Supersymmetric Grand Unified Theories

1 The Standard Model (SM)

1.1 Gauge Group and Particle Content

The Standard Model of particle physics has been tested experimentally in many ways and to high precision during the last decades and turned out to provide a good description of physics at energies below $\sim$ 100 GeV. Its gauge group is $SU(3) \times SU(2)_L \times U(1)_Y$ which is spontaneously broken down to $SU(3) \times U(1)_{EM}$, as described in the Higgs mechanism. Before the symmetry breaking, the model contains 12 massless gauge bosons in the corresponding adjoint representations of the gauge group, three generations of 15 massless fermions and a complex Higgs scalar, whose representations are listed in the table below. [1]

field	l	e^c	q	u^c	d^c	Φ
SU(2)	**2**	**1**	**2**	**1**	**1**	**2**
SU(3)	**1**	**1**	**3**	$\bar{\mathbf{3}}$	$\bar{\mathbf{3}}$	**1**
U(1)	-1	+2	+1/3	-4/3	+2/3	-1

The Higgs mechanism gives masses to three of the four gauge bosons of $SU(2) \times U(1)$, the remaining massless gauge boson is the photon. The fermions also acquire a mass upon electroweak-symmetry breaking, except for the neutrino, which remains massless.

1.2 Why go beyond the SM?

There are several hints that make us believe we have not found the ultimate theory yet. Some are rather obvious, while others require deeper understanding of the features of quantum field theories:

- **Gravitation:** Only three of the four known forces in the universe are described in the SM. The theory of gravity is general relativity (GR). It can be neglected at the energy scale of today's experiments in particle physics but if we want to find a theory of everything, then there should be one model for all of the four forces

- **Gauge coupling unification:** Using the renormalization group equations (RGEs) for the SM to develop the running of the three gauge couplings, one finds that they meet at an energy $\sim 10^{16}$ GeV. This indicates the possible existence of a higher gauge symmetry at high energies, where the three couplings are unified into one.

[1] In terms of the generator Q of electromagnetic $U(1)$ and the third isospin generator T_3, the hypercharge is given by $Y = 2(Q - T_3)$.

- **Hierarchy problem:** The mass of the W-boson has been experimentally determined to be of the order of 80 GeV. Since it acquires its mass via the Higgs mechanism, m_W is proportional to the Higgs mass. But the latter receives radiative corrections that are quadratically divergent. These divergencies have to be cancelled "by hand" in every order in perturbation theory, a procedure referred to as *fine tuning*. Such "accidental" cancellations seem a bit miraculous and give rise to the question whether there might be another reason for the stability of the weak scale.
- **Mass and mixing parameters:** The Standard Model contains a lot of free parameters, e.g. the masses of quarks and leptons and their weak mixing angles. There are correlations between them, that could be explained by additional symmetries, which might also reduce the disturbingly large number of free parameters in the SM. Family symmetries, providing certain possibilities for the form of the PMNS matrix, are not under consideration here, but grand unification will lead to relations among fermion masses.

2 The Minimal Supersymmetric Standard Model (MSSM)

Imposing supersymmetry (SUSY), the gauge fields become components of vector-superfields and the fermionic fields as well as the Higgs become components of chiral superfields. In addition, the other components of the superfields, called gauginos, squarks, sleptons and Higgsinos, appear in the lagrangian. Also, a second Higgs field (accompanied by its superpartner) has to be introduced, in order to give masses to leptons and d-type quarks as well as u-type quarks, [2] and to keep the model anomaly free.
Supersymmetry solves part of the hierarchy problem mentioned above. According to the *non-renormalization theorem*, the radiative corrections to the Higgs mass occurring in the SM will be cancelled by analogous diagrams with superpartners running in the loops. Therefore, if the bare Higgs mass is of the order of 150 GeV, there will be no divergent corrections pushing it up to the scale of a momentum cut-off. SUSY does not explain why the weak scale is at its experimentally determined value, but it does explain why it stays there.
Furthermore, supersymmetry predicts the masses of particles to be the same as those of their superpartners. Since we have not observed any of these superpartners at the energy scales of the SM particle masses, supersymmetry has to be broken. In order not to spoil the achievement of the renormalization theorem, the terms in the lagrangian breaking supersymmetry have to be *soft*, i.e. their mass dimension has to be positive. In most of the models found in the literature, the breaking of SUSY occurs "after" the breaking of the higher gauge symmetry, i.e. at a lower energy scale.
An interesting aspect of some SUSY models is the possibility to obtain a negative squared Higgs mass, necessary for spontaneous symmetry breaking. The

