Pair creation in constant and time-dependent electric fields

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The QED effective Lagrangian

1936 W. Heisenberg and H. Euler: One-loop QED effective Lagrangian in a constant field ("Euler-Heisenberg Lagrangian")

$$\mathcal{L}^{(1)}(a,b) \quad = \quad -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\Big[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2-b^2)T^2 - 1\Big]$$

Here a, b are the two invariants of the Maxwell field, related to **E**, **B** by $a^2 - b^2 = B^2 - E^2, \quad ab = \mathbf{E}\cdot\mathbf{B}$.

1936 V. Weisskopf: Analogously for Scalar QED.

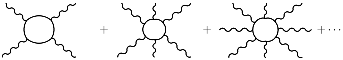
$$\mathcal{L}^{(1)}_{\mathrm{scal}}(a,b) \quad = \quad \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\Big[\frac{(eaT)(ebT)}{\sinh(eaT)\sin(ebT)} + \frac{e^2}{6}(a^2-b^2)T^2 - 1\Big]$$

N-photon amplitudes

The Euler-Heisenberg Lagrangian has the information on the N photon amplitudes in the low energy limit (where all photon energies are small compared to the electron mass, $\omega_i \ll m$). The amplitudes can be constructed explicitly from the weak field expansion coefficients c_{kl}, defined by

$$\mathcal{L}(a,b) = \sum_{k,l} c_{kl}\, a^{2k} b^{2l}$$

Diagrammatically, this corresponds to



Imaginary part of the effective action

If the field has an electric component ($b \neq 0$) there are poles on the integration contour at $ebT = k\pi$ which create an imaginary part. For the purely electric case one gets (J. Schwinger 1951)

$$
\begin{aligned}
\mathrm{Im}\mathcal{L}^{(1)}(E) &= \frac{m^4}{8\pi^3}\beta^2 \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{\pi k}{\beta}\right] \\
\mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{scal}}(E) &= -\frac{m^4}{16\pi^3}\beta^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \exp\left[-\frac{\pi k}{\beta}\right]
\end{aligned}
$$

($\beta = eE/m^2$).

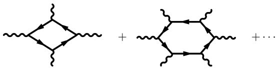
- The kth term relates to coherent creation of k pairs in one Compton volume.
- Weak field limit $\beta \ll 1 \Rightarrow$ only $k = 1$ relevant.
- $\mathrm{Im}\mathcal{L}(E)$ depends on E nonperturbatively, which is a confirmation of the tunneling picture.

Relation to pair creation

For not too strong fields, the imaginary part of the effective action relates to the total pair production probability P as

$$P \approx 2\mathrm{Im}\Gamma(E)$$

This is based on the Optical Theorem, which relates



to the "cut diagrams"

However, the latter individually all vanish for a constant field, which can emit only zero-energy photons. Thus what counts is the asymptotic behaviour for a large number of photons.

Borel dispersion relation

Thus for a constant field we cannot use dispersion relations for individual diagrams; the appropriate generalization is a Borel dispersion relation: define the weak field expansion by

$$\mathcal{L}(E) = \sum_{n=2}^{\infty} c(n) \Big(\frac{eE}{m^2}\Big)^{2n}$$

$$c(n) \overset{n\to\infty}{\sim} c_{\infty} \Gamma[2n-2]$$

(G.V. Dunne & C.S. 1999):

$$\mathrm{Im}\mathcal{L}(E) \sim c_{\infty}\, \mathrm{e}^{-\frac{\pi m^2}{eE}}$$

for $\beta \to 0$.

Laser field configurations

For a constant field the pair creation probability is exponentially small for

$$E \ll E_{\mathrm{crit}} = \frac{m^2}{eE} \approx 10^{18}\mathrm{V/m}$$

Thus to have any chance at seeing pair creation soon, complicated laser configurations must be used to lower the pair creation threshold. For example,

- Counterpropagating lasers beams with linear polarization (M. Ruf, G. R. Mocken, C. Müller, K. Z. Hatsagortsyan & C. H. Keitel 2009).
- Superimposing a plane-wave X-ray beam with a strongly focused optical laser pulse (G.V. Dunne, H. Gies & R. Schützhold 2009).
- . . . (many more).

