(...) an invariance of the theory under interchange of fermions and bosons[9]

Introduction to Supersymmetry

$$\boxed{\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2(\sigma^m)_{\alpha\dot{\alpha}} P_m}$$

Contents

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1 Spinors in four dimensional space time

- Group theory classifies all particles as members of irreducible representations of the underlying symmetry group, i.e. particles are classified according to their transformation law under the symmetry group [1].
- symmetry group of four dimensional space time: Lorentz group $SO(3,1)$ (translations not involved; $\rightarrow$ Poincaré group)
- In what representation of the Lorentz group live fermions?

1.1 Clifford Algebra

- $\{\gamma^\mu\,,\gamma^\nu\} = 2\,g^{\mu\nu}\,\mathbf{1}$; we use the signature $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}\,(-,+,+,+)$.
- special four dimensional representation using (hermitian) Pauli σ matrices: $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}$
  Attention: $\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in the given metric $g^{\mu\nu}$.

1.2 Dirac Spinor Representation of $SO(3,1)$

- $J^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu,\gamma^\nu] = \frac{i}{4}\begin{pmatrix} \sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu & 0 \\ 0 & \overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu \end{pmatrix}$ satisfies Lie Algebra of SO(3,1):
  $[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\nu} + \eta^{\mu\sigma}J^{\nu\rho})$.
  Remark$^{\circledast}$: In all representations of the Lorentz group, the corresponding generators are Bose type symmetry generators, i.e. they act on particles of equal statistics [7].
- Transformation law of a Dirac Spinor $(\Psi_D)_a(x^\mu)$ $a = 1,2,3,4$ [2]:

$$(\Psi_D)_a(x^\mu) \xrightarrow{\Lambda} S(\Lambda)_{ab}(\Psi_D)_b((\Lambda^{-1})_\mu{}^\nu\;\; x^\nu)$$

  with a general Lorentz transformation $\Lambda = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$ and corresponding spinor transformation $S(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$.
  Avoid confusion: A generic element Λ of the Lie group $SO(3,1)$ can always be written as $\Lambda = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$ with $M^{\mu\nu}$ being the generators of the corresponding Lie algebra. $S(\Lambda)$ is a special representation of Λ, the spinor representation, for which the generators are $J^{\mu\nu}$.
- Dirac spinor representation completely reducible because generators $J^{\mu\nu}$ are diagonal.
  Get irreducible representation with the known left- and right-handed projection opertors P_L and P_R:

$$(\Psi_D)_a = \begin{pmatrix} P_L\,(\Psi_D)_a \\ P_R\,(\Psi_D)_a \end{pmatrix} \equiv \begin{pmatrix} \Psi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}.$$

- Ψ_α ; $\alpha = 1,2$ left-handed Weyl spinor ($(\frac{1}{2},0)$ representation of Lorentz group)
$\overline{\chi}^{\dot\alpha}$; $\dot\alpha = 1,2$ right-handed Weyl spinor ($(0,\frac{1}{2})$ representation of Lorentz group)

1.3 Transformation of Weyl Spinors [4]

- v $\in \mathbb{R}^4 \rightarrow \hat{\mathbf{v}} = v_m \sigma^m$ hermitian
- $\mathbf{M} \in SL(2,\mathbb{C}) \rightarrow \hat{\mathbf{v}}' = \mathbf{M}\hat{\mathbf{v}}\mathbf{M}^\dagger$ is hermitian: expansion in σ matrices $\rightarrow$ $\hat{\mathbf{v}}' = v'_m \sigma^m$
$\rightarrow v'^2 = \det\ \hat{\mathbf{v}}' = \det\ \hat{\mathbf{v}} = v^2$, because $\det \mathbf{M} = 1$.
Result: Any $\mathbf{M} \in SL(2,\mathbb{C})$ defines via $\hat{\mathbf{v}} \rightarrow \hat{\mathbf{v}}' = \mathbf{M}\hat{\mathbf{v}}\mathbf{M}^\dagger$ a Lorentz transformation $v_m \rightarrow v'_m = \Lambda_m^{\ n}\ v_n$.
- Weyl spinors transform under $SL(2,\mathbb{C})$
(Weyl spinor has complex components.):

$$\Psi_\alpha \rightarrow \Psi'_\alpha = M_\alpha^{\ \beta}\ \Psi_\beta \ \text{ and } \ \overline{\chi}^{\dot\alpha} \rightarrow \overline{\chi}^{\dot\alpha'} = (M^{*-1})_{\dot\beta}^{\ \dot\alpha}\ \overline{\chi}^{\dot\beta} \quad \circledast\circledast$$

- index zoo:
Lowering and raising indices with antisymmetric tensor $\epsilon = (\epsilon^{\alpha\beta}) = i\sigma_2$:

$$\Psi_\alpha = \epsilon_{\alpha\beta}\Psi^\beta \ \text{ and } \ \overline{\chi}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\overline{\chi}_{\dot\beta} \ .$$

