How to give the Graviton a Mass?

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The usual picture...

Our Universe today:

**Dark Matter**  WIMPS, sterile $\nu$, axions, PBH, etc.

→ Non-particle physics solution?

**Dark Energy**  Cosmological Constant, Quintessence, etc.

→ Connection to the other dark stuff in the Universe?
Motivation

**Dark Matter** modified gravity, UV-compete version of MOND
massive Graviton as DM

**Dark Energy** $\Lambda \sim m_g^2$?

**Theory** perturbation of GR in “theory space”

**test stability of GR predictions**

**WARNING:** I’m not an expert!!!
Dark Matter – galaxy rotation curves

Newton:
\[ a(r) = \frac{v(r)^2}{r} = \frac{G_N M(r)}{r^2} \]

Model galaxy:
\[ M(r) = M_0 \frac{r^3}{(r + r_0)^3} \]

\[ v(r) = \sqrt{\frac{G_N M_0 r^2}{(r + r_0)^3}} \sim \begin{cases} \frac{r}{\sqrt{r}} & r \ll r_0 \checkmark \\ \frac{1}{\sqrt{r}} & r \gg r_0 \times \end{cases} \]

\[ \Rightarrow M(r) \rightarrow M(r) + M_{DM}(r) \text{ or } G_N \rightarrow G_N^0 \left[ 1 + a(r/r_0) + b(r/r_0)^2 \cdots \right] \]
Dark Energy – Degravitation

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \]

⇒ the CC can be viewed as a vacuum energy \( \rho_\Lambda = \frac{\Lambda}{8\pi G_N} \Rightarrow \langle T^0_{\mu\nu} \rangle \sim M^4_{\text{pl}} \)

BUT: \( \rho_\Lambda \sim (10^{-3} \text{ eV})^4! \)

“de-gravitate” the Cosmological Constant:

\[ \Lambda g_{\mu\nu} = 8\pi G_N \langle T^0_{\mu\nu} \rangle \rightarrow 8\pi \langle G_N (m^{-2} \Box) T^0_{\mu\nu} \rangle \]
Why is the vacuum energy so much smaller than expected? Why does the vacuum energy density gravitate so little?

⇒ “high-pass filter” that decouples sources with wavelengths $\gtrsim m_g^{-1}$
usually Why is the vacuum energy so much smaller than expected?

**Degravitation** Why does the vacuum energy density gravitate so little?

⇒ “high-pass filter” that decouples sources with wavelengths $\gtrsim m_g^{-1}$

**Spin-1 analogy – uniform charge distribution** $J_\mu = \delta_\mu^0 \rho$

$$\partial^\mu F_{\mu \nu} = J_\nu \iff \nabla \cdot \vec{E} = \rho \quad \Rightarrow \quad \vec{E} = \vec{x} \rho / 3$$

insert “high-pass filter”:

$$(1 + m_\gamma^2 \Box^{-1}) \partial^\mu \tilde{F}_{\mu \nu} \Rightarrow (\Box + m_\gamma^2) \tilde{A}_\nu = J_\nu \quad (\partial_\alpha \tilde{A}^\alpha = 0)$$

$$\tilde{E} = \frac{\rho}{3} \vec{x} \cos(m_\gamma t)$$
Degravitation – continued

To mimic gravity, one needs to include non-linearities that damp these oscillations: \( \sim \lambda (A_{\mu} A^\mu)^2 \).

For \( t \to \infty \), the potential will settle in a static solution \( A^\infty = (A_0^\infty, \vec{0}) \) with \( m^2 A_0^\infty + \lambda (A_0^\infty)^3 = \rho \). Thus, \( \vec{E}_\infty = \vec{0}! \)

Screening of the homogeneous “vacuum charge density”

Some remarks

- Is this straight-forwardly generalised to Gravity?
- If yes, how does this mechanism operate?
- Does \( m_g \neq 0 \) automatically imply Degravitation?
- Ongoing debate!
Could both approaches be related?

\[ G_N \rightarrow G_N(r) \]
Consistent Massive Gravity

Phenomenology

Conclusions
Consistent Massive Gravity
What’s wrong with $m_g \neq 0$?

Spin-1:

$$\mathcal{L} = a \partial_\mu A^\nu \partial_\nu A^\mu + b (\partial_\mu A^\mu)^2$$

Gauge-invariance removes 1 dof and requires $b = -a = \frac{1}{2}$.

$$A_\mu = A^\bot_\mu + \partial_\mu \chi \Rightarrow \mathcal{L} \supset (a + b)(\Box \chi)^2, \; \partial^\mu A^\bot_\mu = 0.$$  

$\chi$ constitutes 2 ghost dof: kinetic energy $< 0$: INCONSISTENT

Conclusion:

gauge-invariance $\Leftrightarrow$ consistency  

$$b = -a = \frac{1}{2}$$
Now add a mass term

The mass term in

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu$$

breaks gauge-invariance, but the kinetic term is unique by consistency!