Approximation methods for Schwinger pair creation

The calculation of pair creation rates for generic electric fields requires approximative methods:

- Until recent years, practically all such results were obtained using WKB (L. Keldysh 1965, E. Brézin and C. Itzykson 1970, N.B. Narozhnyi and A. I. Nikishov 1970, V.S. Popov 1972, . . .). A more sophisticated version of WKB is the worldline instanton formalism (I.K. Affleck, O. Alvarez and N.S. Manton 1982, G. V. Dunne and C. S. 2005, . . .).
- The quantum kinetic approach, based on some Vlasov-type equation (Y. Kluger et al. 1991, 1992, S.M. Schmidt et al. 1998, R. Alkofer, F. Hebenstreit and H. Gies 2008 . . .).
- The Dirac-Heisenberg-Wigner formalism (F. Hebenstreit, A.Ilderton, M. Marklund and J. Zamanian 2011, . . .).

Purely time-dependent fields

For a purely time-dependent electric field, the spatial momentum **k** is a good quantum number, so that one has a mode decomposition (for a scalar particle at one loop)

$$2\mathrm{Im}\mathcal{L}(t) = \sum_{\mathbf{k}} \ln\big(1 + \mathcal{N}_{\mathbf{k}}(t)\big)$$

The $\mathcal{N}_{\mathbf{k}}(t)$ are densities of created pairs of momentum **k**.
Using the in-out formalism and a Bogoliubov transformation, one can derive the Quantum Vlasov Equation (Y. Kluger et al. 1991, 1992, S.M. Schmidt et al. 1998, R. Alkofer, F. Hebenstreit and H. Gies 2008 . . .).

The Quantum Vlasov equation is an evolution equation at fixed $\mathbf{k}$ for the density of pairs $\mathcal{N}_{\mathbf{k}}(t)$ (scalar case):

$$\dot{\mathcal{N}}_{\mathbf{k}}(t) = \frac{\dot{\omega}_{\mathbf{k}}(t)}{2\omega_{\mathbf{k}}(t)} \int_{t_0}^{t} dt' \frac{\dot{\omega}_{\mathbf{k}}(t')}{\omega_{\mathbf{k}}(t')} (1 + 2\mathcal{N}_{\mathbf{k}}(t')) \cos\left[2 \int_{t'}^{t} dt'' \omega_{\mathbf{k}}(t'')\right]$$

where t_0 is the initital time, usually $-\infty$, and

$$\omega_{\mathbf{k}}^2(t) = (k_{\parallel} - qA_{\parallel}(t))^2 + \mathbf{k}_{\perp}^2 + m^2$$

$\mathcal{N}_{\mathbf{k}}(t)$ is zero at $t = -\infty$, and for $t \to \infty$ turns into the density of created pairs with fixed momentum k.

Alternative Quantum Vlasov Equation

Since (presumably) only the limit of $\mathcal{N}_{\mathbf{k}}(t)$ for $t \to \infty$ has a physical meaning, the evolution equation is not completely fixed. In S.P. Kim and C.S., PRD 84, 125028 (2011) we found (using Lewis-Riesenfeld theory) the alternative evolution equation

$$\begin{aligned} \frac{d}{dt}(1 + 2\tilde{\mathcal{N}}_{\mathbf{k}}(t)) &= \Omega^{(-)}(t) \int_{t_0}^{t} dt' \Big[\Omega^{(-)}(t')(1 + 2\tilde{\mathcal{N}}_{\mathbf{k}}(t')) \\ &\quad \times \cos\Big(\int_{t'}^{t} dt'' \Omega^{(+)}(t'') \Big) \Big] \end{aligned}$$

$$\Omega_{\mathbf{k}}^{(\pm)}(t) = \frac{\omega_{\mathbf{k}}^2(t) \pm \omega_{\mathbf{k}}^2(t_0)}{\omega_{\mathbf{k}}(t_0)}$$