Observe from $\hat{\mathbf{v}}' = \mathbf{M}\hat{\mathbf{v}}\mathbf{M}^\dagger = v_m \mathbf{M}\sigma^m\mathbf{M}^\dagger$ and $\circledast\circledast$ that the index structure of the Pauli σ matrices is given as: $(\sigma^m)_{\alpha\dot\alpha}$ and $(\overline{\sigma}^m)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^m)_{\beta\dot\beta}$ with latin letters as space time indices and greek letters as spinor indices. Attention to dotted and undotted spinor indices: $\Psi^\dagger_\alpha = \overline{\Psi}_{\dot\alpha}$ and $\overline{\chi}^{\dot\alpha\dagger} = \chi^\alpha$.
Kinetic term for spinor

$$i\overline{\Psi}\not\partial\Psi = i\overline{\Psi}_{\dot\alpha}(\overline{\sigma}^m)^{\dot\alpha\beta}\partial_m\Psi_\beta + i\chi^\alpha(\sigma^m)_{\alpha\dot\beta}\partial_m\overline{\chi}^{\dot\beta}$$

results in equation of motion for a massless left-handed Weyl spinor: $i(\overline{\sigma}^m)^{\dot\alpha\beta}\partial_m\Psi_\beta = 0$.

2 ($\mathcal{N}$ =1) Supersymmetry Algebra: From Coleman-Mandula to Haag-Lopuszanski-Sohnius

- Coleman-Mandula no-go theorem for *(...) an invariance of the theory under interchange of fermions and bosons* : Any symmetry group (for which the commutator is the bilinear operation in the Lie algebra) of a quantum field theory (QFT) is locally isomorphic to the direct product of an internal symmetry group, i.e. a group for which the generators have no matrix elements between particles of different spin, and the Poincaré group [6].
$\stackrel{\circledast}{\longrightarrow}$ Symmetry between fermions and bosons impossible?

- Bypass Coleman-Mandula theorem: no-go theorem holds for Lie algebras with commutator as bilinear operation. Consider graded Lie algebras whose generators obey commutator and anticommutator relations.
- new generator ($\mathcal{N}$ =1): Q_α and its hermitian conjugate $\overline{Q}_{\dot{\alpha}}$
- Consider the following graded Lie algebra[1]:
  - vanishing anticommutators (due to spinor character): $\{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0$
  - transformation under Lorentz group: $$[M^{mn}, Q_\alpha] = \underbrace{\frac{i}{4}(\sigma^m\overline{\sigma}^n - \sigma^n\overline{\sigma}^m)_\alpha{}^\beta \; Q_\beta}_{\text{transformation of left-handed Weyl spinor}}$$ Q_α in $(\frac{1}{2}, 0)$ representation, $\overline{Q}_{\dot{\alpha}}$ in $(0, \frac{1}{2})$ representation of Lorentz group
  - All operators should commute with space time translations for energy momentum conversation: $[P^m, Q_\alpha] = 0$.
  - The new anticommutator: $\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2(\sigma^m)_{\alpha\dot{\alpha}} P_m$
    Motivation: $Q_\alpha \overline{Q}_{\dot{\alpha}}$ transforms in $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group what is the spinor description of a four vector that has to be proportional to P^m for energy momentum conversation. The only objects which have space time and spinor indices in their structures are the Pauli σ matrices.
    With $(\overline{\sigma}^m)^{\dot{\alpha}\alpha} \cdot (\sigma^m)_{\alpha\dot{\alpha}} = -4$ the new anticommutator takes on the form $P^m = -\frac{1}{8}(\overline{\sigma}^m)^{\dot{\alpha}\alpha}\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\}$ and we find
    $H = P^0 \underset{\sigma^0 = -\mathbf{1}_{2\times 2}}{\overset{\uparrow}{=}} + \frac{1}{8}\left(Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2\right)$, what shows that the expectation value of the Hamiltonian H is not negative in any state $|\eta\rangle$ constructed by applying creation operators, which are proportional to $\overline{Q}_{\dot{\alpha}}$, to the vaccum state $|\Omega\rangle$: $\langle\eta|H|\eta\rangle \geqq 0$;
    $|\Omega\rangle$ is defined as usual: $Q_\alpha|\Omega\rangle = 0$ with annihilation operators being proportional to Q_α [4].
- Haag, Lopuszanski and Sohnius: The given graded Lie algebra is the unique graded Lie algebra of symmetries consistent with QFT.
  $\rightarrow$ ($\mathcal{N}$ =1) Supersymmetry Algebra [7]
- (global) infinitesimal supersymmetry transformation:

$$\delta_\xi = \xi^\alpha Q_\alpha - \overline{\xi}^{\dot{\alpha}} \overline{Q}_{\dot{\alpha}} = \xi^\alpha Q_\alpha \underset{\text{indices}}{\overset{}{+}} \overline{\xi}_{\dot{\alpha}} \overline{Q}^{\dot{\alpha}}$$

  with spinor parameter ξ.
  The commutator of two supersymmetry transformations leads to space time translations ($P_m = i\partial_m$):

