Vacuum high-harmonic generation in the shock regime

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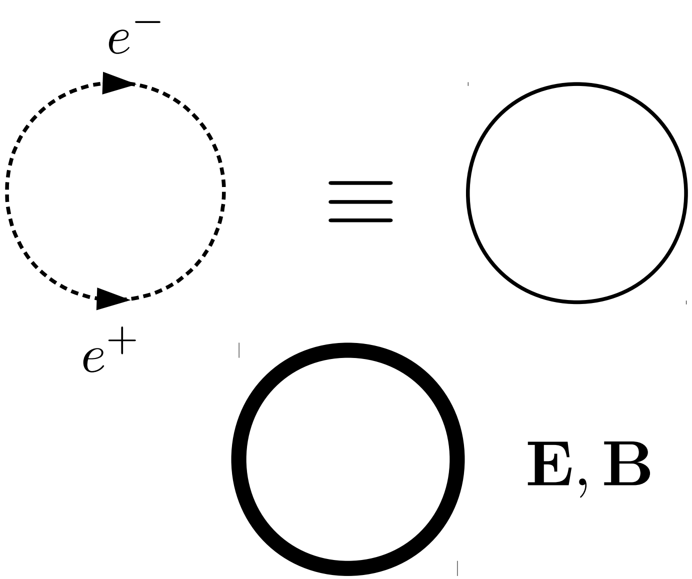
Vacuum electrodynamics

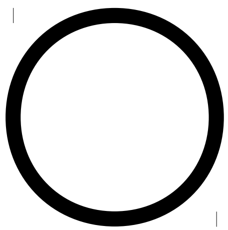
$$\mathcal{L} = \mathcal{L}_{\text{Maxwell}} = (E^2 - B^2)/2$$

$$\Box\,\mathbf{E} = \mathbf{0}$$
$$\Box\,\mathbf{B} = \mathbf{0}$$

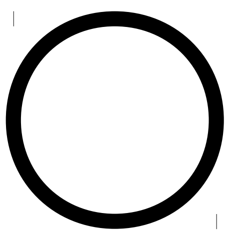


$$\mathcal{L} = \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{vac}}$$

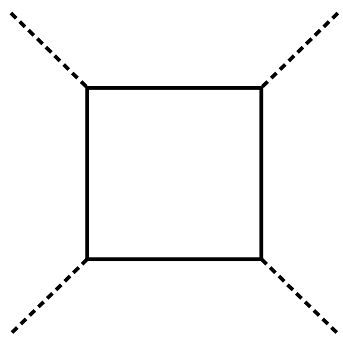
$$\Box\,\mathbf{E} = \partial_t \mathbf{J}_{\text{vac}}\left[\mathbf{E}, \mathbf{B}\right]$$
$$\Box\,\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{J}_{\text{vac}}\left[\mathbf{E}, \mathbf{B}\right]$$



$$\mathcal{L}_{\rm vac} = \mathcal{L}_{\rm HE}\left(\mathbf{E}/E_{\rm cr}, \mathbf{B}/B_{\rm cr}\right) \qquad E_{\rm cr} = \frac{m^2c^3}{e\hbar} = 1.3\times 10^{16}\,\mathrm{Vcm}^{-1}$$

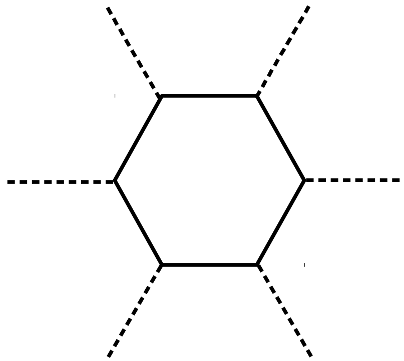


$\sim$



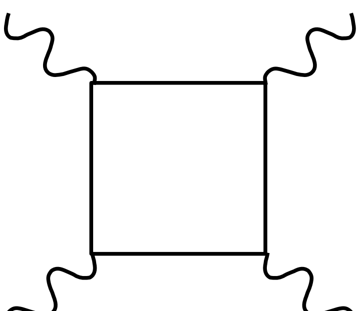
$O\left[(E/E_{\rm cr})^4\right]$

$+$

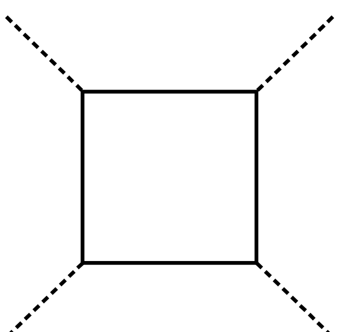


$O\left[(E/E_{\rm cr})^6\right]$

$+ \ \ldots$



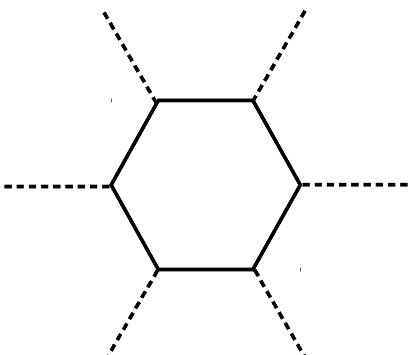
$$\xrightarrow[\frac{\omega}{m} \ll 1]{}$$



R. Karplus, M. Neuman
Phys. Rev. **80**, 380 (1950)

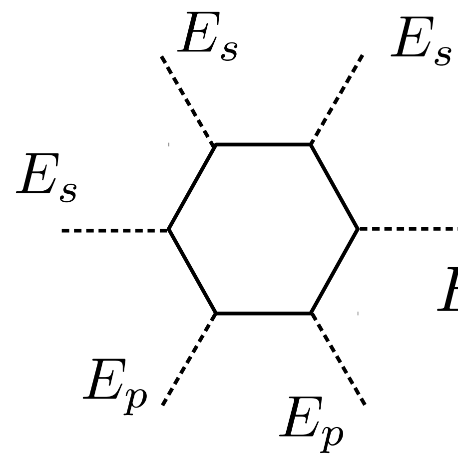
Vacuum harmonic generation

$$\mathbf{E} = \mathbf{E}_s(k_s \cdot x) + \mathbf{E}_p(k_p \cdot x)$$



$$\mathcal{L} \sim (E_s E_p)^3$$

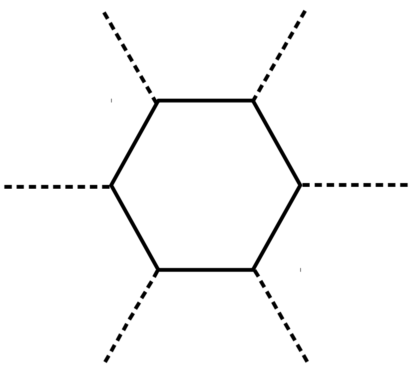
$$J_{\text{vac}} \sim E_s^3 E_p^2,\ E_s^2 E_p^3$$



$$E' \sim E_s^3 E_p^2$$

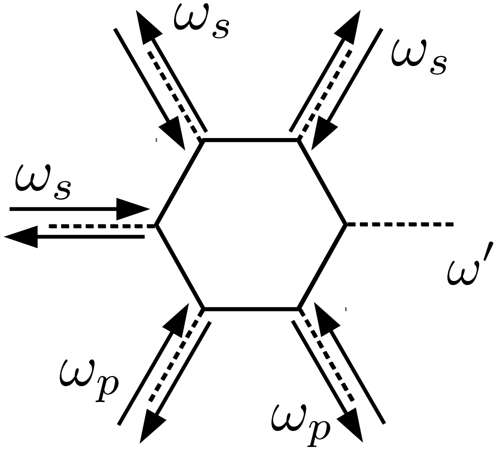
Vacuum harmonic generation

$$\mathbf{E} = \mathbf{E}_s(k_s \cdot x) + \mathbf{E}_p(k_p \cdot x)$$



$$\mathcal{L} \sim (E_s E_p)^3$$

$$J_{\text{vac}} \sim E_s^3 E_p^2,\ E_s^2 E_p^3$$



Vacuum harmonic generation

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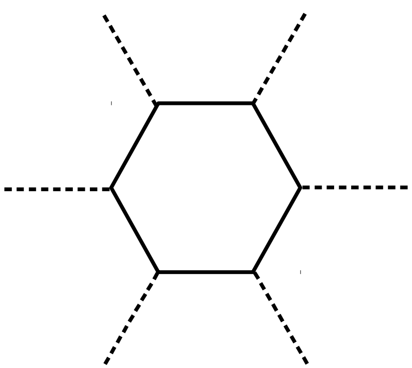
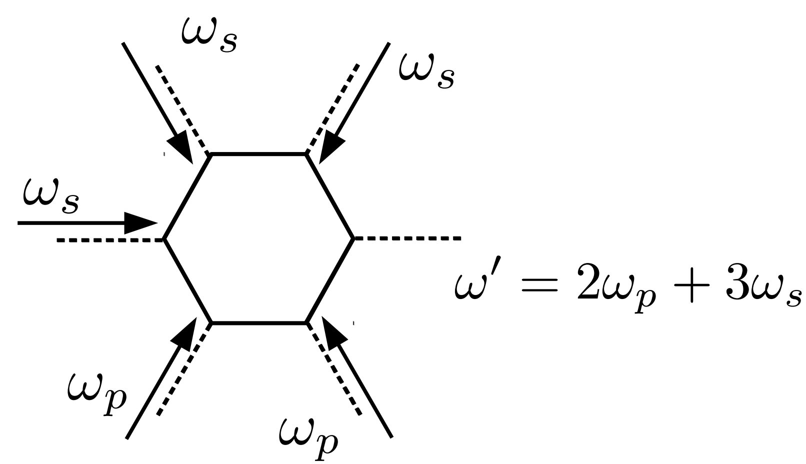
$$\mathbf{E} = \mathbf{E}_s(k_s \cdot x) + \mathbf{E}_p(k_p \cdot x)$$



