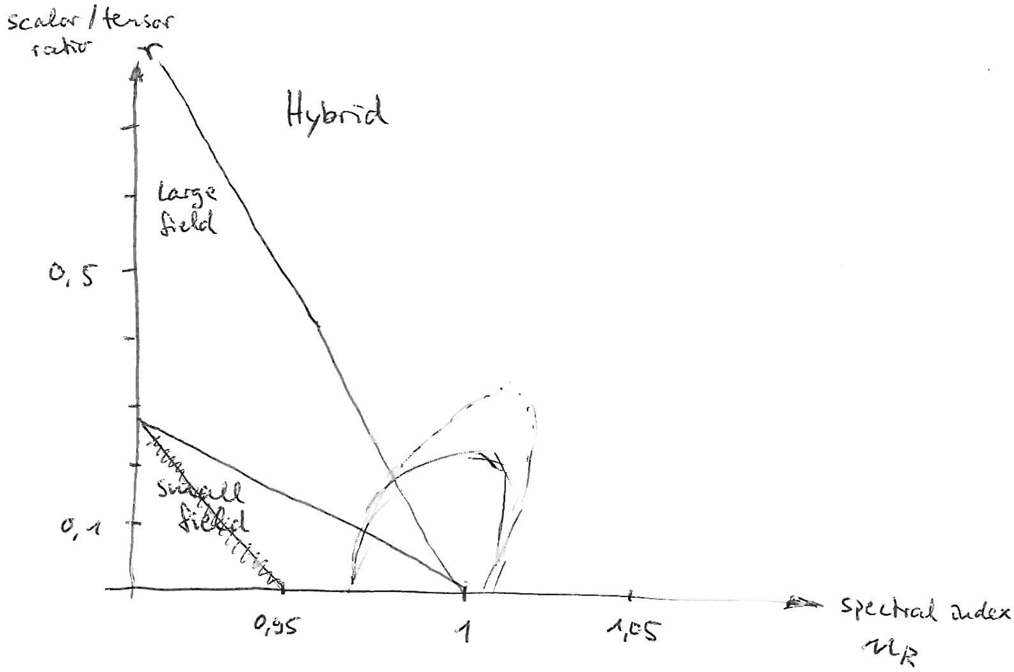


# 4. outlook: single scalar field models

- overview:

| | |
|---|---|
| large field<br>("chaotic inflation": $n=2,4$) | $V(\phi) = \lambda\phi^n$ ⇝ typical $\phi_i \gtrsim 3 M_{Pl}$<br>one free parameter: $\lambda$ |
| small field | $V(\phi) = V_0\left[1-\left(\frac{\phi}{\mu}\right)^n\right]$ ⇝ typical form of the potential from expanding around $\phi = 0$<br>⇝ such potentials arise from spontaneous symmetry breaking<br>$V_0 \sim M_{GUT}^4$, $\mu \sim M_{Pl}$; $\phi_i \lesssim 0{,}1\, M_{Pl}$ ⇒ 2 free parameter |
| Hybrid | one example from [arXiv: gr-qc/9306035]: $V(\phi,\chi) = \frac{\lambda}{4}\left(\chi^2 - \frac{M^2}{\lambda}\right)^2 + \frac{1}{2}g^2\phi^2\chi^2 + \frac{1}{2}m^2\phi^2$<br>→ now 2 fields, but is effectively 1-field inflation<br>→ start with large $\phi^2$ – then $\phi$ rolls down the potential (slow roll – inflation)<br>→ at $\phi < \phi_c := \frac{M}{g}$ the mass term of $\chi$ becomes negative<br>$\left(-\frac{1}{2}\chi^2 M^2 + \frac{1}{2}g^2\underbrace{\left(\frac{M}{g}\right)^2}_{\phi_c^2}\chi^2\right)$ ⇒ now $\chi$ starts to roll down towards $\pm\frac{M}{\sqrt{\lambda}}$<br>⇒ the fast rolling $\chi$ ends inflation<br>4 parameters: $g, \lambda, m, M$ |

- constraints in the plane of the two observables $r$, $n_R$ (schematic picture):




⇝ models with more free parameters are more difficult to exclude (as expected)

⇝ most of the large field models are already excluded. only $n=2$ (i.e. $V=\lambda\phi^2$) for number of e-folds $N \gtrsim 45$ ~~are~~ is still allowed