[1] Action S invariant under supersymmetry transformation $\rightarrow$ get supercurrents J_α^m after applying Noether's theorem $\rightarrow$ supercharges $Q_\alpha = \int d^3x\, J_\alpha^0$ fulfill the given graded Lie algebra [4].

$$[\delta_{\xi_1}, \delta_{\xi_2}] = 2i \left(\xi_1^\alpha \;\; (\sigma^m)_{\alpha\dot\beta} \;\; \overline{\xi}\,_2^{\dot\beta} - \xi_2^\beta \;\; (\sigma^m)_{\beta\dot\alpha} \;\; \overline{\xi}\,_1^{\dot\alpha} \right) \partial_m \quad .$$

Remark: ξ does not depend on space time, i.e. we consider only global supersymmetry. The theory of local supersymmetry transformations with $\xi = \xi(x^m)$, called Supergravity, is beyond the scope of this introduction.

- Find a representation of this algebra on four dimensional space time to describe a QFT !

3 Chiral Field Component Representation

- theory of a complex scalar z and a left-handed Weyl spinor Ψ_α:
  $S = \int d^4x \;\; (-\partial_m \overline{z}\, \partial^m z - i\overline{\Psi}\, \overline{\sigma}^m \partial_m \Psi)$
- supersymmetry transformations: For a supersymmetric theory the action S has to be invariant under a supersymmetry transformation δ_ξ, $\delta_\xi S \overset{!}{=} 0$. Therefore we have to define the following transformations:

$$\begin{aligned} \delta_\xi z &\equiv \sqrt{2}\, \xi^\alpha \Psi_\alpha & \delta_\xi \overline{z} &\equiv \sqrt{2}\, \overline{\xi}\,^{\dot\alpha}\, \overline{\Psi}_{\dot\alpha} \\ \delta_\xi \Psi_\alpha &\equiv \sqrt{2}\, i (\sigma^m)_{\alpha\dot\alpha} \overline{\xi}\,^{\dot\alpha} \partial_m z & \delta_\xi \overline{\Psi}_{\dot\alpha} &\equiv -\sqrt{2}\, i \xi^\alpha (\sigma^m)_{\alpha\dot\alpha} \partial_m \overline{z} \end{aligned}$$

  Observe: In the first row bosons are transformed into fermions, in the second row fermions are transformed into bosons.

- Consistency: The given definitions above must close the supersymmetry algebra. Problem for left-handed Weyl spinor:

$$\begin{aligned} [\delta_{\xi_1}, \delta_{\xi_2}] \Psi_\alpha &= \delta_{\xi_1} \sqrt{2}\, i (\sigma^m)_{\alpha\dot\alpha} \overline{\xi}\,_2^{\dot\alpha} \partial_m z - \delta_{\xi_2} \sqrt{2}\, i (\sigma^m)_{\alpha\dot\alpha} \overline{\xi}\,_1^{\dot\alpha} \partial_m z \\ &= \sqrt{2}\, i (\sigma^m)_{\alpha\dot\alpha} \overline{\xi}\,_2^{\dot\alpha} \partial_m \sqrt{2}\, \xi_1^\beta \Psi_\beta - \sqrt{2}\, i (\sigma^m)_{\alpha\dot\alpha} \overline{\xi}\,_1^{\dot\alpha} \partial_m \sqrt{2}\, \xi_2^\beta \Psi_\beta \\ &= 2i \left((\sigma^m)_{\alpha\dot\alpha} \overline{\xi}\,_2^{\dot\alpha} \xi_1^\beta \partial_m - (\sigma^m)_{\alpha\dot\alpha} \overline{\xi}\,_1^{\dot\alpha} \xi_2^\beta \partial_m \right) \Psi_\beta \\ &\overset{(\star)}{=} 2i \left(-\xi_1^\alpha (\sigma^m)_{\beta\dot\beta} \overline{\xi}\,_2^{\dot\beta} \partial_m \Psi_\beta + \xi_1^\beta (\sigma^m)_{\beta\dot\beta} \overline{\xi}\,_2^{\dot\beta} \partial_m \Psi_\alpha - (1 \leftrightarrow 2) \right) \\ &\overset{(\star\star)}{=} 2i \left(\overline{\xi}\,_2^{\dot\beta} (\overline{\sigma}\,^m)^{\dot\beta\beta} \xi_1^\alpha \partial_m \Psi_\beta + \xi_1^\beta (\sigma^m)_{\beta\dot\beta} \overline{\xi}\,_2^{\dot\beta} \partial_m \Psi_\alpha - (1 \leftrightarrow 2) \right) \\ &= 2i \left(-\xi_1^\alpha \overline{\xi}\,_2^{\dot\beta} (\overline{\sigma}\,^m)^{\dot\beta\beta} \partial_m \Psi_\beta + \xi_1^\beta (\sigma^m)_{\beta\dot\beta} \overline{\xi}\,_2^{\dot\beta} \partial_m \Psi_\alpha - (1 \leftrightarrow 2) \right) \\ &= 2i \left(\xi_1^\beta \;\; (\sigma^m)_{\beta\dot\beta} \;\; \overline{\xi}\,_2^{\dot\beta} - \xi_2^\beta \;\; (\sigma^m)_{\beta\dot\beta} \;\; \overline{\xi}\,_1^{\dot\beta} \right) \partial_m \Psi_\alpha \\ &- 2i \left(\xi_1^\alpha \overline{\xi}\,_2^{\dot\beta} (\overline{\sigma}^m)^{\dot\beta\beta} \partial_m \Psi_\beta - \xi_2^\alpha \overline{\xi}\,_1^{\dot\beta} (\overline{\sigma}^m)^{\dot\beta\beta} \partial_m \Psi_\beta \right) \end{aligned}$$

