Introduction to Inflation

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1 Problems with Big Bang Cosmology

1.1 Horizon Problem

Assuming a radiation- or matter-dominated evolution down to $a = 0$ (Big Bang singularity), one finds that the light cones of many points we observe in the CMB do not intersect. There are almost a million "patches" of the sky that could not have been in causal contact with each other, yet share a surprisingly similar temperature (Fig. 1).

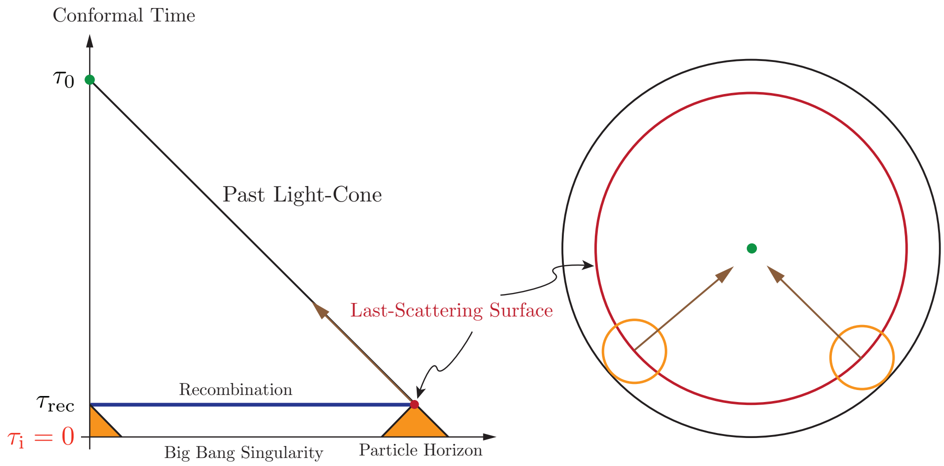

Figure 1: Conformal diagram of Big Bang cosmology. The observed CMB originated from the surface of last-scattering (recombination). The two yellow events are causally disconnected.

1.2 Flatness Problem

Why is our universe so flat, i.e. $\Omega(a)$ so close to unity? Formally we have

$$|1 - \Omega(a)| \sim \frac{1}{(aH)^2} \sim a^{1+3w}\,, \tag{1}$$

which diverges for matter dominated (MD) ($w = 0$) or radiation dominated (RD) ($w = 1/3$) evolution, so $\Omega(a)$ is driven away from flatness. This suggests fine-tuned initial conditions for Ω.

2 Solution via Inflation

Since MD and RD work against us, look for a qualitatively different solution to the Friedmann equations that takes over in the early universe. Already know how a cosmological constant Λ behaves:

$$a(t) \sim e^{Ht}\,, \qquad\qquad H \sim \sqrt{\rho_\Lambda}/M_{\text{Pl}}\,, \tag{2}$$

with equation of state $w = -1$. This solves the flatness problem (1), since $|1-\Omega(a)| \sim e^{-2Ht}$ gets driven to zero as we go forward in time. If inflation goes on for long enough, we can compensate an arbitrary initial Ω. Since horizon and flatness are linked, we are in principle done, for completeness sake we will however explicitly discuss the horizon problem. To that effect, we look at some comoving scale λ (e.g. the distance between two “stars”), which of course is independent of time (or a). To be in causal contact, the two points need to exchange photons, so we will compare λ to the comoving distance a light-ray can travel in the time a to $a+\delta$. In the beginning of inflation (at a_i), this is

$$d(a_i) = \int\limits_{a_i}^{a_i(1+\delta)} \mathrm{d}a\ \frac{1}{Ha^2} = \frac{1}{Ha_i} - \frac{1}{Ha_i(1+\delta)} \approx \frac{\delta}{Ha_i}\,, \tag{3}$$

whereas at the end of inflation (at $a_f \gg a_i$), this distance gets reduced to

$$d(a_f) \approx \frac{\delta}{Ha_f}\,. \tag{4}$$

So the “horizon” shrinks by a factor $d(a_f)/d(a_i) = a_i/a_f \ll 1$, which (as we will see later) is typically of order e^{-60} (see Refs. [1, 2]). That means two points that were in causal contact at the beginning of inflation do not necessarily see each other as inflation ends, at which point the usual RD evolution takes over and the horizon starts to grow again.[1]