[2] This is because the superpotential has to be holomorphic in the scalar field and therefore cannot contain its complex conjugate.

bare squared Higgs mass is assumed to be positive, but then pushed into the negative regime by radiative corrections.[3]
If one writes down all renormalizable terms allowed by Lorentz and gauge invariance as well as supersymmetry, there will be operators leading to proton decay. One can forbid these operators to show up in the lagrangian by imposing an additional symmetry, called R-parity, defined as $R = (-1)^{3(B-L)+2S}$.
The other open problems stated above are not solved by SUSY. In addition, there arise several new aspects that need an explanation, e.g. the origin of the soft-breaking terms and the conservation of R-parity. Grand Unified Theories (GUTs) try to find answers to these open questions.

3 Group Theory

Grand Unification means embedding the SM gauge group into a larger symmetry group, breaking down the symmetry via the Higgs mechanism and obtaining an effective theory at low energies, that contains relics of the structure of the higher symmetry. The mathematical tool for understanding these structures and their backtracking is group theory. Here, only some of the most important aspects for GUT model building are shortly presented.

3.1 Representations

A representation of the group G is a map from G into the group of all invertible, n-dimensional matrices $\mathrm{GL}(n)$,

$$\begin{aligned} R: \quad G &\longrightarrow \mathrm{GL}(n) \\ g &\longmapsto R(g), \end{aligned} \tag{1}$$

with the properties

$$\begin{aligned} R(g \circ h) &= R(g) \cdot R(h) \\ R(e) &= 1, \end{aligned} \tag{2}$$

where e is the identity element in G and 1 is the identity matrix.
A representation is called *reducible*, if there is an invariant subspace, i.e. if P is the projection operator on that subspace, then $\forall g \in G$

$$PR(g)P = R(g)P \tag{3}$$

A representation is *irreducible* if it is not reducible. Irreducible representations are important in physics because they allow for mixing of all components of a field transforming in a specific representation of a symmetry group.

3.2 Decomposition of Representations and Tensor Products

With the technique of Young Tableaux, it is possible to decompose a representation into representations of a subgroup. For example, consider decomposition

[3]This is not in contradiction with the non-renormalization theorem, because soft-breaking terms are not protected from receiving radiative corrections.

of the **5** and **10** of $SU(5)$ under the SM group,

$$\begin{aligned} \bar{\mathbf{5}} &= (\bar{\mathbf{3}}, \mathbf{1}, +2/3) \oplus (\mathbf{1}, \mathbf{2}, -1), \\ \mathbf{10} &= (\mathbf{1}, \mathbf{1}, +2) \oplus (\bar{\mathbf{3}}, \mathbf{1}, -4/3) \oplus (\mathbf{3}, \mathbf{2}, +1/3). \end{aligned} \tag{4}$$

In the same manner, a tensor product of two representations of the group under consideration can be decomposed into representations of this group, e.g. for the **5** and $\bar{\mathbf{5}}$ of $SU(5)$

$$\mathbf{5} \times \bar{\mathbf{5}} = \mathbf{1} + \mathbf{24}. \tag{5}$$

4 Gauge Coupling Unification

The three gauge couplings of the Standard Model, α_1 of $U(1)_Y$, α_2 of $SU(2)_L$ and α_3 of $SU(3)_C$ do not take on a constant value for all energies, but run with the scale according to there beta functions, which in one-loop approximation read

$$\frac{\mathrm{d}\alpha_i}{\mathrm{d}(\ln\mu)} = -\frac{1}{2\pi} b_i \alpha_i^2 \qquad (i = 1, 2, 3) \tag{6}$$

where

$$b_i = 3C_2(G_i) - T_R N_\psi. \tag{7}$$

Here, $C_2(G_i) = \sum_a (T_a^{adj})^2$ is the quadratic Casimir of the group G_i, with T_a^{adj} being the generators in the adjoint representation, and $T_R \delta_{ab} = \mathrm{Tr}(T_a T_b)$, with T_a in the representation R. N_ψ is the number of chiral superfields in the representation R contained in the model. Note that $C_2(SU(N)) = N$ and $C_2(U(1)) = 0$.
Assuming that the three couplings take on the same value α_U at an energy scale M_U, their values at the weak scale M_Z are obtained from the beta-functions [4]:

$$\alpha_i^{-1}(M_Z) = \alpha_U^{-1} + \frac{b_i}{2\pi} \ln\left(\frac{M_U}{M_Z}\right) \tag{8}$$