The algebra closes only for Ψ_β fulfilling the equation of motion (eom): $(\overline{\sigma}^m)^{\dot\alpha\beta} \partial_m \Psi_\beta = 0$.
($\star$) Use Fierz rearrangement idendity:
$\chi_\alpha(\xi\varsigma) = -\xi_\alpha(\varsigma\chi) - \varsigma_\alpha(\chi\xi)$; α fixed, not summed
with $\chi = (\sigma^m)_{\alpha\dot\alpha} \overline{\xi}\,_2^{\dot\alpha}$, $\xi = \xi_1^\beta$ and $\varsigma = \partial_m \Psi_\beta$.
($\star\star$) Use $\chi \sigma^m \xi^\dagger = -\xi^\dagger \overline{\sigma}\,^m$.

- Counting the degrees of freedom (dof): Without use of the eom, there are two independent complex dof for the Weyl spinor, i.e. four independent real fermion dof. For the complex scalar there are two independent real boson dof.
  Same amount of fermion and boson dof in one representation → addition of two real boson dof such that no dof are added with the use of the eom → auxiliary (no kinetic terms) field F which transforms into equation of motion of spinor:

$$\delta_\xi F \equiv \sqrt{2} i \overline{\xi}_{\dot\alpha} (\overline{\sigma}^m)^{\dot\alpha\alpha} \partial_m \Psi_\alpha$$
$$\delta_\xi \Psi_\alpha \equiv \sqrt{2} i (\sigma^m)_{\alpha\dot\alpha} \overline{\xi}^{\dot\alpha} \partial_m z + \sqrt{2} \xi_\alpha F$$

  $S = \int d^4x \ \left(-\partial_m \overline{z}\, \partial^m z - i\overline{\Psi}\, \overline{\sigma}^m \partial_m \Psi + F\, \overline{F}\right)$ is supersymmetric invariant without use of eom.

- z, Ψ_α and F are the components of the chiral multiplet.

4 Superspace

- Supersymmetric transformations on the chiral multiplet in accordance with $\delta_\xi S = 0$.
  Reproduce these transformations from the general supersymmetry transformation $\delta_\xi = \xi^\alpha Q_\alpha - \overline{\xi}^{\dot\alpha} \overline{Q}_{\dot\alpha}$.

- Look at the anticommutator:

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2(\sigma^m)_{\alpha\dot\alpha} P_m$$

  On the left hand side, we have the product of two supersymmetry transformations generated by Q_α and $\overline{Q}_{\dot\alpha}$ which result on the right hand side in translations in four dimensional space time.
  Idea: Construct a space on which Q_α and $\overline{Q}_{\dot\alpha}$ generate translations. Due to the spinor character of Q_α and $\overline{Q}_{\dot\alpha}$ this space must have in additon to the space time coordinates two anticommuting coordinates Θ_α and $\overline{\Theta}_{\dot\alpha}$. This space with coordinates $(x^m, \Theta_\alpha, \overline{\Theta}_{\dot\alpha})$ is called superspace.