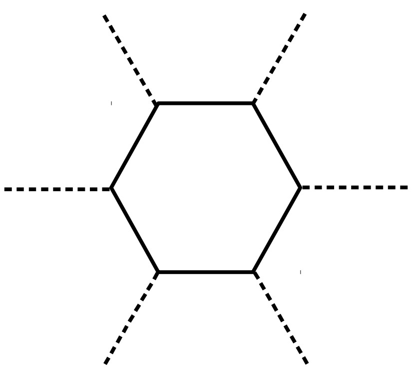
$$\mathcal{L} \sim (E_s E_p)^3$$

$$J_{\text{vac}} \sim E_s^3 E_p^2, \; E_s^2 E_p^3$$



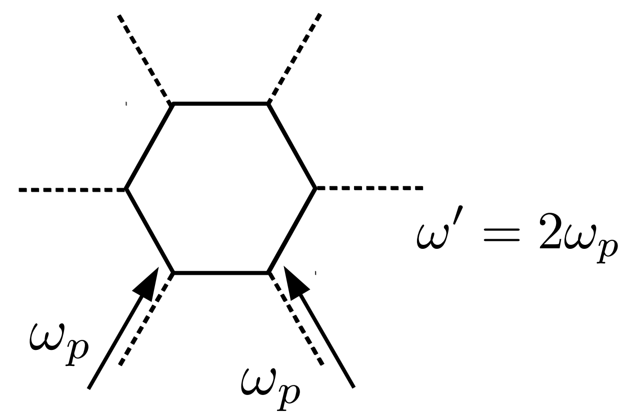
Vacuum harmonic generation

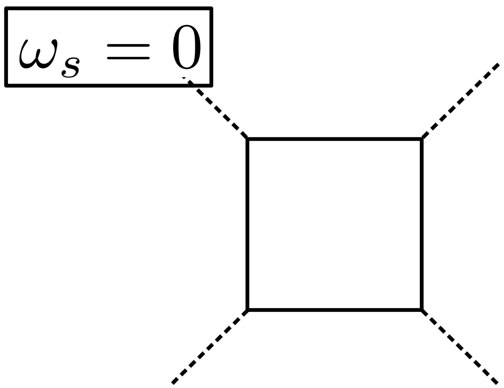
$$\mathbf{E} = \mathbf{E}_s(k_s \cdot x) + \mathbf{E}_p(k_p \cdot x)$$



$$\mathcal{L} \sim (E_s E_p)^3$$

$$J_{\text{vac}} \sim E_s^3 E_p^2,\; E_s^2 E_p^3$$





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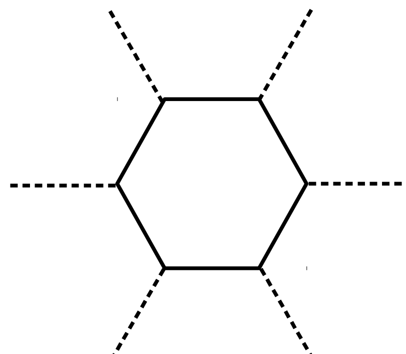
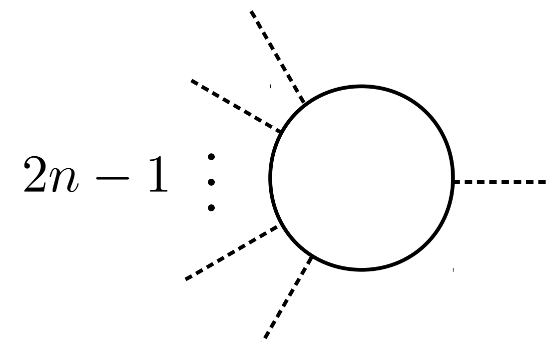


$+ \; \dots \; +$



$\omega_p, 2\,\omega_p$ $\omega_p, 2\,\omega_p, 3\,\omega_p$ $\omega_p, 2\,\omega_p, \dots, n\,\omega_p$

$$\frac{E'}{E_p} \sim \frac{E_p}{E_{\rm cr}}\frac{E_s}{E_{\rm cr}}\,\Phi \qquad \left(\frac{E_p}{E_{\rm cr}}\frac{E_s}{E_{\rm cr}}\right)^2 \Phi \qquad \left(\frac{E_p}{E_{\rm cr}}\frac{E_s}{E_{\rm cr}}\right)^{n-1} \Phi$$

E. Lundström et al.
Phys. Rev. Lett, **96**, 083602 (2006)

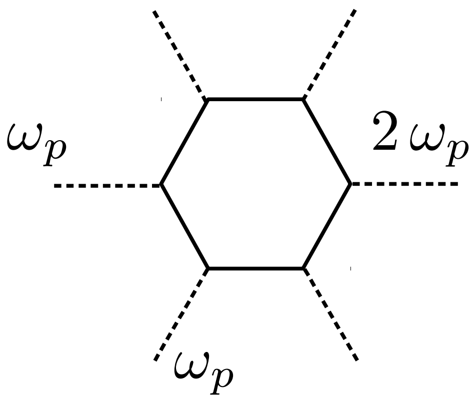
B. King and C. H. Keitel,
New J. Phys. **14,**, 103002 (2012)

A. Di Piazza, K. Hatsagortsyan, C. H. Keitel
Phys. Rev. D **72** (2005)

A. M. Fedotov and N. B. Narozhny,
Phys. Lett. A **362**, 1 (2006)

Vacuum high harmonic generation

$\boxed{\omega_s = 0}$



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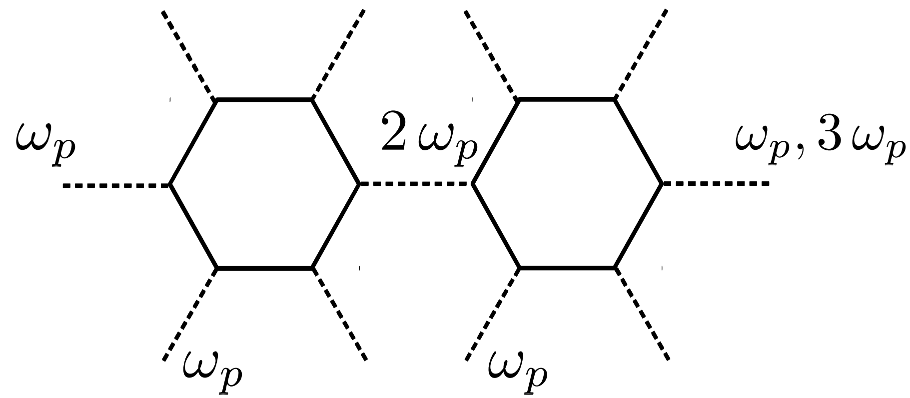
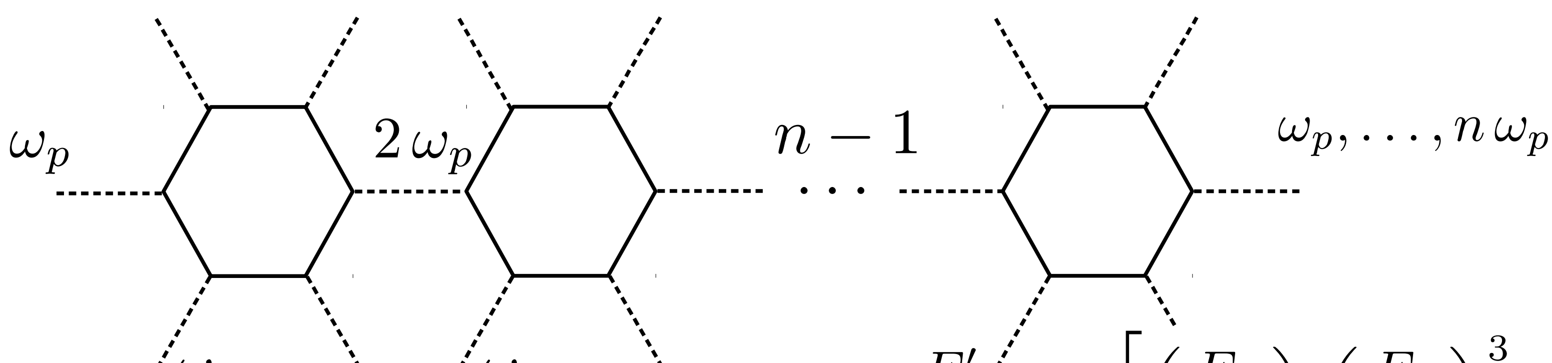


$$\frac{E'}{E_p} \sim \left(\frac{E_p}{E_{\rm cr}}\right)\left(\frac{E_s}{E_{\rm cr}}\right)^3 \Phi$$

$$\frac{E'}{E_p} \sim \left[\left(\frac{E_p}{E_{\rm cr}}\right)\left(\frac{E_s}{E_{\rm cr}}\right)^3 \Phi\right]^2$$

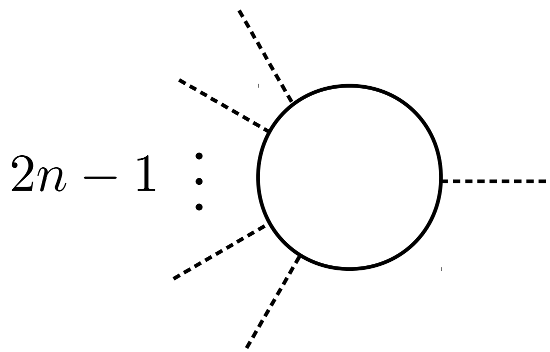
$E_s \gg E_p$

$+$

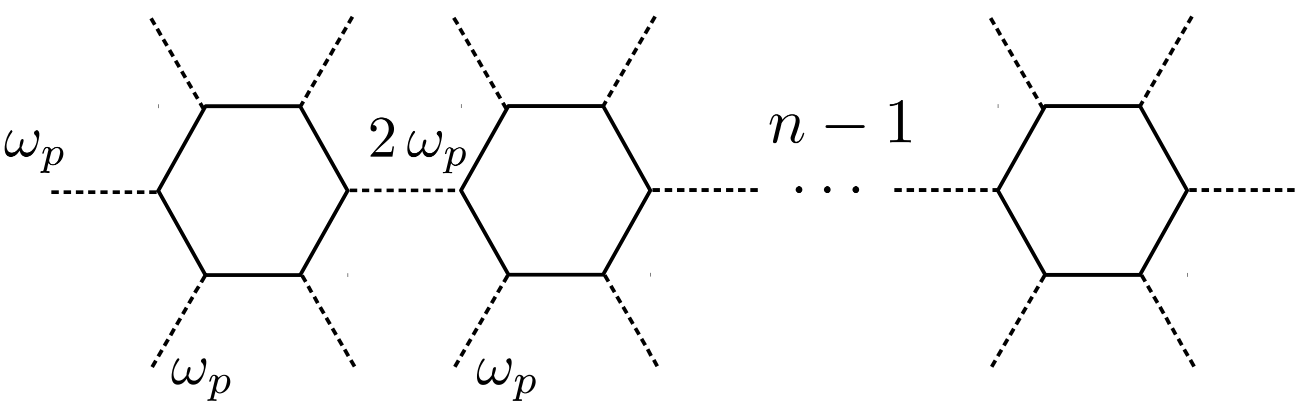


$$\frac{E'}{E_p} \sim \left[\left(\frac{E_p}{E_{\rm cr}}\right)\left(\frac{E_s}{E_{\rm cr}}\right)^3 \Phi\right]^n$$

Vacuum high harmonic generation



$$\frac{E'(n\omega_p)}{E_p(\omega_p)} \sim \left(\frac{E_p}{E_{\rm cr}}\right)^{n-1}\left(\frac{E_s}{E_{\rm cr}}\right)^{n}\Phi$$



$$\frac{E'(n\omega_p)}{E_p(\omega_p)} \sim \Bigg[\underbrace{\left(\frac{E_pE_s^3}{E_{\rm cr}^4}\right)\Phi}_{\to 1}\Bigg]^{n-1}$$

"Shock regime"

Nonlinear vacuum wave equation

$$\Box\,\mathbf{E} = \partial_t \mathbf{J}_{\mathrm{vac}}\left[\mathbf{E},\mathbf{B}\right] \qquad \Box\,\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{J}_{\mathrm{vac}}\left[\mathbf{E},\mathbf{B}\right]$$

Plane wave + const. B field (arbitrary strength)