For the MSSM, the values for b_i are $b_1 = -33/5$, $b_2 = -1$ and $b_3 = 3$, leading to a consistency equation,

$$\Delta\alpha \equiv 5\alpha_1^{-1}(M_Z) - 12\alpha_2^{-1}(M_Z) + 7\alpha_3^{-1}(M_Z) = 0. \tag{9}$$

The experimental values at M_Z yield $\Delta\alpha = -1 \pm 2$, allowing for coupling unification (at least at one loop-order). The unification scale is found to be $M_U \approx 10^{16}$ GeV, and $\alpha_U \approx 1/24$, see *fig.1*.

[4]Here it is assumed, that there are no intermediate scales between M_Z and M_U, where new particles might appear in the spectrum.

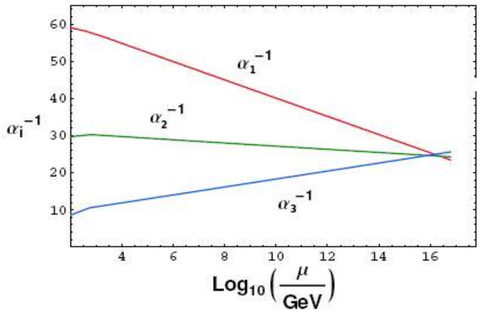

Figure 1: Running of the gauge couplings in the MSSM

5 GUT Model Building

5.1 Steps of Model Building

The building of a SUSY GUT requires several steps, which are listed one by one below. In the subsequent chapters, examples of SUSY GUT models will be given, following each of these steps.

a) Choose a symmetry group G containing the SM gauge group $SU(3) \times SU(2)_L \times U(1)_Y$ as a subgroup. This could either be one single group, e.g. $SU(5)$, $SO(10)$ or E_6, or a product of several groups, such as $SU(3) \times SU(3)$ or $SU(4) \times SU(2)_L \times SU(2)_R$, where the latter is often referred to as *Pati-Salam group*.

b) Specify the particle content of the model. For fermions and Higgs bosons (and, of course, their superpartners) irreducible representations of the gauge group are chosen, while the gauge bosons transform in the adjoint. If the particles of the MSSM do not fit into one representation, they have to be split into several ones or new particles have to be added. A new particle that is welcome in many cases, is the right-handed neutrino, because its existence enlargens the symmetry in the sense that all fermions now appear in a left- and right-handed version. In many cases, there is the need for more than two Higgs fields, in order to break the GUT symmetry and the electroweak symmetry as well as give masses to fermions and gauge bosons, compatible with their experimental values. Furthermore, assigning fermions to certain representations, one has to make sure that the model contains no gauge anomalies.

c) Write down kinetic terms for fermions, Higgses and gauge bosons (and their superpartners, respectively). The covariant derivative (in the case for one single gauge group) reads $D_\mu = \partial_\mu - igA^a_\mu T^a$, where g is the gauge coupling, A^a_μ are the vector gauge bosons and T^a are the generators of the gauge group ($a = 1, \ldots, \mathrm{dimLie}(G)$).

d) Write down the superpotential W for the scalar fields and the Yukawa interactions, keeping in mind that all terms should be renormalizable, Lorentz- and gauge invariant. The superpotential is responsible for breaking the gauge symmetry down to that of the SM and the Yukawa terms will give mass to fermions after symmetry breakdown.

e) Determine the minimum of the scalar potential by imposing F-term flatness ($F_i = \partial W/\partial \Phi_i = 0$, where Φ_i are the scalar fields appearing in W). The F-term has to vanish, if one assumes that SUSY is still present at energies below the GUT-scale. This is a great simplification in calculation provided by SUSY, in non-supersymmetric theories finding the vacuum expectation value (VEV) of the Higgs fields is much more elaborate.

f) Calculate masses and couplings at the weak scale using the RGEs of the theory. Relations between parameters obtained from the theory will be valid at the GUT scale. In order to obtain relations at the weak scale, the running of parameters has to be taken into account. The form of the RGEs is determined by the particle content of the theory. If there are intermediate energy scales where new particles appear, the RGEs will change at these scales. In approximation, this change is often assumed to take place "suddenly", i.e. a theta-function is used.