**Abelian Higgs mechanism:**

$$\mathcal{L}_{\text{Higgs}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (D_\mu \phi)^\dagger (D^\mu \phi) - V(|\phi|)$$

with $\phi = (v + h)e^{i\chi}$ this gives

$$\mathcal{L}_{\text{Higgs}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} g^2 v^2 (A_\mu - \partial_\mu \chi)^2 - \frac{1}{2} (\partial_\mu h)^2 - V'(h)$$

gauge-invariant if $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$ and $\chi \rightarrow \chi + \alpha$. 

consistency implies gauge invariance \[ h_{\mu\nu} = h_{\mu\nu}^\perp + \partial_\mu \chi_\nu + \partial_\nu \chi_\mu \]

There are now 4 possible combinations

no Higgs mechanism known → retain only the Stückelberg fields \( \chi_\mu \)

discontinuity if \( m_g \to 0 \): No analogy for spin-1

\[ \mathcal{L}_{\text{source}} = A_\mu J^\mu = (A_\mu^\perp + \partial_\mu \chi) J^\mu = A_\mu^\perp J^\mu \text{ if } \partial_\mu J^\mu = 0 \]
i.e. \( \chi \) doesn’t couple to conserved, external sources!

**Most importantly:**
In Gravity the consistency (absence of ghosts) is spoiled by non-linearity!
Fierz-Pauli mass term

\[ \mathcal{L}_{\text{FP}} = -\frac{1}{8} m_g^2 \left[ h_{\mu\nu} h^{\mu\nu} - \left( h_{\mu\nu}^{\mu} \right)^2 \right] \quad \text{[Fierz and Pauli, 1939]} \]

can be made gauge-invariant under \( h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_{(\mu} \xi_{\nu)} \) by Stückelberg fields \( \chi^\alpha \):

\[ \mathcal{L}_{\text{FP}}' = -\frac{1}{8} m_g^2 \left[ (h_{\mu\nu} + 2\partial_{(\mu} \chi_{\nu)})^2 - (h_{\mu}^{\mu} + 2\partial_{\alpha} \chi^\alpha)^2 \right], \]

if simultaneously \( \chi^\alpha \rightarrow \chi^\alpha + \frac{1}{2} \xi^\alpha \).

Great! But Gravity is non-linear. This is why it took 70 years to come up with this...
Consistent massive Gravity

The key observation is that we can introduce the Stückelberg fields by redefining the background metric

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + 2\partial_{(\mu} \chi_{\nu)}$$

in the linear case. Extending to the non-linear case is achieved by

$$f_{\mu\nu} = \partial_{\mu} \phi^a \partial_{\nu} \phi^b f'_{ab}$$

for a general background $f$ and 4 Stückelberg fields $\phi^a$.

The FP mass term is then generalised by $h_{\mu\nu} \to (\mathbb{1} - g^{-1}f)_{\mu\nu}$.

**NOT UNIQUE** and **NOT GHOST-FREE**
\[ S_{bi} = \frac{M_g^2}{2} \int d^4 x \sqrt{-|g|} R_g + \int d^4 x \sqrt{-|g|} \mathcal{L}_{\text{matter}} + \\
+ m^2 M_{\text{eff}}^2 \int d^4 x \sqrt{-|g|} \sum_{n=0}^{4} \beta_n e_n(\mathcal{X}) \]

where \( \mathcal{X} = \sqrt{g^{-1} f} \), i.e. \( \mathcal{X}_\mu^\alpha \chi_\nu^\alpha = g^{\mu \alpha} f_{\alpha \nu} \), \( M_{\text{eff}}^2 = \left( \frac{1}{M_g^2} + \frac{1}{M_f^2} \right)^{-1} \), and

\[ e_0 = 1, \ e_1 = \text{tr} (\mathcal{X}), \ e_2 = \frac{1}{2} \left[ \text{tr} (\mathcal{X})^2 - \text{tr} (\mathcal{X}^2) \right], \]

\[ e_3 = \frac{1}{6} \left[ \text{tr} (\mathcal{X})^3 - 3 \text{tr} (\mathcal{X}) \text{tr} (\mathcal{X}^2) + 2 \text{tr} (\mathcal{X}^3) \right], \ e_4 = \det (\mathcal{X}) \]
\[ S_{bi} = \frac{M^2_g}{2} \int d^4x \sqrt{-g} R_g + \int d^4x \sqrt{-g} \mathcal{L}_{\text{matter}} + \]
\[ \quad + m^2 M^2_{\text{eff}} \int d^4x \sqrt{-g} \sum_{n=0}^{4} \beta_n e_n(\mathcal{X}) + \frac{M^2_f}{2} \int d^4x \sqrt{-f} R_f \]