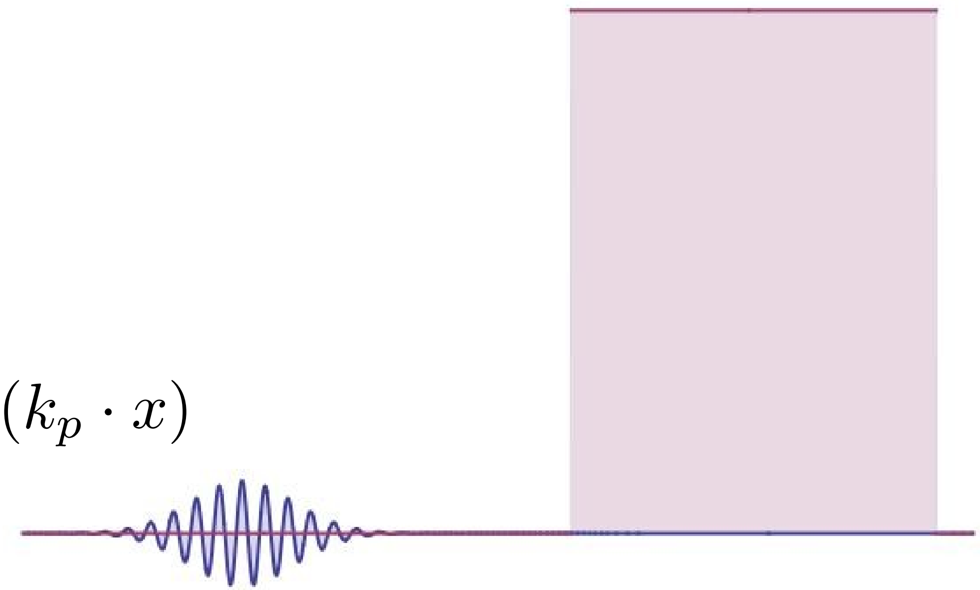
Z. Bialynicka-Birula
Physica **2D**, 513-524 (1981)

Plane wave + const. B field (weak)

V. V. Zheleznyakov and A. L. Fabrikant
Zh. Eksp. Teor. Fiz. **82**, 1366-1374 (1982)

$$\mathbf{E}_s(k_s\cdot x)$$

$$\mathbf{E}_p(k_p\cdot x)$$



Nonlinear wave equation solution

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$$\Box\,\mathbf{E} = \partial_t \mathbf{J}_{\text{vac}}\left[\mathbf{E},\mathbf{B}\right] \qquad \Box\,\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{J}_{\text{vac}}\left[\mathbf{E},\mathbf{B}\right]$$

Solving Maxwell's Equations **numerically** (linear case)

$$\mathbb{I}_4\,\frac{\partial}{\partial t}\begin{pmatrix} E_x \\ E_y \\ B_x \\ B_y \end{pmatrix} + \mathbf{Q}\,\frac{\partial}{\partial z}\begin{pmatrix} E_x \\ E_y \\ B_x \\ B_y \end{pmatrix} = \mathbf{0} \qquad \mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Transformation to advection equation:

$$\partial_t \mathbf{u}(t,z) + \boldsymbol{\Lambda}\,\partial_z \mathbf{u}(t,z) = 0 \qquad \boldsymbol{\Lambda} = \mathbf{S}\mathbf{Q}\mathbf{S}^{-1}$$

$$\mathbf{S} = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \qquad \mathbf{u} := \mathbf{S}\,\mathbf{f} = \frac{1}{\sqrt{2}}\begin{pmatrix} B_y - E_x \\ E_y + B_x \\ E_x + B_y \\ B_x - E_y \end{pmatrix}$$

Pseudocharacteristic Method of Lines + CVODE ODE-Solver from SUNDIALS suite

Nonlinear wave equation solution

$$\Box\,\mathbf{E} = \partial_t \mathbf{J}_{\rm vac}\left[\mathbf{E},\mathbf{B}\right] \qquad \Box\,\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{J}_{\rm vac}\left[\mathbf{E},\mathbf{B}\right]$$

Plane-wave ansatz:

$$\Box\,\mathbf{E}_p = \partial_t \mathbf{J}_{\rm vac}\left[\mathbf{E}_p + \mathbf{E}_s\right]$$

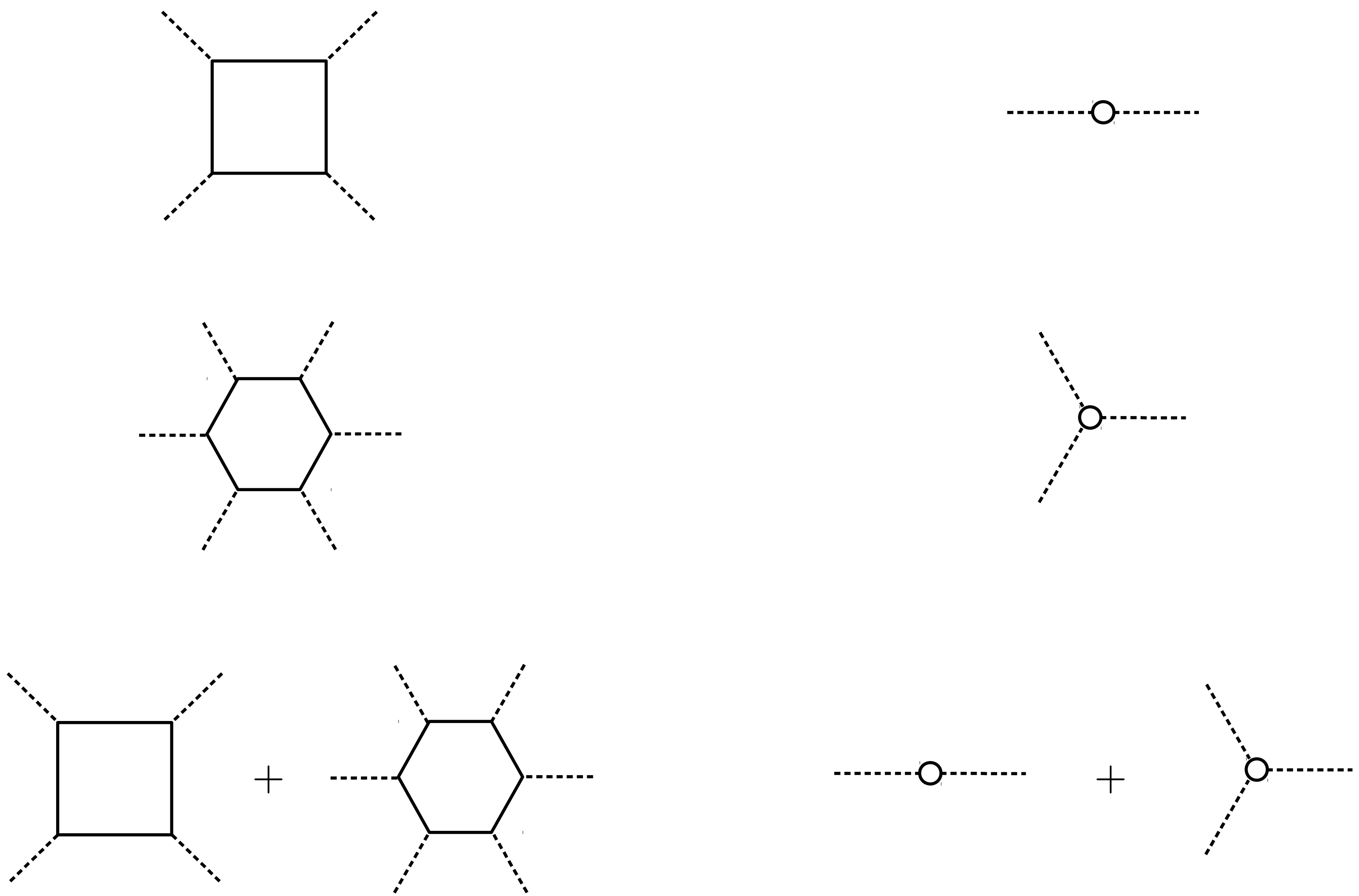
$$\Box\,\mathbf{E}_p^{(n+1)} = \partial_t \mathbf{J}_{\rm vac}\left[\mathbf{E}_p^{(n)} + \mathbf{E}_s^{(0)}\right]$$

$$\mathbf{E}_p^{(n+1)} = \mathbf{E}_p^{(0)} + \Delta\mathbf{E}_p^{(n)}$$

$$\Delta\mathbf{E}_p^{(n)} = \int dt'\,dz'\;G_{\rm R}(t',z')\,\partial_{t'}\mathbf{J}_{\rm vac}\left[\mathbf{E}_p^{(n)}(t',z') + \mathbf{E}_s^{(0)}(t',z')\right]$$

$$G_{\rm R}(t,z) = \frac{1}{2}\theta(t)\theta(t-|z|)$$

Photon scattering processes



Four-photon scattering

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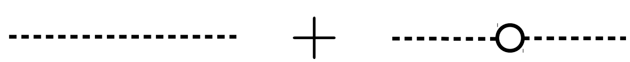
$$\mathbf{E}_p^{(n+1)} = \mathbf{E}_p^{(0)} + \Delta\mathbf{E}_p^{(n)}$$

$E_p^{(0)}$:

$\cos\varphi_p$

$E_p^{(1)}$:



$\cos(\varphi_p) - \upsilon_1 \sin(\varphi_p)$

$E_p^{(2)}$:

$\cos(\varphi_p) - \upsilon_1 \sin(\varphi_p) - \frac{\upsilon_1^2}{2}\cos(\varphi_p)$

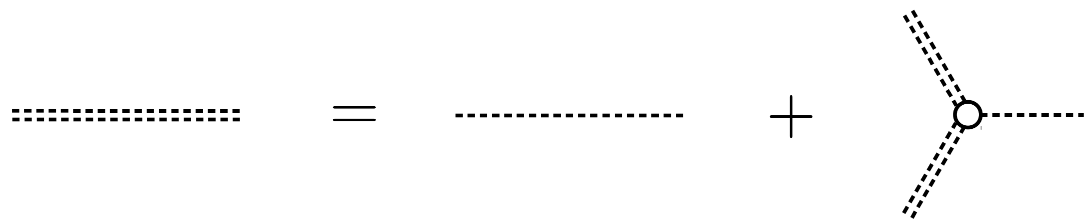
$$\mathbf{E}_p = \lim_{n\to\infty} \mathbf{E}_p^{(n)} = \mathrm{e}^{\upsilon_1 \frac{d}{d\varphi_p}} \mathbf{E}_p^{(0)}(\varphi_p) = \mathbf{E}_p^{(0)}(\varphi_p + \upsilon_1)$$

$$\upsilon_1 = [\mathsf{n}_{\text{vac}} - 1]\,\Phi$$

$$\boxed{\mathsf{n}_{\text{vac}} = 1 + \frac{(11 \pm 3)\alpha}{45\pi}\frac{\mathcal{E}_s^2}{E_{\text{cr}}^2}}$$

Six-photon scattering

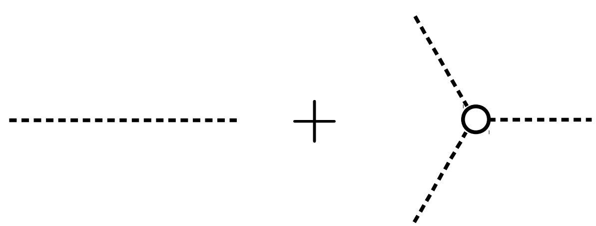
$$\mathbf{E}_p^{(n+1)} = \mathbf{E}_p^{(0)} + \Delta\mathbf{E}_p^{(n)}$$



$E_p^{(0)}$:

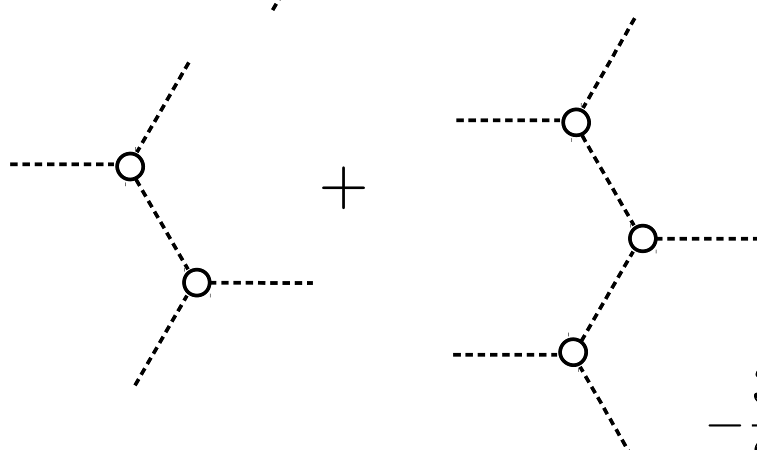
$\cos\varphi_p$

$E_p^{(1)}$:



$$\cos(\varphi_p) - \frac{\nu_2}{2}\sin(2\varphi_p)$$

$E_p^{(2)}$: $E_p^{(1)}$ +

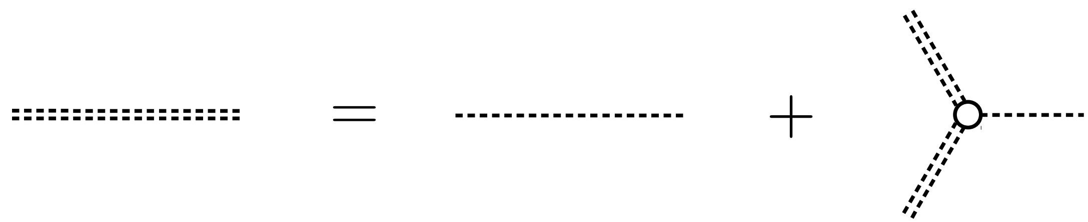


$$E_p^{(1)} - \frac{1}{2}\left(\frac{\nu_2}{2}\right)^2 \cos\varphi_p$$

$$-\frac{3}{2}\left(\frac{\nu_2}{2}\right)^2 \cos 3\varphi_p + \left(\frac{\nu_2}{2}\right)^3 \sin 4\varphi_p$$

Six-photon scattering

$$\mathbf{E}_p^{(n+1)} = \mathbf{E}_p^{(0)} + \Delta\mathbf{E}_p^{(n)}$$



$$E_p(\varphi_p) = \sum_{j=1}^{\infty}\left[(-1)^j \frac{\mathrm{J}_{2j}(2j\nu_2)}{2j\nu_2}\sin 2j\varphi_p + (-1)^{j+1}\frac{\mathrm{J}_{2j-1}[(2j-1)\nu_2]}{(2j-1)\nu_2}\cos(2j-1)\varphi_p\right]$$

$$\nu_2 = \frac{64\alpha}{105\pi}\frac{\mathcal{E}_s^3\mathcal{E}_p}{E_{\mathrm{cr}}^4}\Phi \qquad \nu_2 \to \frac{64\alpha}{105\pi}\frac{\mathcal{E}_s^3\mathcal{E}_p}{E_{\mathrm{cr}}^2}\mathrm{e}^{-\left(\frac{\varphi_p}{\Phi_p}\right)^2}\Phi$$

$$\mathbf{E}_p(\varphi_p) = \mathbf{E}_p^{(0)}\left(\varphi_p + \nu_2^{\parallel}[E_p(\varphi_p)]\right)$$

$$\nu_2^{\parallel} = \left[\mathsf{n}_{\mathrm{vac}}^{\parallel} - 1\right]\Phi \qquad \boxed{\mathsf{n}_{\mathrm{vac}}^{\parallel} = 1 + \frac{64\alpha}{105\pi}\frac{\mathcal{E}_s^2}{E_{\mathrm{cr}}^2}\frac{\mathcal{E}_p}{E_{\mathrm{cr}}}}$$

Six-photon scattering: HHG in vacuum

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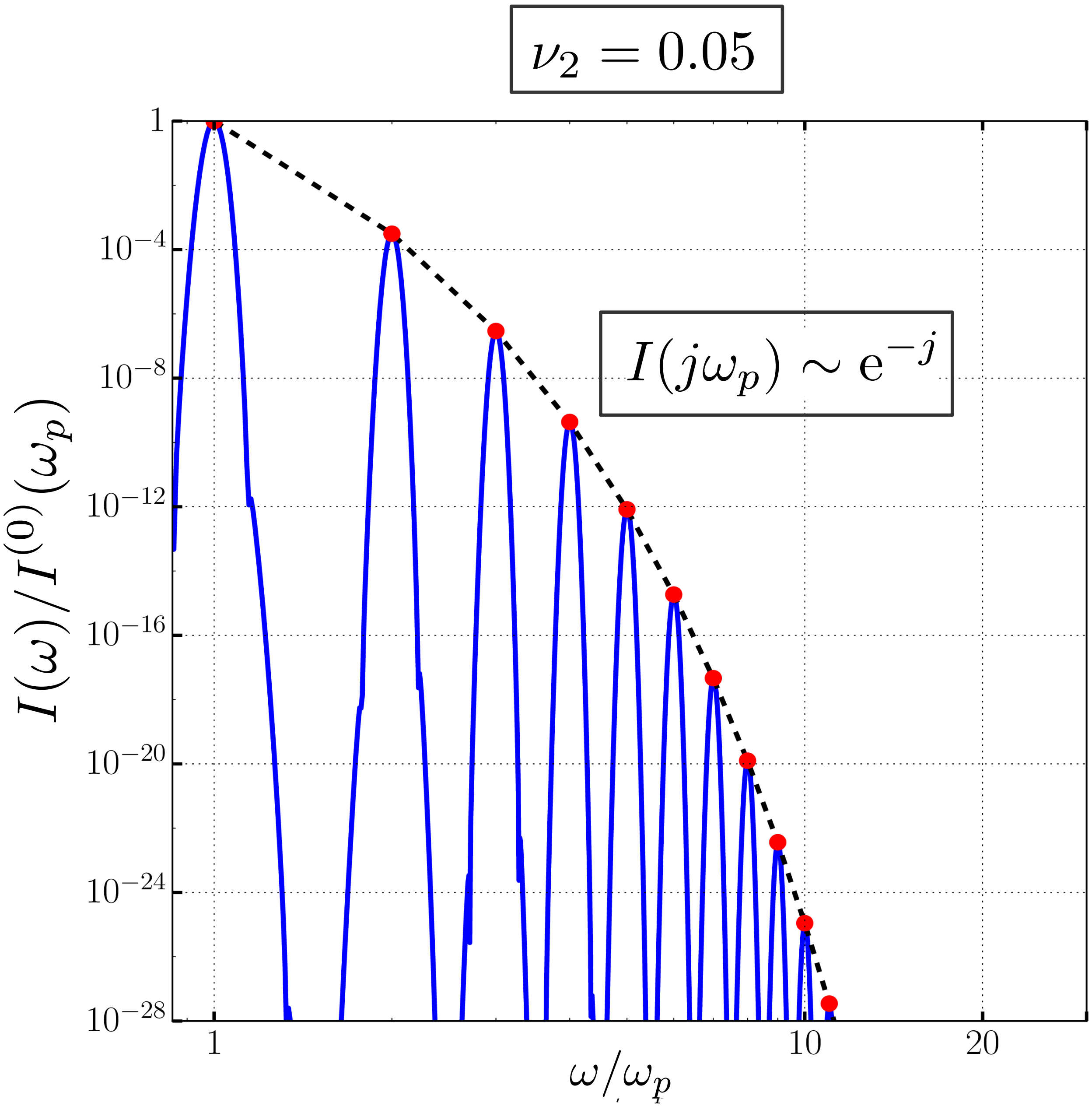
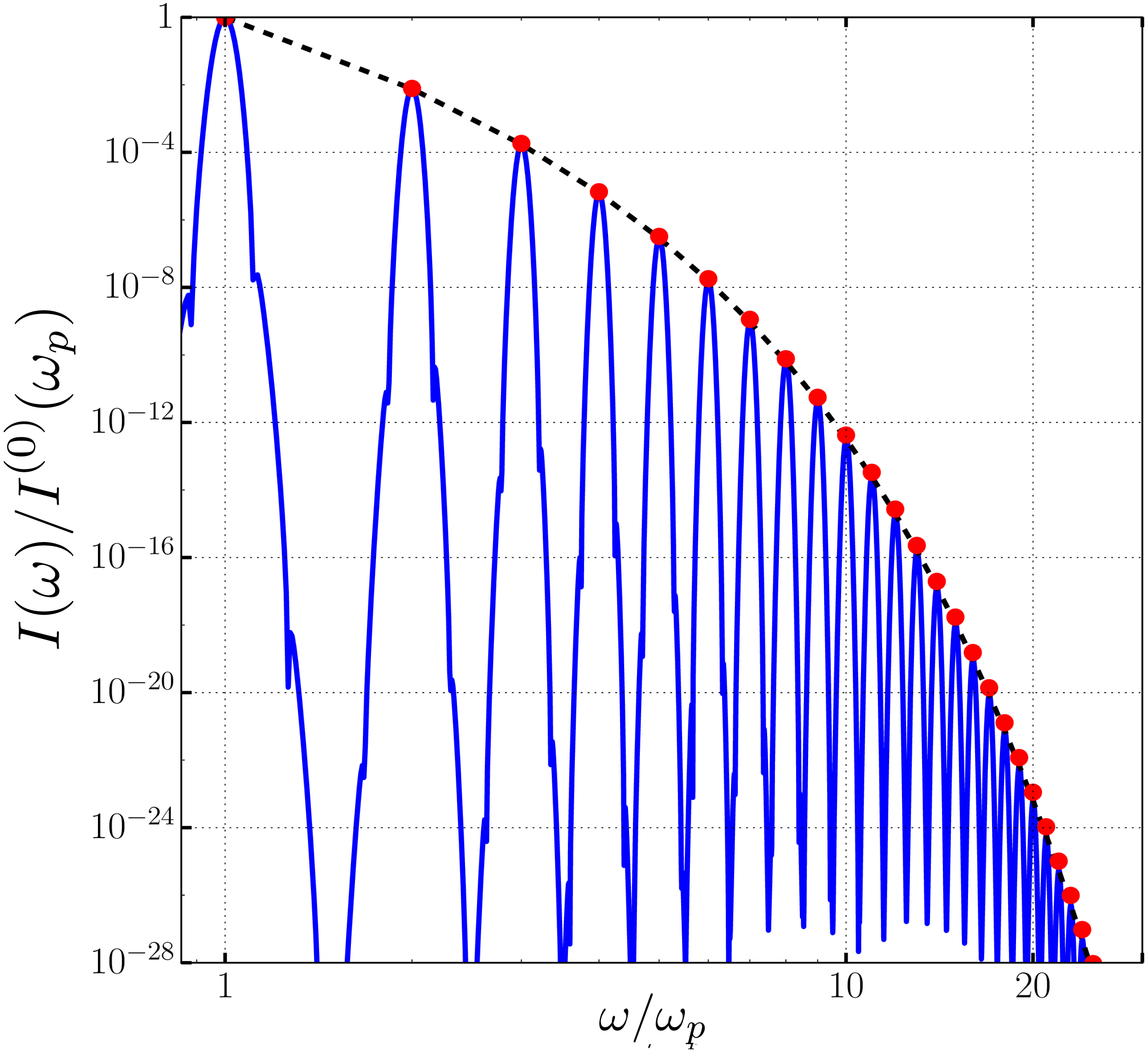