5.2 Achievements and Problems

Grand unified theories, consisting of only one single gauge group, explain the unification of gauge couplings. In supersymmetric versions, the couplings match at $\sim 10^{16}$ GeV within the present experimental errors for the initial values at the weak scale. Furthermore, quarks and leptons are unified being part of the same irreducible representation of the gauge group. This leads to predictions for relations among their masses. Another achievement is the possibility to include the right-handed neutrino in the theory and write down a Majorana mass term, e.g. in $SO(10)$, where the right-handed neutrino can obtain a large Majorana mass upon breaking of left-right symmetry. In these kind of models, the see-saw mechanism can be applied, which explains the smallness of (left-handed) neutrino masses, allowing for a Dirac mass of the order of the weak scale.

Gravity has not been addressed yet. It is a difficult task to combine it with quantum field theory. There are several possibilities to include a spin 2 particle in the model, which can be identified as a graviton, e.g. if one considers local supersymmetry transformations. String theory might provide an adequate framework, but shall not be discussed here.

Even though they propose answers to most of the tasks mentioned in chapter 1, grand unified theories give rise to new questions. One of them is the so-called *doublet-triplet splitting* of the Higgs boson. The Higgs field introduced in a representation of the GUT group contains color triplet and electroweak doublet components. [5] In order to break down also the electroweak symmetry of the standard model at a lower energy scale, one has to decouple the triplet components from the low-energy theory by giving them a large mass, while the doublet components should remain light. This leads to a new fine-tuning problem. A way out of this is provided by introducing an extra dimension, which is compactified into an *orbifold*.

[5]This can be seen by using the branching rules for decomposing the representation.

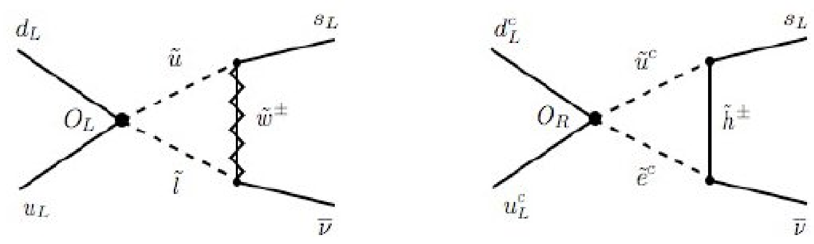

Figure 2: proton decay channels in SUSY models

Another severe problem is the occurence of proton decay, due to the introduction of new gauge bosons. The life-time of the proton provides one of the most reliable tests of grand unified theories. If the model predicts the life-time to be less than $\sim 10^{30}$ years, it is ruled out by experiment. In non-supersymmetric theories, effective operators (induced by the existence of additional gauge bosons at high energies) are at least of mass dimension $d = 6$, see *fig.2*. The dominating decay amplitude is $p \to e^+\pi^0$. In contrast, SUSY models allow for effective $d = 5$ operators, i.e. two fermionic components interacting with two bosonic components of chiral superfields. The bosons then decay into their fermionic superpartners via wino exchange. In this case, the dominating decay mode is $p \to K^+\bar{\nu}_\mu$.

6 SUSY SU(5)

The smallest single gauge group that contains the SM as a subgroup is $SU(5)$. $SU(5)$ is the group of all five-dimensional unitary matrices with determinant one. Its generators are traceless and hermitian (5×5) matrices which can be chosen as follows: Take the 10 matrices with one i above the diagonal, one $-i$ below and zeros everywhere else. Take the 10 matrices with one 1 above the diagonal, one 1 below and zeros everywhere else. Eventually, take the matrices with n ones on the diagonal, followed by a single $-n$ (there are four of these, $n = 1, ..., 4$). These are the 24 generators of $SU(5)$. For $SU(N)$ in general, there are $N^2 - 1$ generators.

6.1 Particle Content

The chiral superfields of the MSSM fit into (three copies of) two representations of $SU(5)$, $\bar{\mathbf{5}}$ and $\mathbf{10}$ in the following way:

$$\psi_{\bar{5}} = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ \nu \end{pmatrix}_L \qquad \psi_{10} = \begin{pmatrix} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ -u_3^c & 0 & u_1^c & u_2 & d_2 \\ u_2^c & -u_1^c & 0 & u_3 & d_3 \\ -u_1 & -u_2 & u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{pmatrix}_L \tag{10}$$