where \( \mathcal{X} = \sqrt{g^{-1} f} \), i.e. \( \mathcal{X}_\alpha^\mu \mathcal{X}_\nu^\alpha = g^{\mu\alpha} f_{\alpha\nu} \), \( M^2_{\text{eff}} = \left( \frac{1}{M^2_g} + \frac{1}{M^2_f} \right)^{-1} \), and

\[ e_0 = 1, \quad e_1 = \text{tr}(\mathcal{X}), \quad e_2 = \frac{1}{2} \left[ \text{tr}(\mathcal{X})^2 - \text{tr}(\mathcal{X}^2) \right], \]
\[ e_3 = \frac{1}{6} \left[ \text{tr}(\mathcal{X})^3 - 3 \text{tr}(\mathcal{X}) \text{tr}(\mathcal{X}^2) + 2 \text{tr}(\mathcal{X}^3) \right], \quad e_4 = \text{det}(\mathcal{X}) \]
Einstein equations

\[ G(g)_{\mu\nu} + m^2 \cos^2(\theta) \sum_{n=0}^{3} \beta_n V^{(n)}(g)_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \]

\[ G(f)_{\mu\nu} + m^2 \sin^2(\theta) \sum_{n=1}^{4} \sqrt{|g|/|f|} \beta_n V^{(n)}(f)_{\mu\nu} = 0, \]

where \( \cos^2(\theta) = \frac{M_{\text{eff}}^2}{M_g^2} \), the \( \sin^2(\theta) = \frac{M_{\text{eff}}^2}{M_f^2} \), and the interaction or mass terms \( V(g/f) \) follow from the variation of the \( e_n \), e.g. \( V_{\mu\nu}^{(0)}(g) = g_{\mu\nu} \).

**Effective CC**

\[ \Lambda_{\text{eff}} = m^2 \cos^2(\theta) \beta_0 \]
Einstein equations – linearised

For a specific choice of $\vec{\beta} = (3, -1, 0, 0, +1)$, one recovers the FP mass term by expanding around a background $\eta$

\[
g_{\mu\nu} = \eta_{\mu\nu} + \frac{\delta g_{\mu\nu}}{M_g}, \quad f_{\mu\nu} = \eta_{\mu\nu} + \frac{\delta f_{\mu\nu}}{M_f}
\]

\[
S_{\text{mass}} = m^2 M_{\text{eff}}^2 \int d^4x \sqrt{-|g|} \sum_{n=0}^{4} \beta_n e_n(\chi)
\]

\[
\simeq - m^2 M_{\text{eff}}^2 \int d^4x \left[ \left( \frac{\delta g}{M_g} - \frac{\delta f}{M_f} \right)_{\mu\nu} \left( \frac{\delta g}{M_g} - \frac{\delta f}{M_f} \right)_{\mu\nu} - \left( \frac{\delta g^\mu}{M_g} - \frac{\delta f^\mu}{M_f} \right)^2 \right]
\]
Therefore, the mass eigenstates are

\[
\begin{pmatrix}
u \\
v'
\end{pmatrix} \equiv \begin{pmatrix}
cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
\delta g \\
\delta f
\end{pmatrix}
\]

\[\Rightarrow S_{\text{mass}} = -\frac{m^2}{8} \int d^4x \left[ u^\mu_\nu u^\mu_\nu - (u^\mu_\mu)^2 \right],\]

while \( v \) remains massless.

**two interesting limits**

\( \theta \rightarrow 0: \) Only massive mode couples to matter

\( \theta \rightarrow \frac{\pi}{2}: \) GR limit, but without the discontinuity!
Starting off with an ansatz

\[
g_{\mu\nu}dx^\mu dx^\nu = -e^{\nu_1(r)} dt^2 + e^{\lambda_1(r)} dr^2 + r^2 d\Omega^2,
\]

\[
f_{\mu\nu}dx^\mu dx^\nu = -e^{\nu_2(r)} dt^2 + e^{\lambda_2(r)} (r + r\mu(r))^2 dr^2 + (r + r\mu(r))^2 d\Omega^2,
\]

we can calculate the “Newtonian” potential in the weak field regime:

\[
\nu_1(r) = \begin{cases} 
-\frac{r_s}{r} - r^2 \frac{\Lambda_{\text{eff}}}{3} & r \ll 3 \sqrt{\frac{r_s}{m_g^2}} \\
-\frac{r_s}{r} \left[ h(\theta) + 2 \cos^2(\theta) e^{-m_g r} \left( 1 + \frac{\Lambda'_{\text{eff}}}{3m_g^2} \right) \right] & 3 \sqrt{\frac{r_s}{m_g^2}} \ll r \ll 3 \sqrt{\frac{r_s}{m_g^2}} \\
-\frac{r^2 \sin^2(\theta) \frac{\Lambda_{\text{eff}}}{3}} & r \gg 3 \sqrt{\frac{r_s}{m_g^2}}
\end{cases}
\]