$$\nu_2 = \frac{64\alpha}{105\pi}\frac{\mathcal{E}_s^3\mathcal{E}_p}{E_{\mathrm{cr}}^4}\Phi$$

$$\Phi = \omega_p T$$

Conversion to higher harmonics: $0.06\,\%$

$\nu_2 = 0.25$



$$\nu_2 = \frac{64\alpha}{105\pi}\frac{\mathcal{E}_s^3\mathcal{E}_p}{E_{\rm cr}^4}\Phi$$

$$\Phi = \omega_p T$$

Conversion to higher harmonics: $1.55\,\%$

Six-photon scattering: HHG in vacuum

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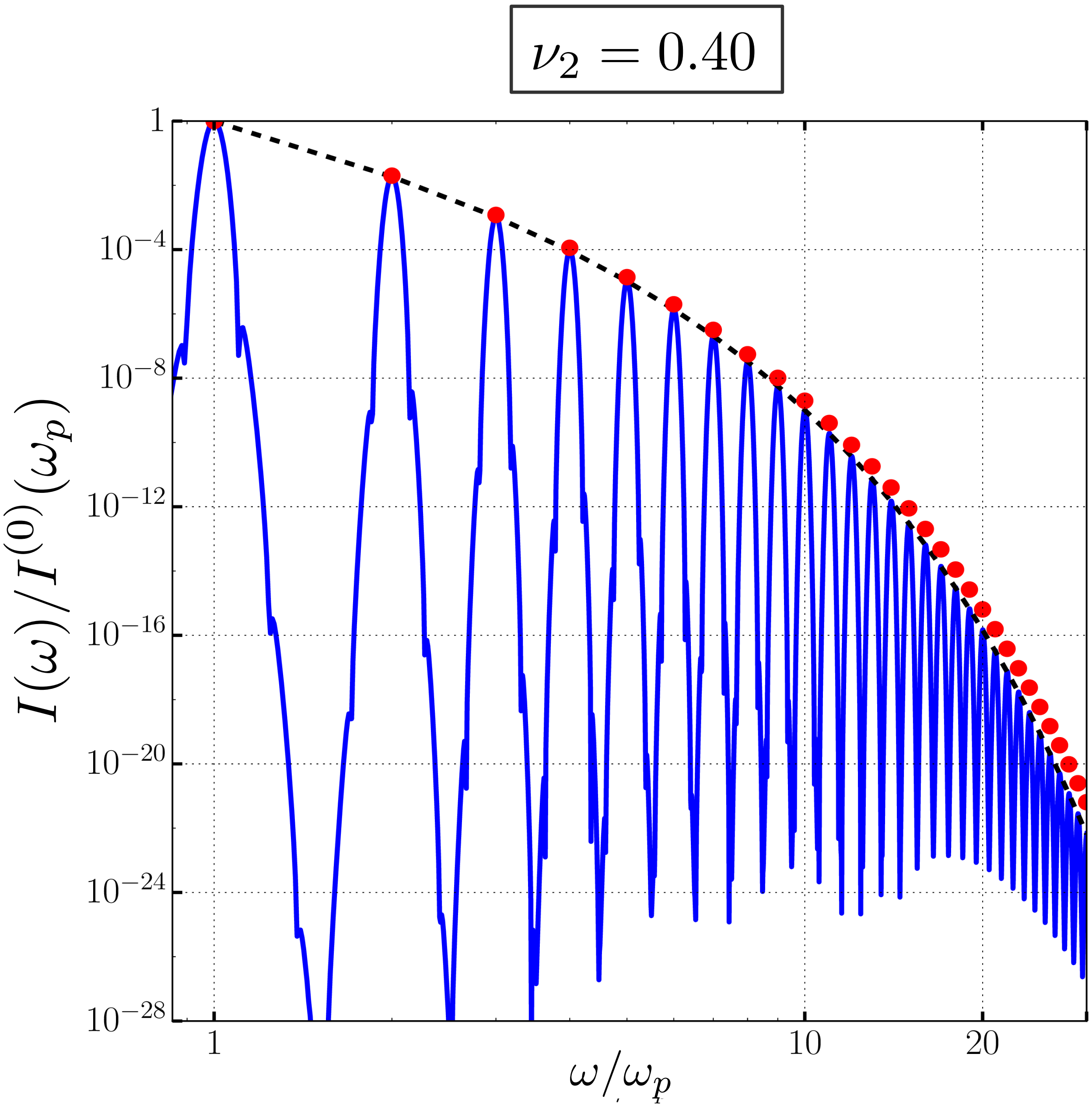

$$\nu_2 = \frac{64\alpha}{105\pi}\frac{\mathcal{E}_s^3\mathcal{E}_p}{E_{\mathrm{cr}}^4}\Phi$$

$$\Phi = \omega_p T$$

Conversion to higher harmonics: $3.93\,\%$

Six-photon scattering: HHG in vacuum

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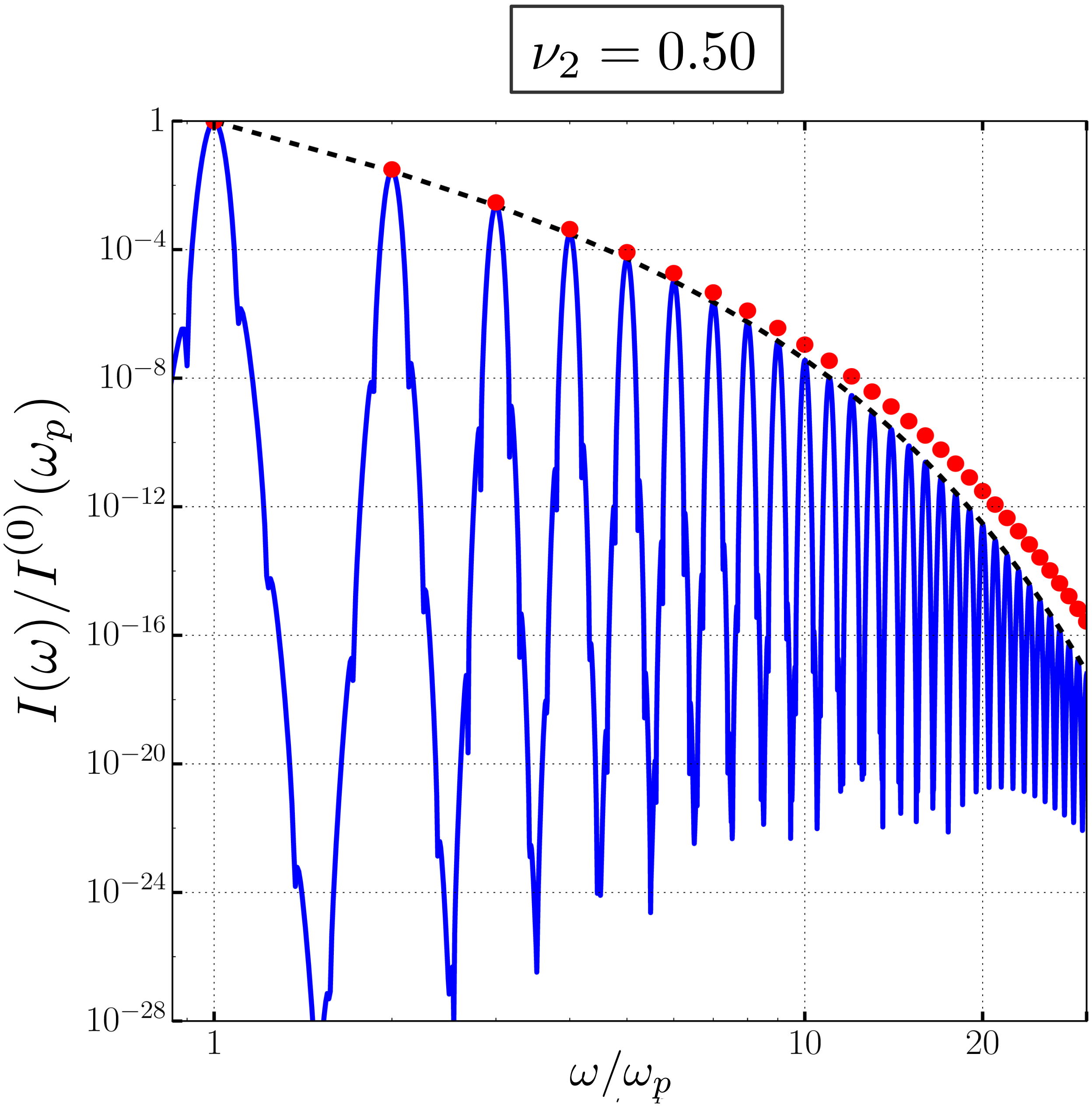

$$\nu_2 = \frac{64\alpha}{105\pi}\frac{\mathcal{E}_s^3\mathcal{E}_p}{E_{\rm cr}^4}\Phi$$

$$\Phi = \omega_p T$$

Conversion to higher harmonics: $6.09\,\%$

Six-photon scattering: HHG in vacuum

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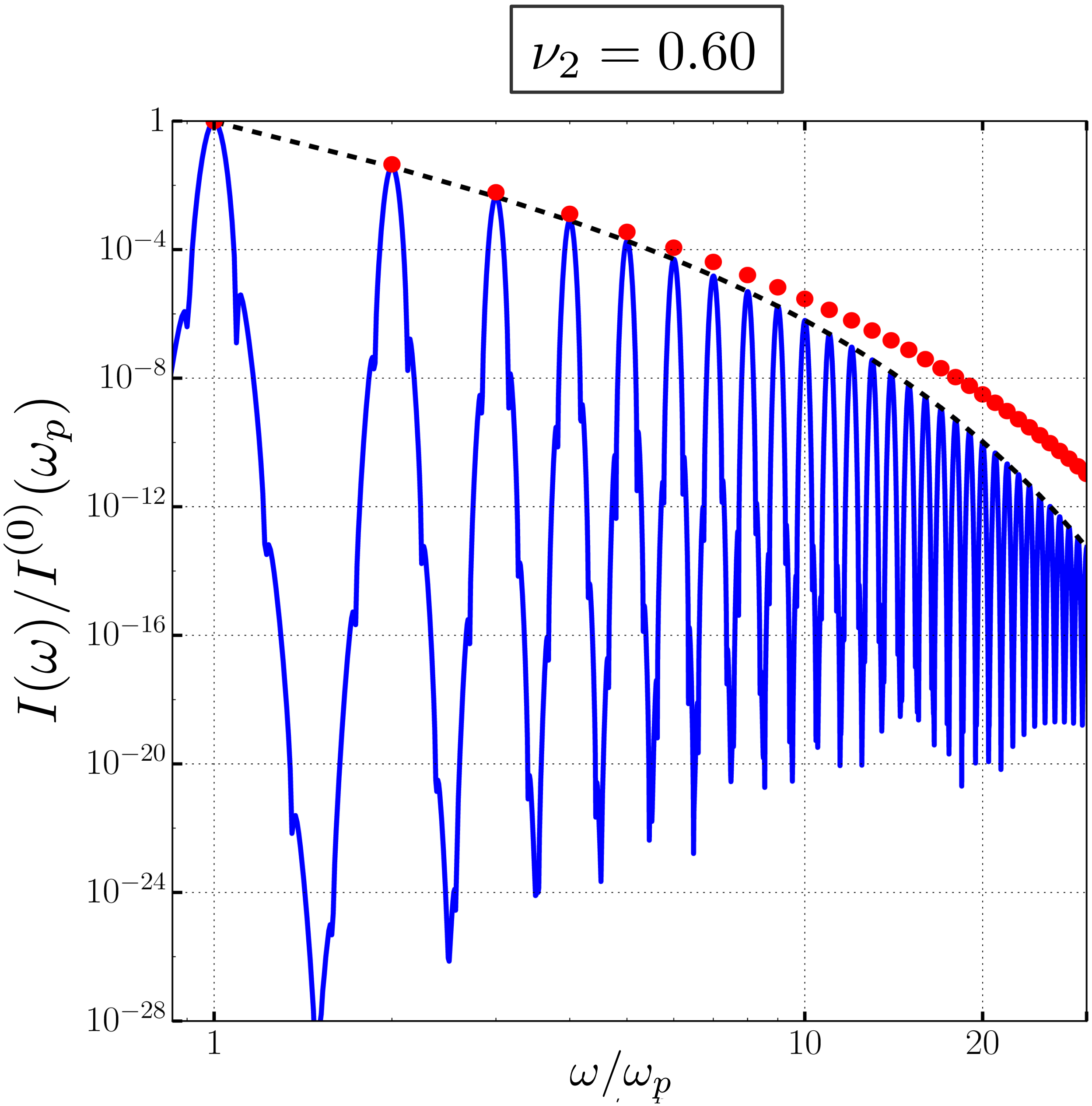