The representations are denoted with an index L, pointing out that all the fields are left-handed. The right-handed fields of the SM have been charge conjugated (denoted by an upper index c) in order to put them together with the left-handed fields in a left-handed representation. The right-handed neutrino does not appear in these representations and has to be included as an extra singlet under $SU(5)$. Taking a closer look at the assignment of particles to the representations, one observes that the $\bar{\mathbf{5}}$ contains the charge conjugate d-quark, the electron and the neutrino, while the **10** comprises the u-quark and its charge conjugate, the d-quark and the charge conjugate electron.
In the Higgs sector, a scalar field Φ in the **24** can be utilized to break the $SU(5)$ down to the SM gauge symmetry and give large masses ($\sim 10^{16}$GeV) to the gauge bosons of $SU(5)/SM$. Furthermore, two superfields H_u and H_d in the **5** and $\bar{\mathbf{5}}$ are included into the model, which will give masses to fermions via Yukawa couplings.
Furthermore, there are the 24 gauge bosons, 12 thereof belonging to the SM gauge group. The remaining 12 gauge bosons acquire a mass upon $SU(5)$ symmetry breaking. They affect the effective low energy theory in the form of higher dimensional operators, inducing, for example, proton decay.

6.2 The Superpotential

Following the steps of GUT model building, the next issue to tackle is the theory's superpotential. It is mostly denoted by W and it can be shown that, in order to preserve supersymmetry, it must be holomorphic in the scalar components of chiral superfields. As mentioned before, this is one of the reason for introducing a second Higgs doublet in the MSSM.
For SUSY $SU(5)$, the superpotential is

$$W = W_h + W_Y + W_{SB}. \tag{11}$$

In the following, the three terms are discussed one by one.
W_h is called the *hidden sector* and consists of soft-terms, that break supersymmetry. W_Y contains the Yukawa couplings,

$$W_Y = y_u^{ab}\psi_{10}^a\psi_{10}^b H_u + y_d^{ab}\psi_{10}^a\psi_{\bar{5}}^b H_d, \tag{12}$$

with y_u and y_d being (3×3) yukawa coupling matrices. Recalling the distribution of SM particles among the two representations, one sees that the first Yukawa term gives mass to the u-quark, while the d-quark and the electron receive their masses from the second term. The neutrino remains massless, unless the right-handed neutrino is included into the model as an extra singlet. If there was such an extra singlet ν_R, one could add a term $\psi_{\bar{5}}\nu_R H_u$, yielding a Dirac mass for the neutrino. Since ν_R is a gauge singlet, it can even get an explicit Majorana mass term $M_R\bar{\nu}_R\nu_R$, providing the background for the see-saw mechanism. W_{SB} is responsible for symmetry breaking,

$$W_{SB} = z\mathrm{Tr}\Phi + x\mathrm{Tr}\Phi^2 + y\mathrm{Tr}\Phi^3 + \lambda H_u \Phi H_d + M H_u H_d. \tag{13}$$

Note that, although Φ is traceless [6], the term $z\mathrm{Tr}\Phi$ appears in the superpotential as a lagrange multiplier. Minimizing the potential, the trace will be constrained to vanish. SUSY requires vanishing of the F-terms:

$$0 = \mathrm{Tr}\left(\frac{\partial W}{\partial \phi^i_j}\right) \tag{14}$$

where

$$\frac{\partial W}{\partial \phi^i_j} = z\delta^j_i + 2x\Phi^i_j + 3y\Phi^i_k\Phi^k_j. \tag{15}$$

Therefore F-term flatness, together with the constraint $\mathrm{Tr}\Phi = 0$, yields the relation

$$z = -\frac{3}{5}y\mathrm{Tr}\Phi^2. \tag{16}$$

Now there are three solutions to the equation $\mathrm{Tr}\left(\partial W/\partial\phi^i_j\right) = 0$, one of them corresponding to unbroken $SU(5)$, another one corresponding $SU(4) \times U(1)$ as residual symmetry, and the last one corresponding to the Standard Model $SU(3) \times SU(2) \times U(1)$. The latter is obtained by the VEV

$$\langle \Phi^i_j \rangle = \begin{pmatrix} \frac{4x}{3y} & & & & \\ & \frac{4x}{3y} & & & \\ & & \frac{4x}{3y} & & \\ & & & -\frac{2x}{y} & \\ & & & & -\frac{2x}{y} \end{pmatrix}. \tag{17}$$

The MSSM Higgs doublets are contained in H_u and H_d and have to be separated from the triplets by finetuning the parameters x, y and M, to ensure the doublet masses remain light.