This alternative Vlasov equation

- shows a surprising relation to the Korteweg-de-Vries equation
- can be solved explicitly for a family of “solitonic” time-dependent fields
- but seemed to be inequivalent: $\mathcal{N}_{\mathbf{k}}(t)$ and $\tilde{\mathcal{N}}_{\mathbf{k}}(t)$ do not always have the same limit for $t \to \infty$!

Relating both Vlasov equations

A. Huet, S.P. Kim and C.S., PRD 90, 125033 (2014):

- The difference between both Vlasov equations is just between the in-out ($\mathcal{N}_{\mathbf{k}}(t)$) and in-in ($\tilde{\mathcal{N}}_{\mathbf{k}}(t)$) formalisms.
- Asymptotically, solutions of the two equations are related by

$$2\mathcal{N}_{\mathbf{k}} + 1 = 2\frac{\omega_{\mathbf{k}}(t_0)\omega_{\mathbf{k}}(t_\infty)}{\omega_{\mathbf{k}}^2(t_0) + \omega_{\mathbf{k}}^2(t_\infty)}\langle 2\tilde{\mathcal{N}}_{\mathbf{k}} + 1\rangle$$

where $\langle\rangle$ means taking the asymptotic time average.

TIme-like Sauter field

Example: the *time-like Sauter field* $E(t) = E_0 \operatorname{sech}^2(t/\tau)$

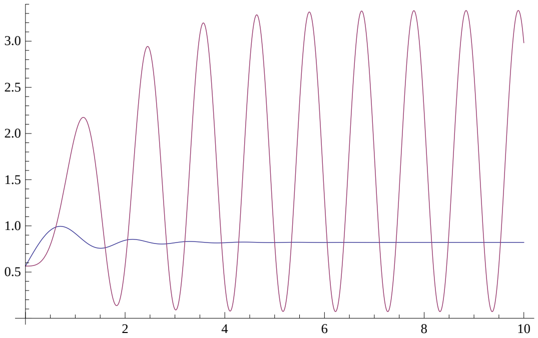


Figure: A comparison of $\mathcal{N}(t)$ (blue curve) against $\tilde{\mathcal{N}}(t)$ (red curve) for the timelike Sauter field.

Alternative Vlasov equation and KdV

Inspection shows, that the general solution of the alternative Vlasov equation can be parameterized by a function $f(t)$ fulfilling the integral equation.

$$\dot{f}(t) = \frac{\Omega^{(-)}(t)}{\omega_0} - 2\int_{t_0}^{t} dt' f(t')(\omega^2(t) + \omega^2(t')) \tag{1}$$

with the initial condition $f(t_0) = \dot{f}(t_0) = 0$. Knowing $f(t)$, $\tilde{\mathcal{N}}_{\mathbf{k}}(t)$ can be recovered as

$$1 + 2\tilde{\mathcal{N}}_{\mathbf{k}} = 1 + \omega_0 \int_{t_0}^{t} dt' f(t')\Omega^{(-)}(t')$$

Ansatz: $f(t) = \frac{(\omega^2)^{\cdot}(t)}{8\omega_0^4}$, $\quad F(t) = \frac{\omega^2(t) - \omega_0^2}{8\omega_0^4}$

Defining $r(t) := \omega^2(t)/\omega_0^2$ and then $u(x,t) := -r(x - 10t)$, one can show that for (1) to be fulfilled u must solve the Korteweg-de Vries equation,

$$u_{xxx} - 6uu_x + u_t = 0$$

Thus we can use solutions of the KdV equation to define gauge fields which lead to a solvable Vlasov equation.