4.1 Representation of supersymmetry generators

- On superspace we define $\partial_\alpha \equiv \frac{\partial}{\partial \Theta^\alpha}$ and represent Q_α and $\overline{Q}_{\dot\alpha}$ as translation operators [4]:

$$Q_\alpha \equiv \partial_\alpha - i(\sigma^m)_{\alpha\dot\alpha} \overline{\Theta}^{\dot\alpha} \partial_m$$
$$\overline{Q}_{\dot\alpha} \equiv -\overline{\partial}_{\dot\alpha} + i\Theta^\alpha (\sigma^m)_{\alpha\dot\alpha} \partial_m \quad .$$

  → Supersymmetry algebra motivated in section 2 justified:
  $\{Q_\alpha, Q_\beta\} = 0$ and $\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2(\sigma^m)_{\alpha\dot\alpha} P_m$

4.2 Superfields

- General complex superfield $G(x^m, \Theta_\alpha, \overline{\Theta}_{\dot\alpha})$ with supersymmetry transformation
$\delta_\xi G = \left(\xi^\alpha Q_\alpha + \overline{\xi}_{\dot\alpha} \overline{Q}^{\dot\alpha}\right) G$.

- actions for superfields → How to integrate over superspace?
The integration of a function $g = g(\Theta)$ depending on an anticommuting number Θ is the same operation as differentiation with respect to Θ: $\int d\Theta^1 \widehat{=} \partial_1$.
fact: $\partial^2\Theta^2 = -4$ [3]
normalization: $\int d^2\Theta = -\frac{1}{4}\partial^2 \to \int d^2\Theta \;\; \Theta^2 = 1$.
action: $S = \int d^4x\, d^2\Theta\, d^2\overline{\Theta} \;\; G(x, \Theta, \overline{\Theta}) = \frac{1}{16}\int d^4x \partial^2\overline{\partial}^2 \;\; G(x, \Theta, \overline{\Theta})$

- achievment: action S for general superfield G is automatically supersymmetric invariant:

$$
\begin{aligned}
\delta_\xi S &= \int d^4x\, d^2\Theta\, d^2\overline{\Theta} \;\; \left(\xi^\alpha Q_\alpha - \overline{\xi}^{\dot\alpha}\overline{Q}_{\dot\alpha}\right) G \\
&= \int d^4x\, d^2\Theta\, d^2\overline{\Theta} \left(\underbrace{-\partial_\alpha(\xi^\alpha G) - \overline{\partial}_{\dot\alpha}(\overline{\xi}^{\dot\alpha} G)}_{\text{total derivative in } \Theta \text{ and } \overline{\Theta}} \;\; \underbrace{-\, i(\xi\sigma^m\overline{\Theta} - \Theta\sigma^m\overline{\xi})\partial_m G}_{\text{total derivative in } x} \right) \\
&= 0 \;\; .
\end{aligned}
$$

- components of a general superfield G
→ How to differentiate on superspace [4]?

$$
\begin{aligned}
D_\alpha &\equiv \partial_\alpha + i(\sigma^m)_{\alpha\dot\alpha}\overline{\Theta}^{\dot\alpha}\partial_m \\
\overline{D}_{\dot\alpha} &\equiv -\overline{\partial}_{\dot\alpha} - i\Theta^\alpha(\sigma^m)_{\alpha\dot\alpha}\partial_m
\end{aligned}
$$

with $\{D_\alpha, \overline{D}_{\dot\alpha}\} = -2i(\sigma^m)_{\alpha\dot\alpha}\partial_m\,;\; \{D_\alpha, Q_\beta\} = \{\overline{D}_{\dot\alpha}, Q_\beta\} =$
$= \{\overline{D}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0$ and $(Q_\alpha G)|_{\Theta=\overline{\Theta}=0} = (D_\alpha G)|_{\Theta=\overline{\Theta}=0}$.
Attention: $\partial^*_\alpha = -\overline{\partial}_{\dot\alpha}$ [5].

- Define components of G via derivatives [3]:

$$
\begin{aligned}
&C = G|_{\Theta=\overline{\Theta}=0} && \overbrace{(\sigma^m)_{\alpha\dot\alpha}A_m = -\frac{1}{2}[D_\alpha, \overline{D}_{\dot\alpha}]G|_{\Theta=\overline{\Theta}=0}}^{\text{components of vector multiplet}} \\
&\Psi_\alpha = \tfrac{1}{\sqrt{2}}D_\alpha G|_{\Theta=\overline{\Theta}=0} && \lambda_\alpha = -\tfrac{i}{4}\overline{D}^2 D_\alpha G|_{\Theta=\overline{\Theta}=0} \\
&\overline{\Psi}_{\dot\alpha} = \tfrac{1}{\sqrt{2}}\overline{D}_{\dot\alpha}G|_{\Theta=\overline{\Theta}=0} && \overline{\lambda}_{\dot\alpha} = -\tfrac{i}{4}D^2\overline{D}_{\dot\alpha}G|_{\Theta=\overline{\Theta}=0} \\
&F = -\tfrac{1}{4}D^2 G|_{\Theta=\overline{\Theta}=0} && \tilde{D} = \tfrac{1}{8}D^\alpha\overline{D}^2 D_\alpha G|_{\Theta=\overline{\Theta}=0} \\
&\overline{F} = -\tfrac{1}{4}\overline{D}^2 G|_{\Theta=\overline{\Theta}=0} &&
\end{aligned}
$$

Higher derivatives vanish due to anticommuting variables and other orderings of derivatives not independent of the chosen ones.