So, during inflation, a scale k might transition from inside to outside the horizon (“horizon exit”) and enter it only much later during the RD or MD era. This means the scales that enter our horizon now could have been in causal contact before or during inflation, allowing the whole CMB to be at the same temperature (as long as inflation goes on for long enough). This is depicted in Fig. 2.

2.1 Conditions for Inflation

The deSitter solution $a \sim e^{Ht}$ is not the sole inflationary model, from Eq. (1) we see that an equation of state with $1+3w<0$ solves the flatness problem (and also the horizon problem). This can be expressed in numerous ways:

$$w < -\frac{1}{3}\,, \qquad \frac{\mathrm{d}^2 a(t)}{\mathrm{d}t^2} > 0\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0\,. \tag{5}$$

[1] One can also argument with physical distances just by multiplying the comoving distances by the scale factor. In this point of view, the distance between the two points grows exponentially during inflation, while the horizon is constant.

In analogy to the deSitter case, we define the number of e-folds N via

$$a(t) \sim \exp\left(\int \mathrm{d}t \; H\right) = e^{-N(t)} \,, \tag{6}$$

i.e. $\mathrm{d}N = -H\mathrm{d}t = -\mathrm{d}\ln a$. This sign convention counts the number of e-folds till the end of inflation [3], so $N(t_f) = 0$ and $N(t_i) \sim 60$. This allows us to define yet another inflationary condition:

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\mathrm{d}\ln H}{\mathrm{d}N} \overset{!}{<} 1 \,. \tag{7}$$

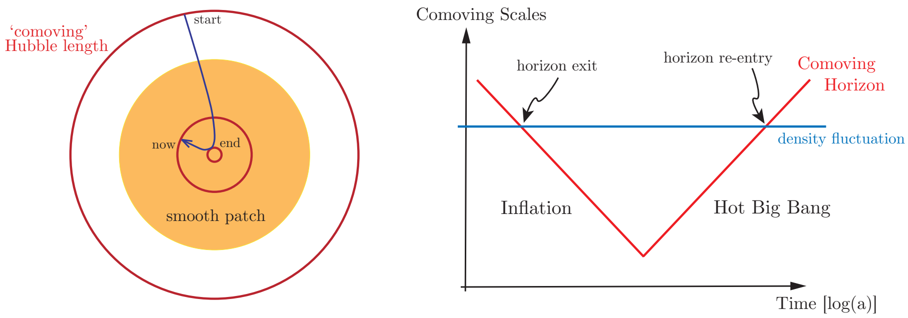

Figure 2: Solution of the horizon problem: Scales (here density fluctuations) exit the horizon during inflation and re-enter it much later during the RD or MD era (from Ref. [4]).

2.2 How much Inflation is enough?

We will now briefly derive a bound on the number of e-folds N. Discussions based on different methods (e.g. entropy conservation) can be found in the first four references, we will only give a simplified argument.

To solve the horizon problem, the largest scales observed today $\lambda(t_0) \approx 1/H_0$ should be within the horizon at the beginning of inflation, i.e.

$$\frac{1}{a_0 H_0} < \frac{1}{a_i H_i} \,, \tag{8}$$

which can be rewritten as

$$\frac{1}{H_0}\frac{a_f}{a_0}\frac{a_i}{a_f} < \frac{1}{H_i} \,, \tag{9}$$

where $a_i/a_f = \exp(-N)$. Since the temperature of photons drops with $T \sim 1/a$, we can express $a_f/a_0 = T_0/T_f$ through the CMB temperature today T_0 and the temperature after reheating T_f. Solving for N, we find

$$N > \ln\left(T_0/H_0\right) + \ln\left(H_i/T_f\right) \approx 67 + \ln\left(H_i/T_f\right) \,. \tag{10}$$

The limits on N are always dependent on the energy scale of inflation, one typically assumes (motivated for example by GUT and actually constrained through the tensor/scalar ratio) $H_i \sim 10^{15}\,\mathrm{GeV}$. The reheating temperature depends typically on the decay rate of the inflaton, but is of similar order $T_f \sim 10^8 - 10^{12}\,\mathrm{GeV}$, so the second term in Eq. (10) will be of order $\mathcal{O}(10)$.