Solitonic example

Example: choose the following soliton-type solution of the KdV equation

$$u(x,t) = -1 - \frac{2}{\cosh^2(x - 10t)}$$

which corresponds to

$$r(t) = \frac{\omega^2(t)}{\omega_0^2} = 1 + \frac{2}{\cosh^2(\omega_0 t)}, \quad F(t) = \frac{1}{4\omega_0^2 \cosh^2(\omega_0 t)}$$

The gauge potential is

$$qA(t) = k_\parallel - \sqrt{k_\parallel^2 + \frac{2\omega_0^2}{\cosh^2(\omega_0 t)}}.$$

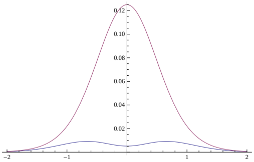
Non pair-creation

The exact solutions of the Vlasov equation and the alternative Vlasov equation are

$$\mathcal{N}_{\mathbf{k}}(t) = \frac{4 + \mathrm{sech}^4(\omega_0 t)(1 + 2\cosh(2\omega_0 t))}{8\sqrt{1 + 2\mathrm{sech}^2(\omega_0 t)}} - \frac{1}{2}$$

$$\tilde{\mathcal{N}}_{\mathbf{k}}(t) = \frac{1}{8\cosh^4(\omega_0 t)}$$

Both $\tilde{\mathcal{N}}_{\mathbf{k}}(t)$ and $\mathcal{N}_{\mathbf{k}}(t)$ vanish for $t \to \infty$, thus there is no pair creation *at that particular momentum* $\mathbf{k}$.

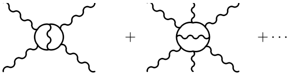


Higher-loop corrections to Schwinger's formula

So far all our discussion was at the one-loop level. Higher-loop corrections are not likely to be measured any time soon, but of great theoretical interest.

Two loop (one-photon exchange) corrections:

Euler-Heisenberg Lagrangian:



Schwinger pair creation:

2-Loop Euler-Heisenberg Lagrangian

V. I. Ritus 1975, S.L. Lebedev & V.I. Ritus 1984, W. Dittrich & M. Reuter 1985, M. Reuter, M.G. Schmidt & C.S. 1997: The two-loop correction $\mathcal{L}^{(2)}(E)$ to the Euler-Heisenberg Lagrangian leads to rather intractable integrals. However, the imaginary part $\mathrm{Im}\mathcal{L}^{(2)}(E)$ becomes extremely simple in the weak-field limit:

Weak field limit:

$$\mathrm{Im}\mathcal{L}^{(1)}(E) + \mathrm{Im}\mathcal{L}^{(2)}(E) \overset{\beta\to 0}{\sim} \frac{m^4\beta^2}{8\pi^3}\left(1+\alpha\pi\right)\mathrm{e}^{-\frac{\pi}{\beta}}$$

The exponentiation conjecture

S.L. Lebedev & V.I. Ritus 1984: Assuming that higher orders will lead to exponentiation

$$\mathrm{Im}\mathcal{L}^{(1)}(E) + \mathrm{Im}\mathcal{L}^{(2)}(E) + \mathrm{Im}\mathcal{L}^{(3)}(E) + \dots \quad \overset{\beta\to 0}{\sim} \quad \frac{m^4 \beta^2}{8\pi^3}\, \mathrm{e}^{\alpha\pi}\, \mathrm{e}^{-\frac{\pi}{\beta}}$$

For Scalar QED the corresponding conjecture was established already two years earlier by I.K. Affleck, O. Alvarez, N.S. Manton (1982) using worldline instantons.

Remarkable:

- True all-loop result, receives contributions from an infinite set of graphs of arbitrary loop order
- Includes mass renormalization
- Implausible: an all-order loop summation has produced the factor $e^{\alpha\pi}$ which is analytic in α!