$$\nu_2 = \frac{64\alpha}{105\pi}\frac{\mathcal{E}_s^3\mathcal{E}_p}{E_{\rm cr}^4}\Phi$$

$$\Phi = \omega_p T$$

Conversion to higher harmonics: $8.67\,\%$

Six-photon scattering: HHG in vacuum

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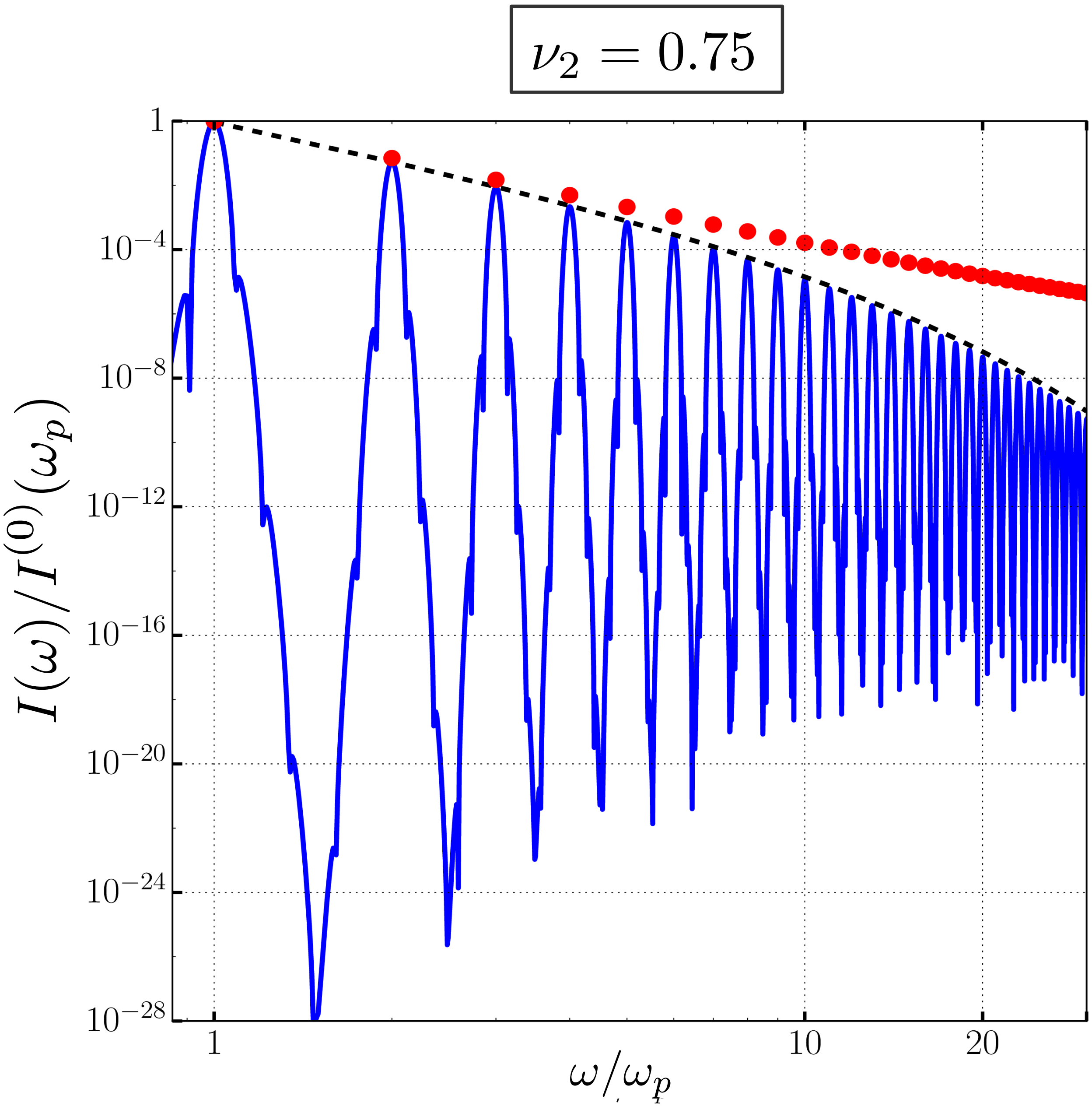

$$\nu_2 = \frac{64\alpha}{105\pi} \frac{\mathcal{E}_s^3 \mathcal{E}_p}{E_{\rm cr}^4} \Phi$$

$$\Phi = \omega_p T$$

Conversion to higher harmonics: $13.27\,\%$

Six-photon scattering: HHG in vacuum

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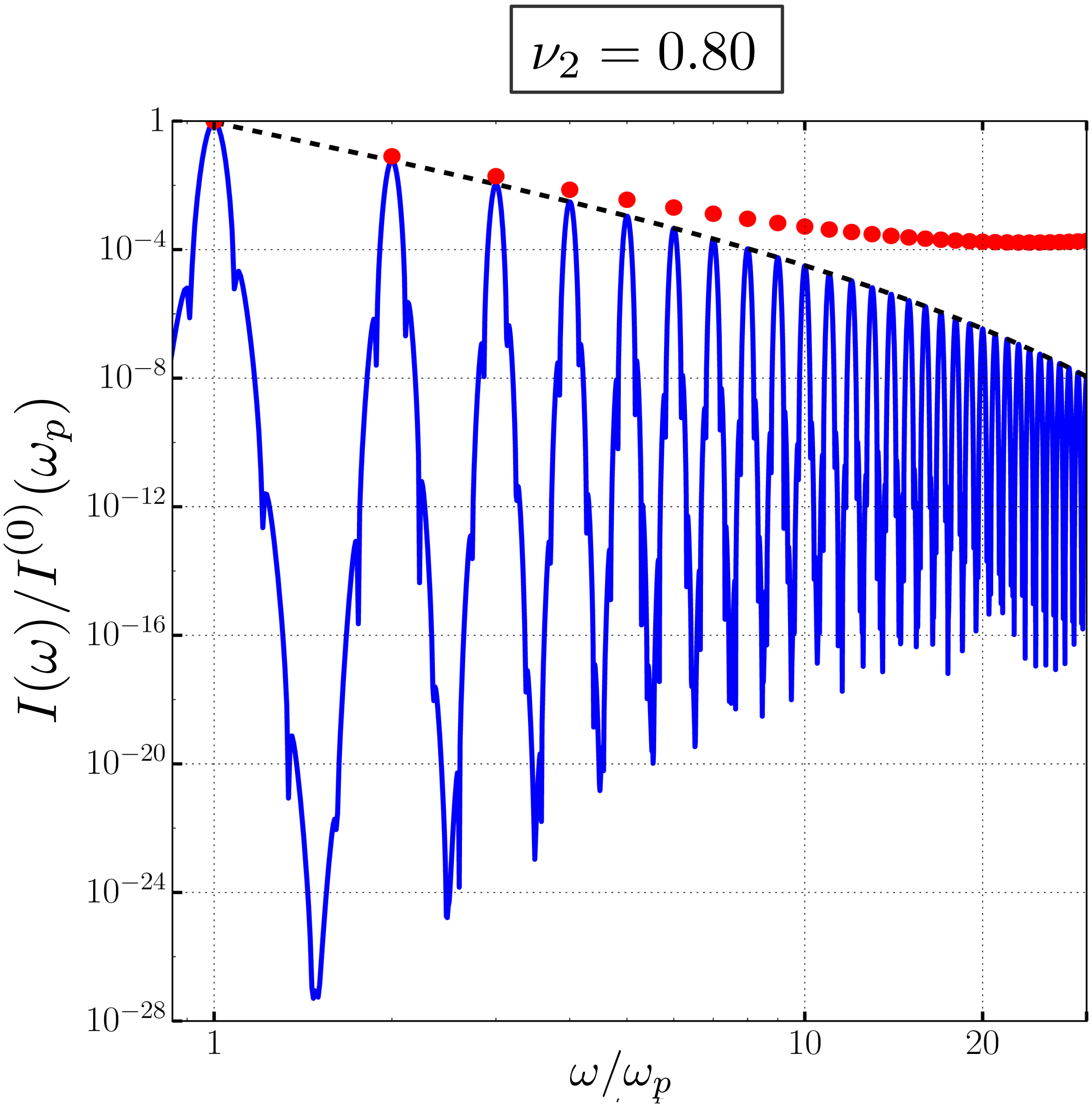

$$\nu_2 = \frac{64\alpha}{105\pi}\frac{\mathcal{E}_s^3\mathcal{E}_p}{E_{\rm cr}^4}\Phi$$