Degravitation for \( \sin^2(\theta) \to 0! \)
Phenomenology
Previous analyses [Brownstein and Moffat, 2006]
Previous analyses [Brownstein and Moffat, 2006]
Bounds on the graviton mass

solar system tests $\lambda_g > 2.8 \cdot 10^{12} \text{ km}, \ m_g < 7.2 \cdot 10^{-23} \text{ eV}$

weak lensing $\lambda_g > 2 \cdot 10^{21} \text{ km}, \ m_g < 6 \cdot 10^{-32} \text{ eV}$

rely on a Yukawa potential $\propto e^{-m_g r}$

GW150914 $\lambda_g > 4.2 \cdot 10^{11} \text{ km}, \ m_g < 1.2 \cdot 10^{-22} \text{ eV}$

due to a modified dispersion relation $v_g = \sqrt{1 - \frac{m_g^2}{E^2}}$

But do these bounds apply here?
A word of honesty – the bullet cluster

Bullet cluster favours particle DM
Need extra, non-baryonic degrees of freedom!
Conclusions
Summary

- We know how to give the graviton a mass! ✓
- Non-linearities make this a difficult, yet interesting problem!
- Bigravity is a consistent framework which allows to study some of the effects
  - ⇒ solve (in part) the DM problem ✓
  - ⇒ Degravitation seems to work ✓
  - ⇒ $\Lambda \sim m_g^2$ ✓
- Many open questions and lots of work to be done!
  - ⇒ CMB, GW, cosmological implications etc.
Thank you!

(and please comment!)
Back-up slides
Recall that $\mathcal{X}^{\mu}_{\nu} = \left(\sqrt{g^{-1}f}\right)^{\mu}_{\nu}$:

$V^{(0)}(g)^{\mu}_{\nu} = \delta^{\mu}_{\nu},$

$V^{(1)}(g)^{\mu}_{\nu} = \text{tr} \left( \mathcal{X} \right) \delta^{\mu}_{\nu} - \mathcal{X}^{\mu}_{\nu},$

$V^{(2)}(g)^{\mu}_{\nu} = (\mathcal{X}^2)^{\mu}_{\nu} - \text{tr} \left( \mathcal{X} \right) \mathcal{X}^{\mu}_{\nu} + \frac{\delta^{\mu}_{\nu}}{2} \left[ \text{tr} \left( \mathcal{X} \right)^2 - \text{tr} \left( \mathcal{X}^2 \right) \right],$

$V^{(3)}(g)^{\mu}_{\nu} = - \left( \mathcal{X}^3 \right)^{\mu}_{\nu} + \text{tr} \left( \mathcal{X} \right) \left( \mathcal{X}^2 \right)^{\mu}_{\nu} - \frac{1}{2} \left[ \text{tr} \left( \mathcal{X} \right)^2 - \text{tr} \left( \mathcal{X}^2 \right) \right] \mathcal{X}^{\mu}_{\nu} +$

$+ \frac{\delta^{\mu}_{\nu}}{6} \left[ \text{tr} \left( \mathcal{X} \right)^3 - 3 \text{tr} \left( \mathcal{X} \right) \text{tr} \left( \mathcal{X}^2 \right) + 2 \text{tr} \left( \mathcal{X}^3 \right) \right]$
Recall that $\mathcal{X}^{\mu}_{\nu} = \left( \sqrt{g^{-1} f} \right)^{\mu}_{\nu}$:

$V^{(1)}(f)^{\mu}_{\nu} = \mathcal{X}^{\mu}_{\nu},$

$V^{(2)}(f)^{\mu}_{\nu} = - \left( \mathcal{X}^2 \right)^{\mu}_{\nu} + \text{tr}(\mathcal{X}) \mathcal{X}^{\mu}_{\nu},$

$V^{(3)}(f)^{\mu}_{\nu} = \left( \mathcal{X}^3 \right)^{\mu}_{\nu} + \text{tr}(\mathcal{X}) \left( \mathcal{X}^2 \right)^{\mu}_{\nu} + \frac{1}{2} \left[ \text{tr}(\mathcal{X})^2 + \text{tr}(\mathcal{X}^2) \right] \mathcal{X}^{\mu}_{\nu},$

$V^{(4)}(f)^{\mu}_{\nu} = \delta^{\mu}_{\nu}$