Field dependence of the electron mass

- In QED at tree-level, many arguments have been given for a field-dependence of the electron mass (N.D. Sengupta 1952, D.Volkov 1953, H. Reiss 1962, A.I. Nikishov & V.I. Ritus 1964, T.W.B. Kibble 1965 . . .).
- This mass-shift has been confirmed so far only indirectly (through the change in frequency of the radiation emitted by the electron).
- It is always positive but otherwise far from universal, depending on both intensity and pulse-shape (C. Harvey, T. Heinzl, A. Ilderton & M. Marklund 2012).

Ritus' "classical" one-loop mass shift

V.I. Ritus 1978: electron mass shift from the one-loop propagator in a constant electric field.

In the weak-field limit,

$$m(E) \approx m - \frac{\alpha}{2}\frac{eE}{m} + O(\hbar)$$

This mass shift is negative, and has a "classical" part that does not vanish for $\hbar \to 0$.

Lebedev-Ritus mass shift

S.L. Lebedev & V.I. Ritus 1984: assuming the exponentiation

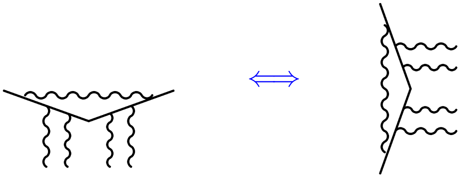
$$\sum_{l=1}^{\infty} \mathrm{Im}\mathcal{L}^{(l)} \quad \overset{\beta\to 0}{\sim} \quad -\frac{m^4\beta^2}{16\pi^3}\exp\Big[-\frac{\pi}{\beta}+\alpha\pi\Big]$$

then the result can be interpreted in the tunneling picture as the corrections to the Schwinger pair creation rate due to the pair being created with a negative Coulomb interaction energy

$$m(E) \approx m - \frac{\alpha}{2}\frac{eE}{m}$$

Ritus vs. Lebedev-Ritus

This mass shift is identical with the Ritus mass shift, as it should, since the processes are related by crossing:



This lends further support to the exponentiation conjecture.

QED in 1+1 dimensions

The exponentiation conjecture has been verified at two loops in Scalar and Spinor QED. A three-loop check is in order, but calculating the three-loop Euler-Heisenberg Lagrangian in $D = 4$ is too difficult.

M. Krasnansky 2005: Studied the EHL in $D = 2, 4, 6$.

$$\begin{aligned} \mathcal{L}^{(2)(2D)}_{\mathrm{scal}}(\kappa) &= -\frac{e^2}{32\pi^2}\left(\xi_{2D}^2 - 4\kappa\xi'_{2D}\right), \\ \xi_{2D} &= -\Bigl(\psi(\kappa+\frac{1}{2}) - \ln(\kappa)\Bigr) \end{aligned}$$

($\psi(x) = \Gamma'(x)/\Gamma(x)$, $\kappa = m^2/(2ef)$, $f^2 = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$).

$\rightarrow$ Suggests to establish and verify the above predictions for $2D$ QED.

Correspondences 4D - 2D

$$
\begin{array}{rcl}
4D\ \ QED & \leftrightarrow & 2D\ \ QED \\
\alpha = \dfrac{e^2}{4\pi} & \leftrightarrow & \tilde{\alpha} = \dfrac{2e^2}{\pi m^2} \\
\mathrm{Im}\Gamma^{D=4} \sim \mathrm{e}^{-\frac{m^2\pi}{eE}+\alpha\pi} & \leftrightarrow & \mathrm{Im}\Gamma^{D=2} \sim \mathrm{e}^{-\frac{m^2\pi}{eE}+\tilde{\alpha}\pi^2\kappa^2} \\
\lim_{n\to\infty}\dfrac{c_{4D}^{(l)}(n)}{c_{4D}^{(1)}(n)} = \dfrac{(\alpha\pi)^{l-1}}{(l-1)!} & \leftrightarrow & \lim_{n\to\infty}\dfrac{c_{2D}^{(l)}(n)}{c_{2D}^{(1)}(n+l-1)} = \dfrac{(\tilde{\alpha}\pi^2)^{l-1}}{(l-1)!}
\end{array}
$$