$$\Phi = \omega_p T$$

Conversion to higher harmonics: $14.97\,\%$

Six-photon scattering: HHG in vacuum

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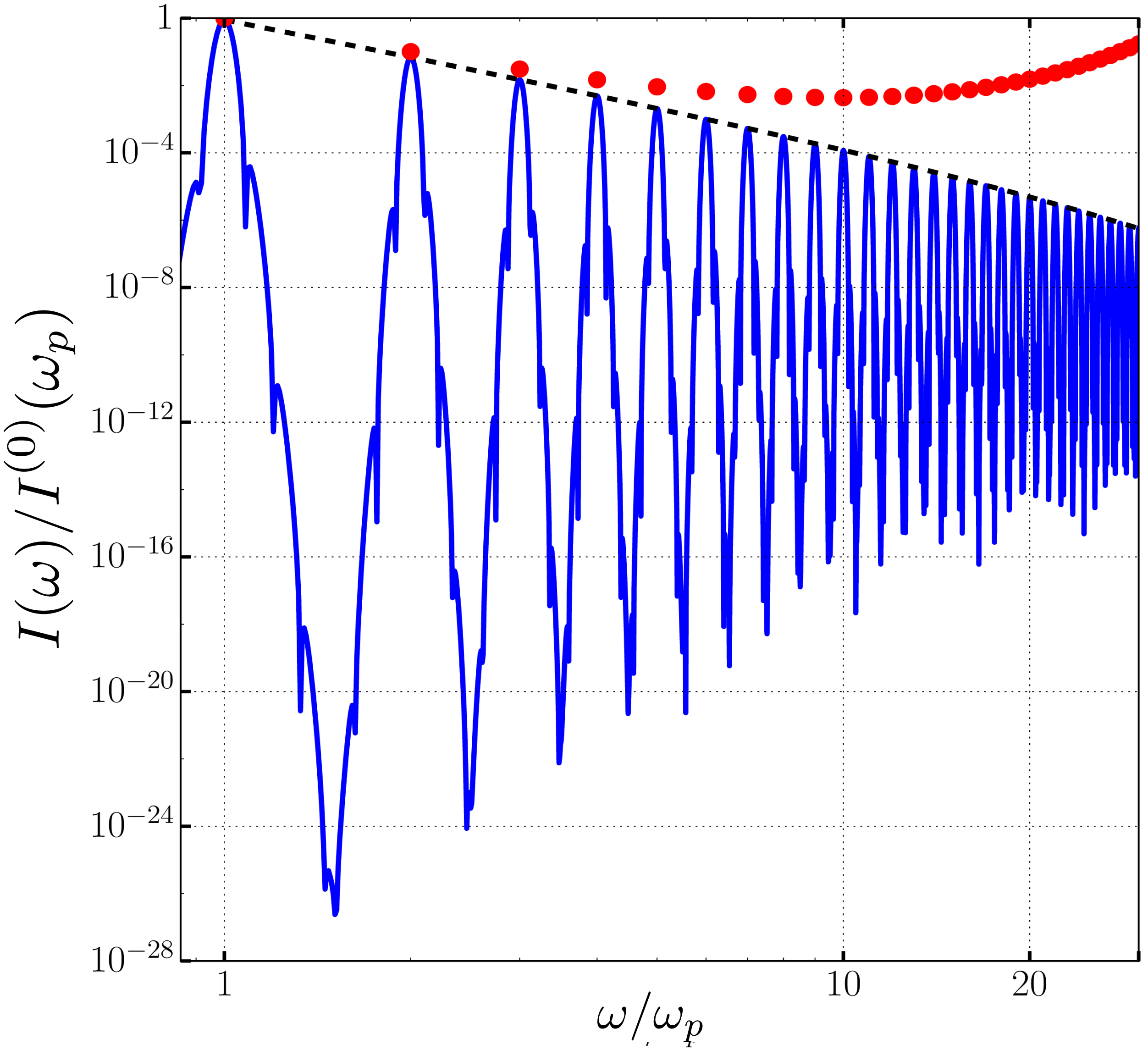
$\nu_2 = 0.90$



$$\nu_2 = \frac{64\alpha}{105\pi}\frac{\mathcal{E}_s^3\mathcal{E}_p}{E_{\rm cr}^4}\Phi$$

$$\Phi = \omega_p T$$

Conversion to higher harmonics: $18.62\,\%$

Six-photon scattering: HHG in vacuum

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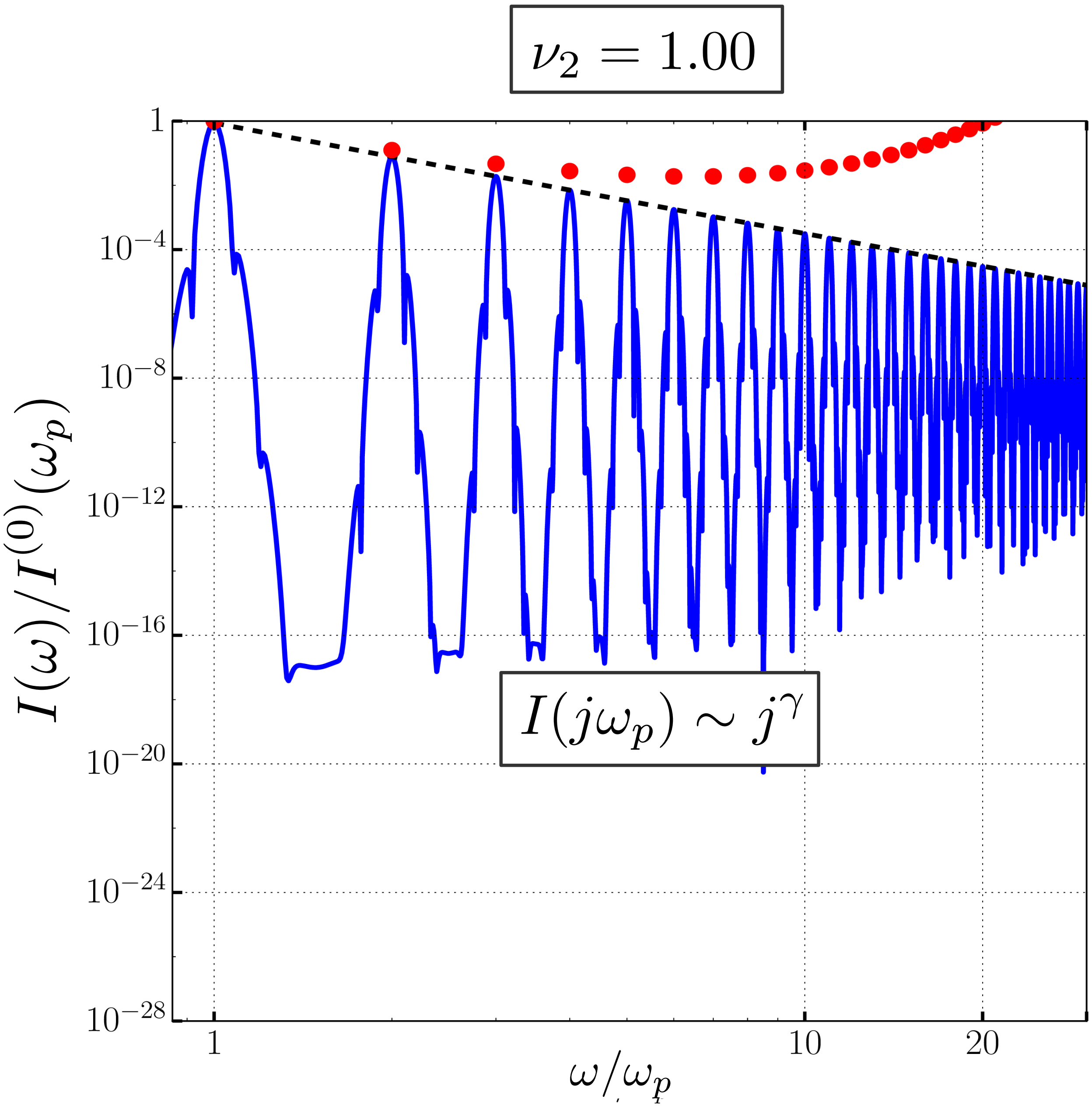
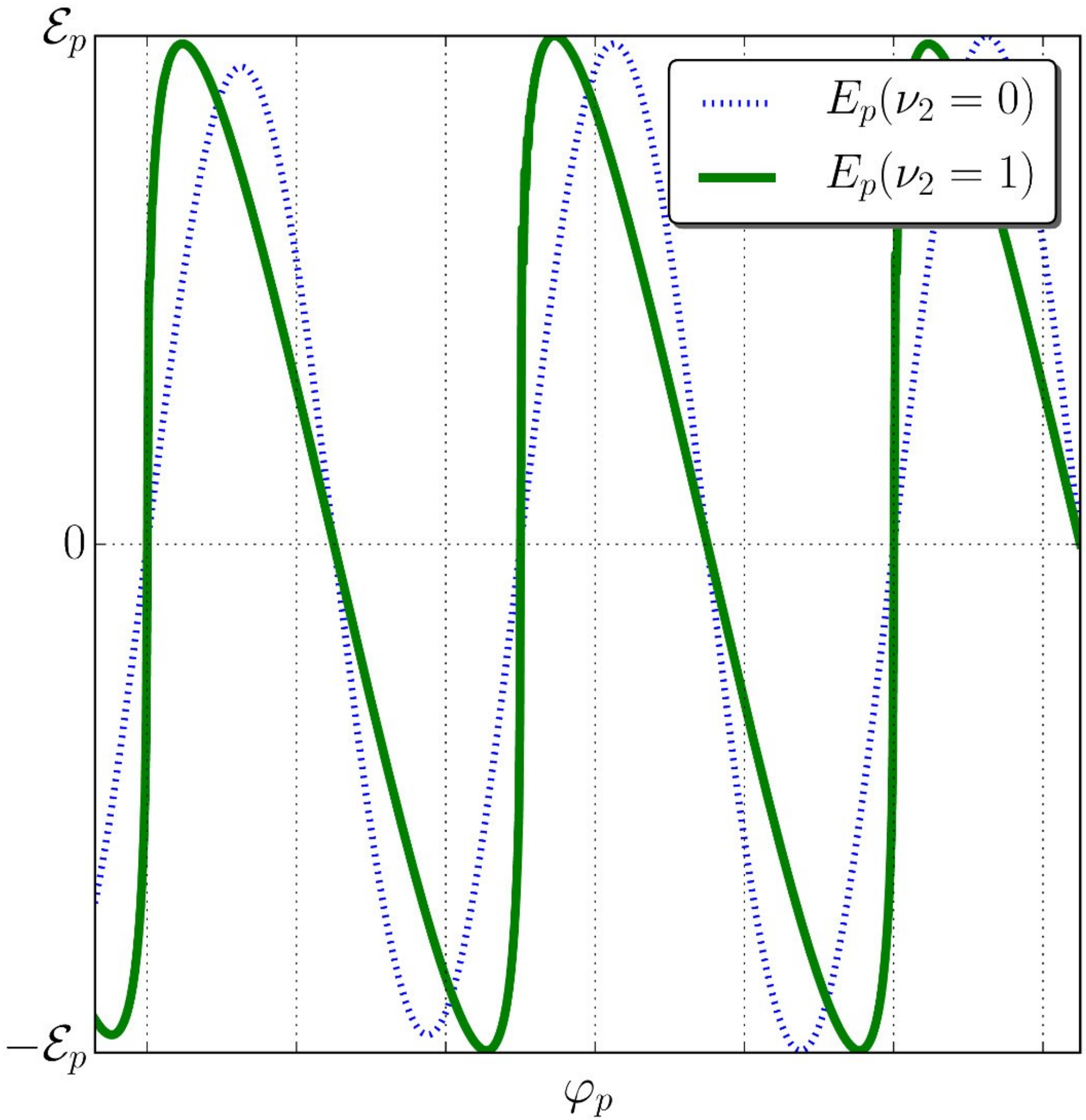


$$\nu_2 = \frac{64\alpha}{105\pi}\frac{\mathcal{E}_s^3\mathcal{E}_p}{E_{\mathrm{cr}}^4}\Phi$$