6.3 Fermion masses

The unification of quarks and leptons into the same representations of $SU(5)$ leads to relations among their masses. From W_Y it is obvious that the d-quark and the electron receive the same mass, since they share the same Yukawa coupling matrix. Of course, this equality is only valid at the GUT scale. If the experimentally determined values for the masses at the weak scale are taken as initial values of the RGEs, one finds for the masses at the GUT scale approximately $m_d \approx 3m_e$. This shows that the $SU(5)$ prediction $m_d = m_e$ is a little too strict. Another result extracted from this model is the scale independent relation $\frac{m_e}{m_\mu} = \frac{m_d}{m_s}$, which is also in conflict with the experimental values.

[6]The **24** of SU(5) is written as a hermitian (5×5) matrix with vanishing trace.

7 SUSY SO(10)

SO(10) is the group of all orthogonal ten-dimensional matrices with determinant one. Its generators are antisymmetric, hermitian (10×10) matrices which can be chosen to be the 45 matrices with one i above the diagonal, one $-i$ below and zeros everywhere else. For $SO(N)$ in general, there are $N(N-1)/2$ generators. An important difference between $SO(10)$ and $SU(5)$ is the rank of their Lie algebras. While the Lie algebra of $SO(10)$ contains 5 commuting generators, that of $SU(5)$ has only 4, i.e. it has the same rank as the Lie algebra of the Standard Model gauge group. Since the rank of $SO(10)$ is higher than that of the SM, there are different ways to break down the symmetry. Specifically, there are different possible intermediate symmetries of rank 5, e.g. $SU(5)\times U(1)$, $SU(4) \times SU(2)_L \times SU(2)_R$ or $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)$.

7.1 Spinors

$SO(10)$ is very attractive as a GUT group since all chiral superfields of the MSSM, together with the right-handed neutrino, fit into (three copies of) its **16** spinor representation. An elegant way to represent spinors in $SO(2N)$ is to express them in $SU(N)$ basis. This is possible because $SU(N)$ is a subgroup of $SO(2N)$. To see how this is done, consider N operators χ_i, $\chi_i^\dagger$, obeying

$$\begin{aligned} \{\chi_i, \chi_j^\dagger\} &= \delta_{ij} \\ \{\chi_i, \chi_j\} &= 0 \\ \{\chi_i^\dagger, \chi_j^\dagger\} &= 0. \end{aligned} \tag{18}$$

The $SO(2N)$ Clifford algebra consists of $2N$ gamma matrices,

$$\begin{aligned} \Gamma_{2i} &= \frac{1}{\sqrt{2}}\left(\chi_i + \chi_i^\dagger\right) \\ \Gamma_{2i-1} &= \frac{i}{\sqrt{2}}\left(\chi_i - \chi_i^\dagger\right), \end{aligned} \tag{19}$$

fulfilling

$$\{\Gamma_\mu, \Gamma_\nu\} = \delta_{\mu\nu} 1. \tag{20}$$

Explicitly, for $N = 5$, the spinor representation now may be obtained by defining

$$\begin{aligned} |\psi\rangle = {} & |0\rangle\, \psi_0 + \chi_i^\dagger\, |0\rangle\, \psi_i + \frac{1}{2}\chi_i^\dagger \chi_j^\dagger\, |0\rangle\, \psi_{ij} + \frac{1}{12}\epsilon^{ijklm}\chi_k^\dagger \chi_l^\dagger \chi_m^\dagger\, |0\rangle\, \bar{\psi}_{ij} \\ & + \frac{1}{24}\epsilon^{ijklm}\chi_j^\dagger \chi_k^\dagger \chi_l^\dagger \chi_m^\dagger\, |0\rangle\, \bar{\psi}_i + \chi_1^\dagger \chi_2^\dagger \chi_3^\dagger \chi_4^\dagger \chi_5^\dagger\, |0\rangle\, \bar{\psi}_0, \end{aligned} \tag{21}$$

or in vector notation

$$\psi = \begin{pmatrix} \psi_0 \\ \psi_i \\ \psi_{ij} \\ \bar{\psi}_{ij} \\ \bar{\psi}_i \\ \bar{\psi}_0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 5 \\ 10 \\ \bar{10} \\ \bar{5} \\ \bar{1} \end{pmatrix} \quad \text{of } SU(5). \tag{22}$$