The 2D Spinor QED EHL at one, two and three loops

I. Huet, D.G.C. McKeon and C.S., JHEP 12 (2010) 036

$$\mathcal{L}^{(1)}(\kappa) = -\frac{m^2}{4\pi}\frac{1}{\kappa}\Big[\ln\Gamma(\kappa) - \kappa(\ln\kappa - 1) + \frac{1}{2}\ln\Big(\frac{\kappa}{2\pi}\Big)\Big]$$

$$\mathcal{L}^{(2)}(f) = \frac{m^2}{4\pi}\frac{\tilde{\alpha}}{4}\Big[\tilde{\psi}(\kappa) + \kappa\tilde{\psi}'(\kappa) + \ln(\lambda_0 m^2) + \gamma + 2\Big]$$

where

$$\tilde{\psi}(x) \equiv \psi(x) - \ln x + \frac{1}{2x}$$

Weak-field expansion coefficients

Explicit formulas not only for $c^{(1)}_{2D}(n)$ but $c^{(2)}_{2D}(n)$:

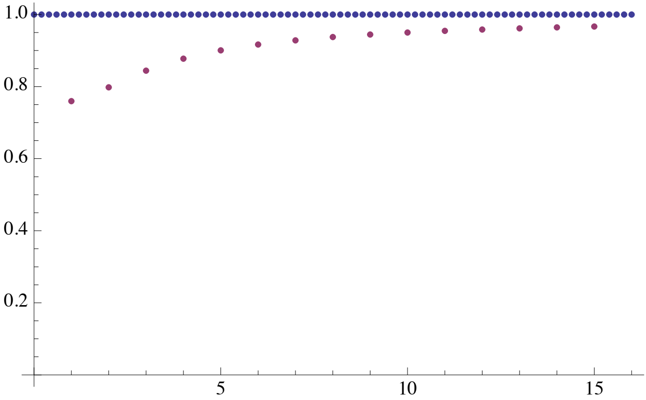
$$
\begin{aligned}
c^{(1)}(n) &= (-1)^{n+1}\frac{B_{2n}}{4n(2n-1)} \\
c^{(2)}(n) &= (-1)^{n+1}\frac{\tilde{\alpha}}{8}\frac{2n-1}{2n}B_{2n}
\end{aligned}
$$

Using properties of the Bernoulli numbers B_n, we can easily verify that

$$
\lim_{n\to\infty}\frac{c^{(2)}(n)}{c^{(1)}(n+1)} = \tilde{\alpha}\pi^2
$$

Two-loop EHL in 2D Spinor QED

Rapid convergence of $c^{(2)}(n)$ to the asymptotic prediction:



Three-loop EHL in 2D Spinor QED

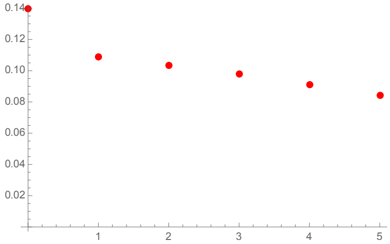
I. Huet, M. Rausch de Traubenberg and C.S. 2015:
We computed the first coefficient analytically,

$$c^3(0) = \Big(-\frac{3}{2} + \frac{7}{4}\zeta(3)\Big)\frac{\tilde{\alpha}^2}{64}$$

and five more coefficients numerically. Using these in

$$\lim_{n\to\infty} \frac{c_{2D}^{(3)}(n)}{c_{2D}^{(1)}(n+2)} = \frac{(\tilde{\alpha}\pi^2)^2}{2!}$$

Exponentiation at three loops



we fall even below the asymptotic prediction!
Exponentiation does not work in $D = 2$.

Thank you for your attention!