- Use $\{\overline{D}_{\dot\alpha}, Q_\beta\} = 0$ to show $\overline{D}_{\dot\alpha}\,\delta_\xi G = \delta_\xi \overline{D}_{\dot\alpha} G$.
  If $\overline{D}_{\dot\alpha} G = 0$, then $\overline{D}_{\dot\alpha}\,(\delta_\xi G) = 0$.
  $\to$ All superfields G with $\overline{D}_{\dot\alpha} G = 0$ form a subrepresentation of the supersymmetry algebra because after a supersymmetry transformation δ_ξ of G, we still have $\overline{D}_{\dot\alpha}\,(\delta_\xi G) = 0$ [5].

5 Chiral Superfields

- superfield Φ chiral $\Leftrightarrow \overline{D}_{\dot\alpha}\Phi = 0$
- Remaining independent components of Φ (see section 4.2): C, Ψ_α and F, i.e. the members of the chiral multiplet from the component approach in section 3.
- achievement: Supersymmetry transformations of the components of a chiral superfield are now calculated rather than assuming them to obtain a supersymmetric action (cf. section 3).
- Special importance has supersymmetry transformation of F term of a chiral superfield Φ:

$$\begin{aligned}
\delta_\xi F &= -\frac{1}{4}\left(\xi^\alpha D_\alpha - \overline{\xi}^{\,\dot\alpha}\overline{D}_{\dot\alpha}\right) D^2 G|_{\Theta=\overline{\Theta}=0} \\
&= \frac{1}{4}\overline{\xi}^{\,\dot\alpha}\overline{D}_{\dot\alpha} D^2 G|_{\Theta=\overline{\Theta}=0} \quad \text{, because } D_\alpha D^2 = 0 \\
&= -\frac{1}{4}\overline{\xi}^{\,\dot\alpha}[D^2, \overline{D}_{\dot\alpha}]G|_{\Theta=\overline{\Theta}=0} \quad \text{, because G is chiral} \\
&\overset{(\star)}{=} i(\sigma^m)_{\alpha\dot\alpha}\overline{\xi}^{\,\dot\alpha}\partial_m D^\alpha G|_{\Theta=\overline{\Theta}=0} \\
&= \sqrt{2}\,i(\sigma^m)_{\alpha\dot\alpha}\overline{\xi}^{\,\dot\alpha}\partial_m \Psi^\alpha \\
&= \sqrt{2}\,i\epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^m)_{\alpha\dot\alpha}\overline{\xi}_{\dot\beta}\partial_m\Psi_\beta \\
&= \sqrt{2}\,i\epsilon^{\dot\beta\dot\alpha}\epsilon^{\beta\alpha}(\sigma^m)_{\alpha\dot\alpha}\overline{\xi}_{\dot\beta}\partial_m\Psi_\beta \\
&= \sqrt{2}\,i\,\overline{\xi}_{\dot\beta}(\overline{\sigma}^m)^{\dot\beta\beta}\partial_m\Psi_\beta \quad \circledast\circledast\circledast \quad .
\end{aligned}$$

Observation: F term changed by total derivative after supersymmetry transformation. $\to$ Action for F term of any chiral superfield Φ is supersymmetric invariant [8].
($\star$) Use $[D^2, \overline{D}_{\dot\alpha}] = -4i\,(\sigma^m)_{\alpha\dot\alpha}D^\alpha\partial_m$.

5.1 Kähler potential

- Consider two superfields Φ and $\overline{\Phi}$ with $\overline{D}_{\dot\alpha}\Phi = 0$ (chiral) and $D_\alpha\overline{\Phi} = 0$ (antichiral) and calculate the action for $\overline{\Phi}\Phi$.