2.3 End of Inflation

We have yet to specify how inflation comes to an end, allowing for the RD and MD universe we observe. Taking the deSitter solution $a \sim e^{-Ht}$ with $H \sim \sqrt{\rho_\Lambda}/M_{\rm Pl}$, we see that we need a cosmological "constant" that goes to zero as time goes on. The simplest guess would be to replace the constant ρ_Λ with the potential energy density $V(\phi)$ of some field ϕ that goes to zero at a_f: $V(\phi(a_f)) = 0$. To introduce such a field, one has to take the kinetic energy into account as well, leading to the classical inflaton field.

3 The Inflaton Field

A scalar field ϕ, minimally coupled to gravity has the action[2]

$$S = \int \mathrm{d}^4x \, \sqrt{-g} \left[\frac{1}{2} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] , \tag{11}$$

leading to the energy-momentum tensor of a perfect fluid with

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} , \tag{12}$$

which can take either sign and value, depending on $\dot{\phi}^2$ and $V(\phi)$. Inflation requires $w_\phi < -1/3$, so the potential has to dominate over the kinetic energy at early times. Ignoring the position-dependence of ϕ, we get the dynamics:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 , \qquad H^2 = \frac{1}{3} \left(\frac{1}{2} \dot{\phi}^2 + V \right) . \tag{13}$$

This gives the slow-roll parameter $\varepsilon = \frac{1}{2} \frac{\dot{\phi}^2}{H^2}$ and for $\dot{\phi}^2 \ll V$ we obtain the desired quasi deSitter solution $H \approx \sqrt{V(\phi)}$. If V changes only slowly, this gives inflation, stopping as the kinetic energy catches up and the condition $w_\phi < -1/3$ is violated.

We define a second slow-roll parameter that is connected to the duration of inflation:

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} = \varepsilon - \frac{1}{2\varepsilon} \frac{\mathrm{d}\varepsilon}{\mathrm{d}N} , \tag{14}$$

a small $|\eta| \ll 1$ just ensures long inflation.

More convenient are slow-roll parameters that only depend on the form of the potential

$$\varepsilon_V \equiv \frac{1}{2} \left(\frac{V_{,\phi}}{V} \right)^2 , \qquad \eta_V \equiv \frac{V_{,\phi\phi}}{V} , \tag{15}$$

slow-roll conditions are again $\varepsilon_V, |\eta_V| \ll 1$, and we have in the slow-roll regime the connection to the old parameters $\varepsilon \approx \varepsilon_V$ and $\eta \approx \eta_V - \varepsilon_V$. Inflation ends when $\varepsilon(\phi_f) \approx 1$, the number

[2] We set the reduced Planck mass $M_{\rm Pl} = 1/\sqrt{8\pi G}$ to one from here on out.

of e-folds between ϕ and ϕ_f is

$$N(\phi) = -\int \mathrm{d}t\ H = -\int \mathrm{d}\phi\ \frac{H}{\dot{\phi}} \approx -\int_{\phi}^{\phi_f} \mathrm{d}\phi\ \frac{V}{V_{,\phi}}\,. \tag{16}$$

The fluctuations in the CMB are created roughly $40 - 60$ e-folds before inflation ended, implicitly giving the field value ϕ_{CMB} via

$$\int_{\phi_f}^{\phi_{\mathrm{CMB}}} \mathrm{d}\phi\ \frac{V}{V_{,\phi}} \approx 40 - 60\,. \tag{17}$$

3.1 What is left to do?

Reheating How does the energy of the inflaton gets converted into radiation at the end of inflation, i.e. how does ϕ couple to the standard model? Highly model dependent, we will not get into it (see for example Ref. [5]).