$$\Phi = \omega_p T$$

Conversion to higher harmonics: $22.54\,\%$

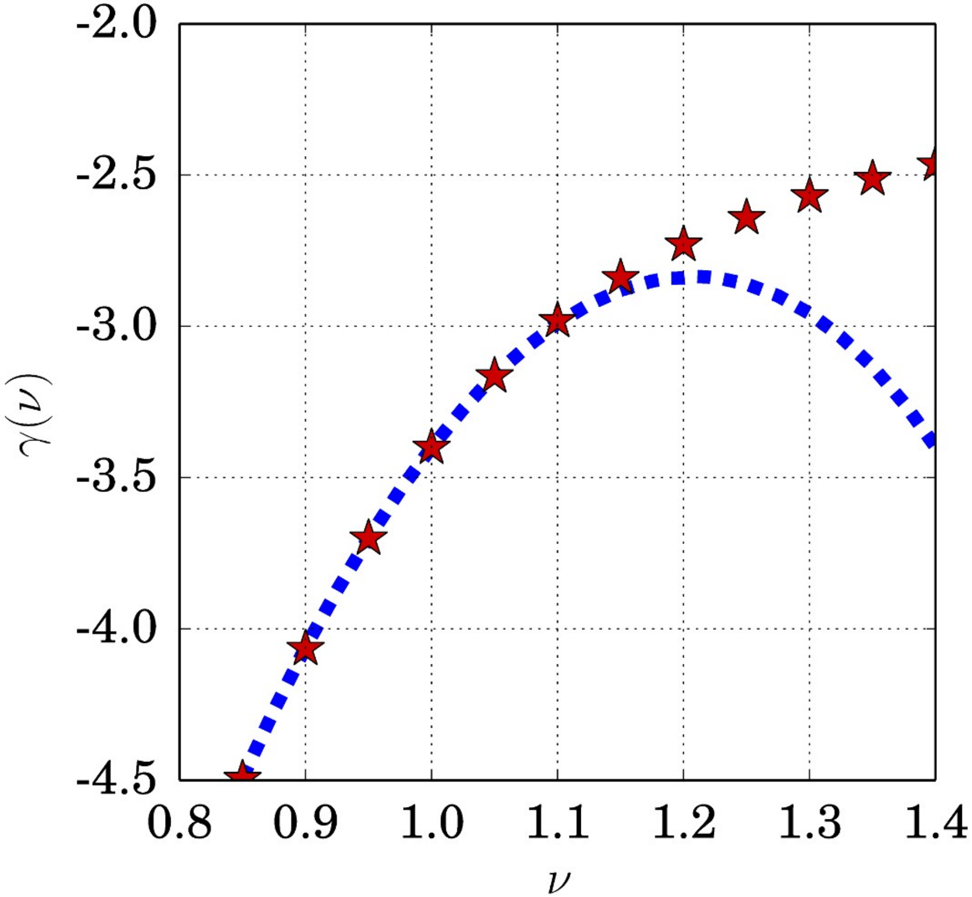
Vacuum shocks



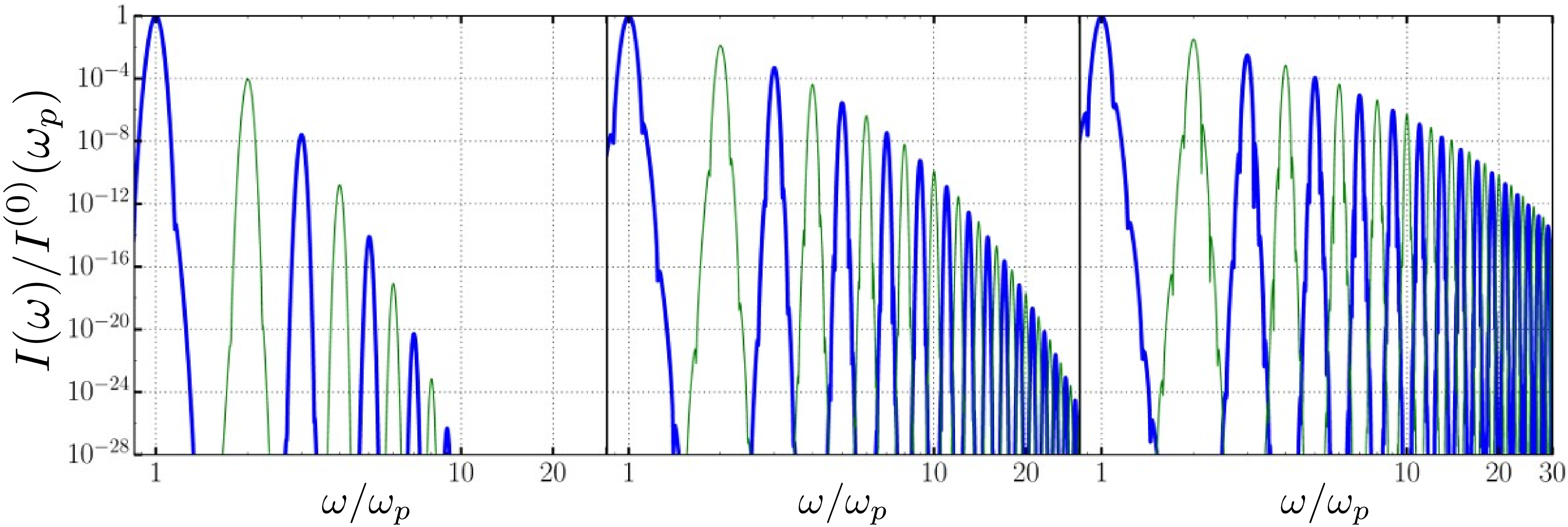
$$\mathbf{E}_{\mathrm{saw}}(\varphi) = \varepsilon\mathcal{E}\sum_{j=1}^{\infty}(-1)^j\,\frac{\sin 2j\varphi}{2j}$$

$$I_{\mathrm{saw}}(j\omega) \sim j^{-2}$$

$$\mathbf{E}_{\mathrm{saw}}(\varphi) = \varepsilon\mathcal{E}\sum_{j=1}^{\infty}(-1)^j\,\frac{\sin 2j\varphi}{2j} \qquad\qquad I_{\mathrm{saw}}(j\omega) \sim j^{-2}$$

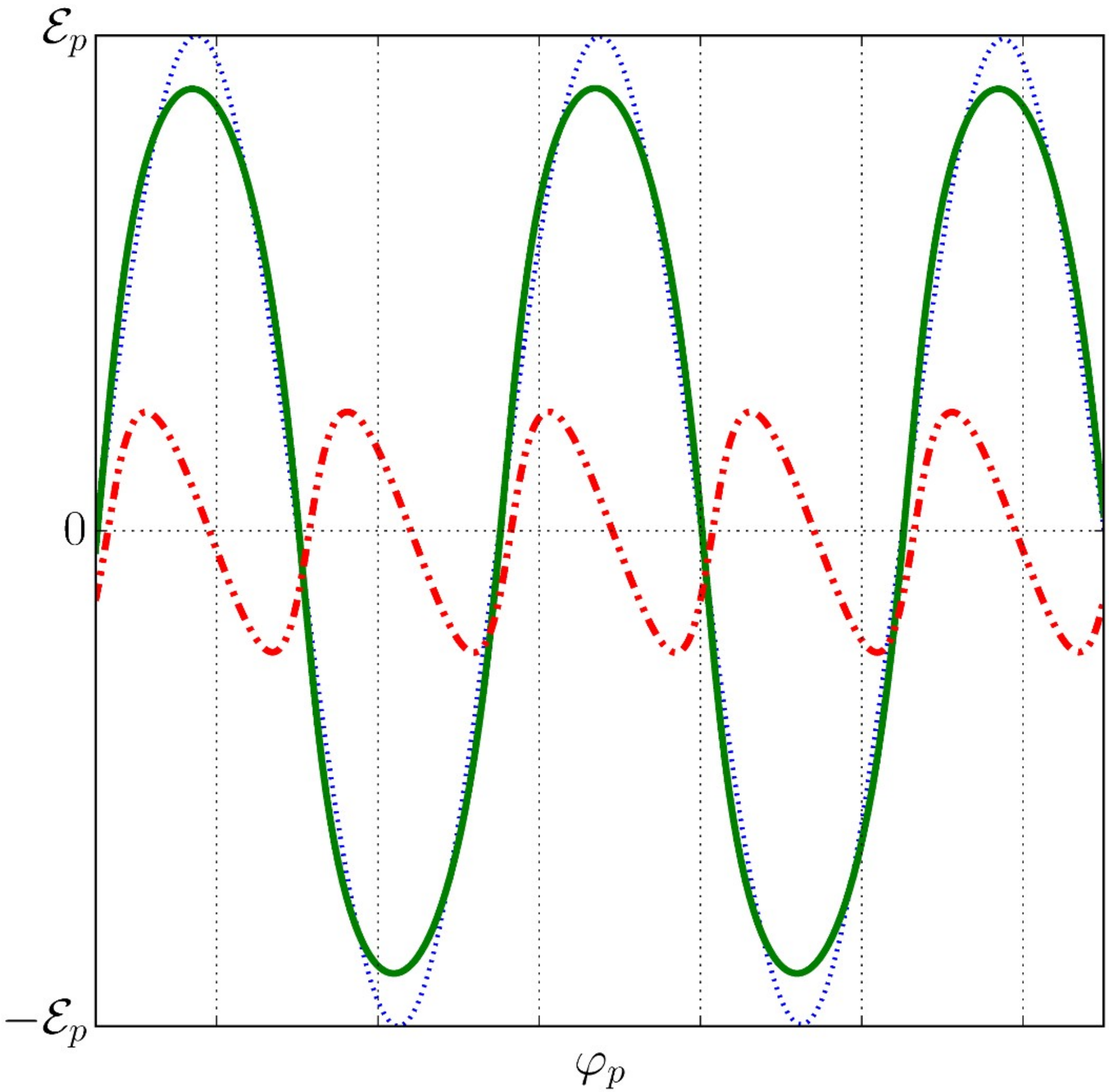


Six-photon scattering: perpendicular pols.



$\nu_2 = 0.05$ $\nu_2 = 0.6$ $\nu_2 = 1.0$

Shock wave: perpendicular set-up



$$\mathbf{E}_{\mathrm{sq.}}(\varphi) = \varepsilon \mathcal{E} \sum_{j=1}^{\infty} (-1)^j \frac{\cos(2j-1)\varphi}{2j-1}$$

VHHG in shock regime: Summary

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- The polarisability of the "vacuum plasma" allows for high harmonic generation in a pump-probe set-up.

- There exists a "shock regime" of harmonic generation where multiple four- and six-photon scattering becomes more efficient at generating higher harmonics

- For large enough shock parameter, the spectrum takes on a power-law rather than exponential behaviour and all orders of scattering are relevant.

$$I(j\omega_p) \sim j^{\gamma} \qquad -4 \lesssim \gamma \lesssim -2.4$$

$$\mathrm{n}^{\|}_{\mathrm{vac.}} = 1 + \frac{\alpha}{\pi}\frac{\mathcal{E}_s^2}{E_{\mathrm{cr}}^2}\left[\frac{8}{45} + \frac{64}{105}\frac{\mathcal{E}_s}{E_{\mathrm{cr}}}\frac{\mathcal{E}_p}{E_{\mathrm{cr}}} + \frac{2048}{315}\left(\frac{\mathcal{E}_s}{E_{\mathrm{cr}}}\frac{\mathcal{E}_p}{E_{\mathrm{cr}}}\right)^2 + \ldots\ldots\right]$$

P. Böhl, BK, H. Ruhl,
Vacuum high harmonic generation in the shock regime, arXiv:1503.05192

BK, P. Böhl, H. Ruhl,
Phys. Rev. D **90**, 065018 (2014)