$$
\begin{aligned}
S &= \int d^4x\, d^2\Theta\, d^2\overline{\Theta} \quad \left[\overline{\Phi}\Phi\right] |_{\Theta=\overline{\Theta}=0} \\
&= \frac{1}{16}\int d^4x\, \overline{D}^2\, D^2 \quad \left[\overline{\Phi}\Phi\right] |_{\Theta=\overline{\Theta}=0} \\
&= \frac{1}{16}\int d^4x\, \overline{D}_{\dot\alpha}\, \overline{D}^{\dot\alpha}\, \left[\overline{\Phi} D^2\Phi\right] |_{\Theta=\overline{\Theta}=0} \quad ; D_\alpha D^\alpha \overline{\Phi} = 0 \text{ because } \overline{\Phi} \text{ is antichiral} \\
&= \frac{1}{16}\int d^4x\, \left[(\overline{D}_{\dot\alpha}(\overline{D}^{\dot\alpha}\overline{\Phi})(D^2\Phi)) \underset{\overline{\Theta}=\Theta=0}{\underset{\uparrow}{+}} \overline{D}_{\dot\alpha}(\overline{\Phi}\,\overline{D}^{\dot\alpha} D^2\Phi) \right] |_{\Theta=\overline{\Theta}=0} \\
&= \frac{1}{16}\int d^4x\, \left[(\overline{D}^2\overline{\Phi})(D^2\Phi) \underset{\{\overline{D}_{\dot\alpha},\overline{D}^{\dot\alpha}\}=0}{\underset{\uparrow}{-}} (\overline{D}^{\dot\alpha}\overline{\Phi})\overline{D}_{\dot\alpha}(D^2\Phi) + (\overline{D}_{\dot\alpha}\overline{\Phi})\overline{D}^{\dot\alpha}(D^2\Phi) + \overline{\Phi}\,\overline{D}^2 D^2\Phi \right] |_{\Theta=\overline{\Theta}=0} \\
&= \frac{1}{16}\int d^4x\, \left[(\overline{D}^2\overline{\Phi})(D^2\Phi) - (\overline{D}^{\dot\alpha}\overline{\Phi})\overline{D}_{\dot\alpha}(D^2\Phi) \underset{\text{indices}}{\underset{\uparrow}{-}} (\overline{D}^{\dot\alpha}\overline{\Phi})\overline{D}_{\dot\alpha}(D^2\Phi) + \overline{\Phi}\overline{D}^2 D^2\Phi \right] |_{\Theta=\overline{\Theta}=0} \\
&= \frac{1}{16}\int d^4x\, \left[(\overline{D}^2\overline{\Phi})(D^2\Phi) - 2(\overline{D}^{\dot\alpha}\overline{\Phi})\overline{D}_{\dot\alpha}(D^2\Phi) + \overline{\Phi}\,\overline{D}^2 D^2\Phi \right] |_{\Theta=\overline{\Theta}=0} \\
&\overset{(\star)}{=} \frac{1}{16}\int d^4x\, \left((-4)(-4)\,\overline{F}F + 16\,\overline{\Phi}\partial^m\partial_m\Phi - 2\sqrt{2}\frac{1}{\sqrt{2}}\,\overline{D}^{\dot\alpha}\overline{\Phi}\, 4i\,(\sigma^m)_{\alpha\dot\alpha}\partial_m\sqrt{2}\frac{1}{\sqrt{2}}\, D^\alpha\Phi \right) \\
&= \frac{1}{16}\int d^4x\, \left(16\,\overline{F}F \underset{\text{part. int.}}{\underset{\uparrow}{-}} 16\,\partial^m\overline{\Phi}\partial_m\Phi - 16i\,\overline{\Psi}^{\dot\alpha}(\sigma^m)_{\alpha\dot\alpha}\partial_m\Psi^\alpha \right) \\
&= \int d^4x\, \left(\overline{F}F - \partial^m\overline{z}\partial_m z - i\overline{\Psi}\sigma^m\partial_m\Psi \right)
\end{aligned}
$$

→ The free action of a complex scalar z, a left-handed Weyl spinor Ψ and an auxiliary field F results from integrating the product of the corresponding chiral and antichiral superfield over superspace.
($\star$) Use chirality and the following relations:

$$
\begin{aligned}
\{\overline{D}^2, D^2\} &= 2\,\overline{D}_{\dot\alpha}D^2\overline{D}^{\dot\alpha} + 16\,\partial^m\partial_m \\
[D^2, \overline{D}_{\dot\alpha}] &= -4i\,(\sigma^m)_{\alpha\dot\alpha}D^\alpha\partial_m\,.
\end{aligned}
$$

- Generalizing the approach of getting a supersymmetric invariant action from a real valued function $K(\overline{\Phi}, \Phi)$ of an antichiral superfield $\overline{\Phi}$ and a chiral superfield Φ leads to Kähler geometries and the Kähler potential [3].
  $K(\overline{\Phi}, \Phi) = \overline{\Phi}\,\Phi$ is just a special form of the Kähler potential for one antichiral superfield $\overline{\Phi}$ and one chiral superfield Φ.
- Remark: It can be shown that for a general superfield G the only component whose supersymmetric transformation is a total derivative and hence

whose action is supersymmetric invariant is the $\tilde{D}$ component (cf. section 4.2). Since the action of the Kähler potential is supersymmetric invariant, the Kähler potential must be the $\tilde{D}$ term of a general superfield G [8].