Potential We have to choose a potential V and see if long-enough inflation is possible and how the parameters of the potential can be constrained by cosmology. Suitable potentials are

- Small field inflation: $V \sim (1 - (\phi/\mu)^p)$ (Fig. 3 (left)) or Coleman-Weinberg-type: $V \sim \phi^4 \ln \phi$
- Large field inflation (chaotic inflation): $V \sim \phi^p$ (Fig. 3 (right))
- Natural inflation: $V \sim 1 + \cos(\phi/f)$ (inflaton $\widehat{=}$ axion)
- Hybrid inflation: models with a non-zero potential-value at the minimum, e.g. $V \sim (\phi^2 + \mu^2)^2$ (see Refs. [6, 7])

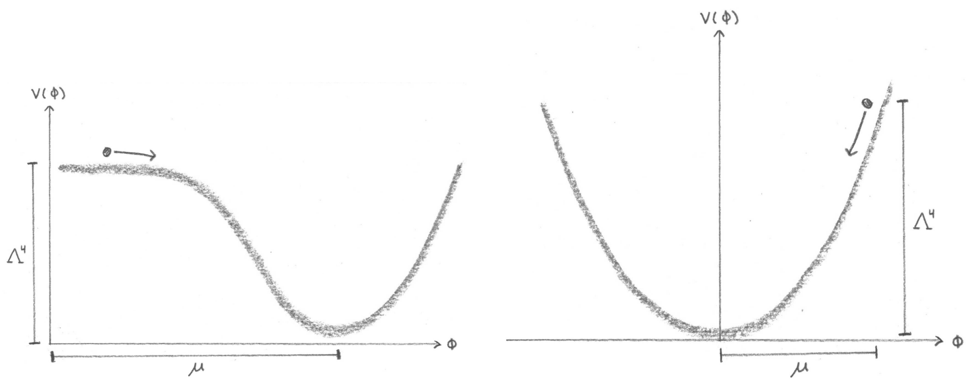

Figure 3: Small field (left) and large field (right) potentials for inflation (from Ref. [3]).

3.2 Case Study: $V \sim \phi^2$

Since the large field potential is not obviously a solution (how is it slow-roll?), we will consider it as a simple example and explicitly derive the conditions for inflation. Taking

$V(\phi) = \frac{1}{2}m^2\phi^2$, we obtain the slow-roll parameter

$$\varepsilon_V(\phi) = \eta_V(\phi) = \frac{2}{\phi^2}\,, \tag{18}$$

so starting with high field values (larger than $M_{\rm Pl}$!) gives indeed inflation, which ends as $\phi \to \sqrt{2}$. So the number of e-folds is

$$N(\phi) \approx -\int_{\phi}^{\phi_f=\sqrt{2}} \mathrm{d}\phi\, \frac{V}{V_{,\phi}} = \frac{1}{4}\phi^2 - \frac{1}{2}\,, \tag{19}$$

so for CMB we need field values $\phi_{\rm CMB} \approx 2\sqrt{40-60} \approx 13-16$. Constraints on m^2 come from the power spectrum (next lecture).

4 Conclusion

A sufficiently long period of accelerated expansion can solve the horizon- and the flatness-problem, i.e. replace fine-tuned initial conditions by a quite general dynamical process. The actual realization of the inflaton is highly model dependent, even for a single inflaton field there are a lot of valid possible potentials. Allowing for more fields or a composite inflaton further expands the landscape.

In the next talks we will see that inflation is falsifiable; it makes model-dependent predictions about e.g. density fluctuations. Even though inflation is designed to explain the smoothness of the CMB, it also sheds light on the small inhomogeneities (as a proper quantum mechanical treatment of the inflaton shows).

References

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