The chirality operator in $SO(2N)$ is defined as

$$\frac{1}{2}(1 \pm \Gamma_0), \quad \text{where } \Gamma_0 \equiv i^N \Gamma_1 \Gamma_2 \cdots \Gamma_N, \tag{23}$$

providing the possibility to define chiral fields,

$$\begin{aligned} \psi_\pm &= \frac{1}{2}(1 \pm \Gamma_0)\psi, \\ \psi_+ &= \begin{pmatrix} \psi_0 \\ \psi_i \\ \psi_{ij} \\ \mathbf{0} \end{pmatrix}, \quad \psi_- = \begin{pmatrix} \mathbf{0} \\ \bar{\psi}_{ij} \\ \bar{\psi}_i \\ \bar{\psi}_0 \end{pmatrix}. \end{aligned} \tag{24}$$

7.2 Particle Content

The fact that all MSSM chiral superfields fit into the **16** spinor of $SO(10)$ can be seen by simple counting: There are 3 up- and 3 down quarks (one for each color), 1 charged and 1 neutral lepton, adding up to 8 fields. Multiplication by two, accounting for the charge conjugates, yields 16. In the $SU(5)$ basis introduced in the previous section, the way in which the particles appear in the representation becomes clear. $\bar{\psi}_i$, corresponding to $\bar{\mathbf{5}}$, and ψ_{ij}, corresponding to **10**, contain the fields as in $SU(5)$, the singlet ψ_0 is the right-handed neutrino. Via the branching rule for the **16**, this translates into SM language,

$$\begin{aligned} 16 &= (3,2,1/3)_{q_L} \oplus (1,2,-1)_{l_L} \oplus (\bar{3},1,-4/3)_{u_L^c} \oplus (\bar{3},1,2/3)_{d_L} \\ &\oplus (1,1,0)_{\nu_L^c} \oplus (1,1,2)_{e_L^c}. \end{aligned} \tag{25}$$

The Higgs sector is much more elaborate than in $SU(5)$. There are many possibilities to choose representations for Higgs fields, yielding various predictions for fermion masses. Of course, one wants to keep the number of free parameters in the model as low as possible to obtain high predictivity, but since all fermions are combined in one representation, a small number of Higgs fields usually results in mass relations that are too restrictive and not compatible with the experimental values.

The minimal scenario that has been discussed extensively in the literature is the $SO(10)$ gauge symmetry breaking via a Higgs field in the **210** representation down either to the Pati-Salam group $SU(4) \times SU(2)_L \times SU(2)_R$, to $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ or to $SU(3) \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$. The next stage of symmetry breaking occurs due to two Higgses in the **126** and $\overline{\mathbf{126}}$, after what the remaining gauge group is that of the Standard Model. The electroweak symmetry breaking is then induced by a linear combination of doublet components of the **210**, **126** and $\overline{\mathbf{126}}$ as well as another Higgs in the **10** representation (see *fig.3*). Apparently, the doublet-triplet splitting becomes an involved task. In fact, this is the model with the minimal Higgs content.
On the other hand, an advantage of this model is the automatic conservation of R-parity down to low energies. This is due to the fact, that the $\overline{\mathbf{126}}$ breaks $B-L$ symmetry in such a way, that $B-L$ always changes by two units, leaving $R = (-1)^{3(B-L)+2S}$ invariant.

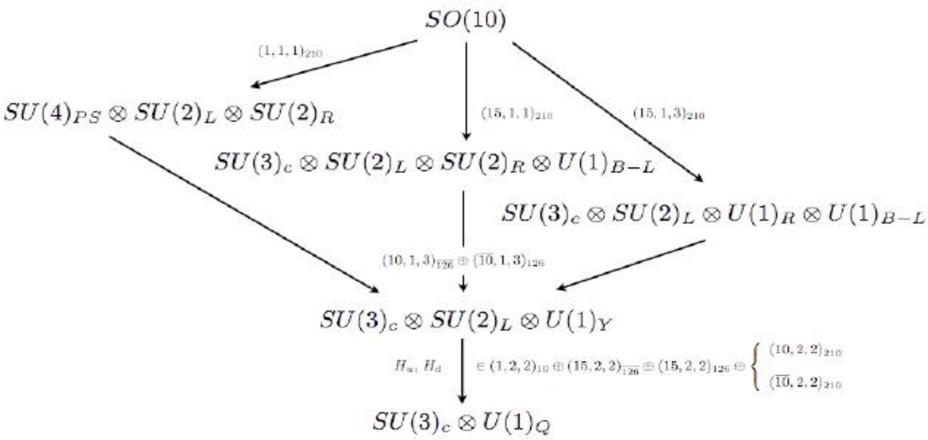

Figure 3: SO(10) breaking chains in the minimal model

7.3 Fermion Masses

The superpotential of the theory, as in $SU(5)$, contains Yukawa terms, interactions of Higgs fields and SUSY soft-breaking terms. The requirement of F-term flatness, i.e. preserving SUSY below the GUT scale, yields the minimum of the scalar potential and thus the VEVs of the Higgs fields, whose electroweak doublet components give masses to the fermions. All Dirac masses are obtained from linear combinations of the VEVs of components of the **10** and the **126**, while the Majorana mass for the right-handed neutrino is provided by a linear combination of **10** and $\overline{\mathbf{126}}$. The VEVs giving mass to d-quarks and charged leptons point in the same direction, and, in case of a dominating **126** component, it is possible to obtain the relation