5.2 Superpotential

- G general superfield $\rightarrow$ action for $\tilde{D}$ component supersymmetric invariant $\rightarrow$ Kähler potential
  Φ chiral superfield $\xrightarrow{\circledast\circledast\circledast}$ action for F component supersymmetric invariant $\rightarrow$? potential ?
- Observe from definition of chiral superfield:
  $\Phi_i\,;\, i = 1, \ldots, n$ chiral superfields $\Rightarrow \sum_i \Phi_i$ and $\prod_i \Phi_i$ chiral superfields.
  $\rightarrow$ Any holomorphic function $W(\Phi)$ of chiral superfields $\Phi = \Phi_i\,;\, i = 1, \ldots, n$ is a chiral superfield, i.e. $\overline{D}_{\dot{\alpha}} W(\Phi) = 0$.
- $W(\Phi)$ no $\overline{\Theta}$ dependence $\rightarrow$ only $d^2\Theta$ integration for action S:

$$
\begin{aligned}
S &= \int d^4x\, d^2\Theta\, W(\Phi) \\
\delta_\xi S &= -\frac{1}{4} \int d^4x\, D^2 \xi^\alpha D_\alpha\, W(\Phi) \\
&= \int d^4x\, \xi^\alpha D_\alpha \left(-\frac{1}{4} D^2 W(\Phi) \right) \\
&= 0 \;\;, \text{ because } D_\alpha D^2 = 0
\end{aligned}
$$

Recall from section 4.2: $[W(\Phi)]_F = -\frac{1}{4} D^2 W(\Phi)|_{\Theta=\overline{\Theta}=0}$
$\rightarrow$ The action for the F term of the chiral superfield $W(\Phi)$ is supersymmetric invariant. The holomorphic function $W(\Phi)$ is called the superpotential.

6 Guide to the Literature

1. *A Modern Introduction to Quantum Field Theory*; M. Maggiore

   - Chapter 2 implements the basics of group theory and the representation of the Lorentz group.

2. *Quantum Field Theory I*; Lecture Notes by A. Hebecker

   - Chapter 8 reveals $SL(2,\mathbb{C})$ as the true symmetry group of four dimensional space time (universal covering groups).

3. *Supersymmetry*; Lecture Notes by G. Nibbelink

   - Chapter 1 to chapter 6 are the main input for this talk.
   - Knowledge of QFT is presumed.

4. *Supersymmetry and Supergravity*; J.Wess, J. Bagger

   - THE introduction to supersymmetry and beyond.
   - few text, mainly calulations, therefore very formal and technical
   - useful review about spinors and their index structure in APPENDIX A and APPENDIX B

5. *Physics Beyond The Standard Model*; Lecture Notes by A. Hebecker

   - short introduction into supersymmetry in chapter 4

6. *All Possible Symmetries of the S Matrix*; S. Coleman, J. Mandula; Physical Review **159**, 5, (1967)

   - mathematical formulation and proof of the no-go theorem

7. *All Possible Generators Of Supersymmetries of the S-Matrix*; R.Haag, J.T. Lopuszanski, M. Sohnius; Nuclear Physics **B88** (1975) 257-274

   - motivation for the supersymmetric algebra and its extension to $\mathcal{N}$=L
   - general classification of Bose and Fermi type symmetry generators

8. *The Quantum Theory of Fields, Volume III Supersymmetry*; S. Weinberg

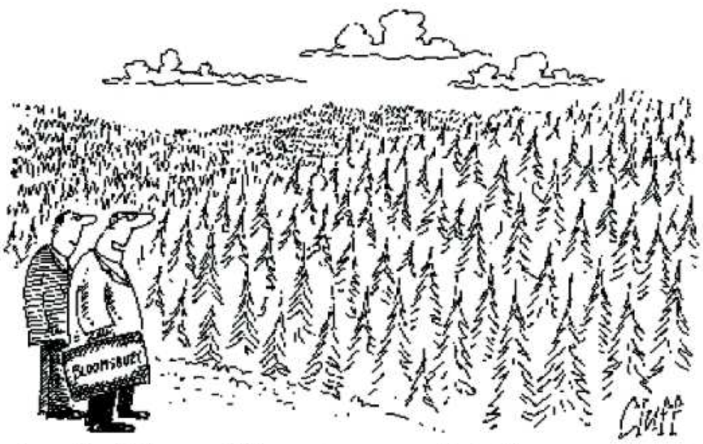
   - Chapter 26.3 and chapter 26.4 provide explanation of D and F terms.
   - One must get used to the the author's unique notation.

9. *The Supersymmetric World. The Beginnings of the Theory*; G. Kane, M. Shifman

   - story of the historical development of supersymmetry seen through the eyes of the contributors

$\hookrightarrow$

Outlook



"One day, all of these will be supersymmetric phenomenology papers."