$$m_d = 3m_e \tag{26}$$

at the GUT scale, which is in good agreement with the experimental values at the electroweak scale. Here, the 3 in front of the charged lepton mass is a Clebsch-Gordan coefficient, a remnant of the decomposition of the **16** into color triplets and singlets.

8 Conclusion and Outlook

Supersymmetric grand unified theories provide numerous possible answers to questions posed by the Standard Model. They explain gauge coupling unification and can be useful in obtaining relations between fermion masses. Supersymmetry solves (at least part of) the hierarchy problem and leads to models with particle content symmetric in fermionic and bosonic degrees of freedom. On the other hand, SUSY GUT models give rise to new challenges, such as the doublet-triplet splitting in the Higgs sector or avoiding large proton decay amplitudes. Also the number of free parameters has not been reduced remarkably

compared to the Standard Model. The $SO(10)$ model with the minimal Higgs content introduced in the last section has 23 real parameters, while the SM with additional massive (Majorana) neutrinos contains 27. Besides, gravity is not yet included in the theory and neither have family symmetries been addressed.
Solving the problems associated to SUSY GUTs is a new challenge and has been pursuited in the recent years. Additional symmetry groups (discrete and continuous) have been introduced to accommodate family symmetries. In supergravity, local SUSY invariance leads to the appearance of a spin 2 particle, which can be identified as the graviton. Furthermore, extra dimensions have been considered in order to achieve the doublet-triplet splitting or explain the hierarchy between the weak and the Planck scale. But all these models are accompanied by new open questions and we are still far from finding the ultimate theory.

References

[1] R.N. Mohapatra, *Supersymmetric Grand Unification (Lectures at TASI 97)*, (1998), arXiv: hep-ph/9801235v2
The content of Mohapatra's lectures is pretty much the same as the chapters 5,7 and 13 in his book, see below.

[2] R.N. Mohapatra, *Unification and Supersymmetry*, Third Edition, Springer (2003)
This book provides a good introduction into GUTs and also SUSY Guts, but one should be careful, since I found some mistakes reading it again, which I had missed when I read it the first time.

[3] D.Bailin, A. Love, *Introduction to Gauge Field Theory*, Revised Edition, Taylor & Francis Group (1993)
Only one chapter was of interest here, where SU(5) is discussed in great detail. Especially helpful if one wants to do some actual calculations.

[4] M. Malinsky, *Quark and Lepton Masses and Mixing in Supersymmetric Grand Unified Theories*, PhD Thesis (2005)
This thesis is very useful for the insight into SO(10) models. Several models with different Higgs contents are discussed and masses and mixing angles are calculated. I found the pictures showing Feynman diagrams of proton decay and SO(10) breaking chains in this text.

[5] R. Slansky, *Group Theory for Unified Model Building*, Physics Reports 79, No.1 (1981) 1-128
This report is actually a huge table of coefficients arising in group theory. If you need specific Clebsch-Gordan coefficients, branching rules or Dynkin indices, it is great, but it is not at all pedagogical.

[6] S.P. Martin, *A Supersymmetry Primer*, version 5 (2008) arXiv: hep-ph/9709356
This lecture is good for beginners in SUSY, especially because it focusses on the physics, avoiding the superfield formalism.

[7] K.S. Babu, *TASI Lectures on Flavor Physics*, (2009), arXiv: hep-ph/0910.2948v1
These lectures focus on mass hierarchies of leptons and quarks, which can be explained in a SUSY SU(5) model furnished with the Frogatt-Nielsen mechanism. Very interesting literature, although the main focus is not on SUSY GUTs. I list it here, because the picture displaying gauge coupling unification is taken from that script.

[8] H. Georgi, *Lie Algebras in Particle Physics*, Second Edition, Westview Press (1999)
Best book ever! If you want to learn more about group theory, read this!