

IONIZATION OF HYDROGEN BY TWO-COLOR PULSES IN THE X-RAY AND SOFT X-RAY DOMAINS

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A more appropriate title:

STIMULATED COMPTON SCATTERING CONTRIBUTION TO THE TWO-COLOR IONIZATION OF HYDROGEN IN THE X-RAY AND SOFT X-RAY DOMAINS

Theoretical investigation based on the integration of TDSE and predictions from LOPT.

- The system. The Hamiltonian
- The "one-color" pulse case
- The model for a two-color pulse of radiation
- What stimulated Compton scattering (SCS) is
- The two-color case I - (55 a.u., 50 a.u.) and (55 a.u., 54 a.u.)
 - The electron spectrum and the processes revealed
 - Zoom on SCS peak
 - Comparison of TDSE with LOPT
 - PADS

Outline(continued)

- Details on LOPT for CS and SCS
 - The relevant equations for LOPT **rates**
 - Extracting **rates** from the electron spectrum
- The two-color case II - (15 a.u., 14 a.u.) and (10 a.u., 9 a.u.)
 - The electron spectrum
 - Comparison of TDSE with LOPT
- Results from LOPT at the limit of zero electron energy
- Details on numerical solving of TDSE
- Details on Lauricella functions
- Conclusions

The system. The Hamiltonian.

All our calculations refer to the hydrogen atom with a fixed nucleus and are nonrelativistic (NR)

Exact Hamiltonian in the Coulomb gauge

$$\mathcal{H} = \frac{1}{2m_e} (\mathbf{P} - e\mathbf{A})^2 - \frac{Ze^2}{r} = H_0 - \frac{e}{m_e} \mathbf{A} \cdot \mathbf{P} + \frac{e^2}{2m_e} \mathbf{A}^2$$

$\mathbf{A}(\mathbf{r}, t)$, $\mathbf{A} \cdot \mathbf{P}$ and \mathbf{A}^2 terms.

$\mathbf{A}(\mathbf{r}, t) \rightarrow \mathbf{A}(t)$ - dipole approximation (DA). LOPT formulas we use are based on the exact hamiltonian.

The integration of TDSE was performed with an **approximate Hamiltonian** $\tilde{\mathcal{H}}$, containing first order ($1/c$) retardation corrections

$$\tilde{\mathcal{H}} = H_0 + H_{DA}^{(1)} + H_{\text{ret}}^{(1)} + H_{\text{ret}}^{(2)}$$

$$H_{DA}^{(1)} = -\frac{e}{m_e} \mathbf{A}(t) \cdot \mathbf{P}$$

The Hamiltonian - the retardation corrections

$H_{\text{ret}}^{(1)}$ and $H_{\text{ret}}^{(2)}$ represent retardation corrections. They come, respectively, from the terms $\mathbf{A} \cdot \mathbf{P}$ and \mathbf{A}^2 of the Hamiltonian. They change with the concrete expression of $\mathbf{A}(\mathbf{r}, t)$ (one-color or two-color pulses) and will be given later.

The contribution of \mathbf{A}^2 is a retardation effect.
It plays an important role in SCS

One-color pulse

THE PULSE - only **one central frequency**, 0x - direction of propagation

$$\mathbf{A}(\mathbf{r}, t) = A\left(t - \frac{x}{c}\right) \mathbf{e}_z \approx \left[A(t) + \frac{x}{c} F(t)\right] \mathbf{e}_z \quad F(t) = -\frac{dA(t)}{dt}$$

$$H_{\text{ret}}^{(1)} = -\frac{e}{mc} F(t) \times P_z \quad H_{\text{ret}}^{(2)} = \frac{e^2}{m_e c} F(t) A(t) \times$$

$$A(t) = \frac{\sqrt{I}}{\omega} f(t) \cos \omega t$$

$I = I_0$. \cos^2 profile for the pulse envelope $f(t)$, defined on the symmetric time interval $|t| < \tau/2$. A total pulse duration τ of 20 oscillations of the field (80 as at 1 keV).

For the numerical procedure of integrating TDSE, see *M. Dondera and H. Bachau, Phys. Rev. A* **85**, 013423 (2012).

THE TWO-COLOR PULSE

$$\mathbf{A}(\mathbf{r}, t) = A(\mathbf{r}, t) \mathbf{e}_z = \left[A_1\left(t - \frac{\mathbf{r} \cdot \mathbf{n}_1}{c}\right) + A_2\left(t - \frac{\mathbf{r} \cdot \mathbf{n}_2}{c}\right) \right] \mathbf{e}_z, \quad \cos \beta = \mathbf{n}_1 \cdot \mathbf{n}_2$$

Two central frequencies $\omega_1 > \omega_2$, different directions of propagation $\mathbf{n}_1, \mathbf{n}_2$, same linear polarization along \mathbf{e}_z , ($\mathbf{n}_1 \cdot \mathbf{e}_z = \mathbf{n}_2 \cdot \mathbf{e}_z = 0$).

$$A(\mathbf{r}, t) \approx A(t) + \frac{1}{c} [F_a(t)x + F_b(t)y], \quad A(t) = A_1(t) + A_2(t)$$

$$F_i(t) = -\frac{dA_i}{dt}, \quad i = 1, 2, \quad F_a(t) = F_1(t) + F_2(t) \cos \beta, \quad F_b(t) = F_2(t) \sin \beta,$$

$$\rightarrow H_{\text{ret}}^{(1)} = -\frac{e}{mc} (F_a(t)x + F_b(t)y) P_z$$

$$H_{\text{ret}}^{(2)} = \frac{e^2}{mc} [F_a(t)x + F_b(t)y] A(t).$$

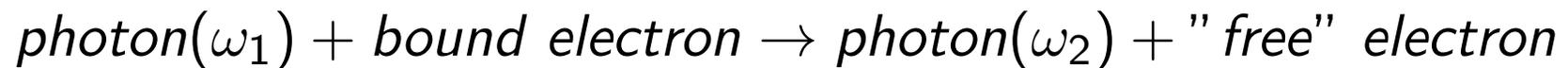
The same \cos^2 envelope for the shape of the pulses. The two pulses have the same duration and they exactly overlap in time.

$\tau = 6\pi \rightarrow 150$ oscillations at $\omega_1 = 50$ a.u., 165 at $\omega_2 = 55$ a.u.

$$\tilde{\mathcal{H}} = H_0 + H_{DA}^{(1)} + H_{\text{ret}}^{(1)} + H_{\text{ret}}^{(2)}$$

WHAT SCS IS

Compton scattering on an atom (CS)- mechanism of ionization in competition with photoeffect. The elementary process due to electron-photon interaction



Energy conservation $E = \hbar\omega_1 - \hbar\omega_2 - E_B \geq 0$, $E_B = 0.5$ a.u.

Simultaneously the photon ω_1 is absorbed and the photon ω_2 is **spontaneously emitted**. The emitted photon can have any energy between 0 and $\hbar\omega_2^{\max} = \hbar\omega_1 - E_B$, any direction and any polarization.

Stimulated Compton scattering (SCS). In the presence of photons with a frequency ω_2 , fixed direction of propagation and polarization, the emission of the photon with **the frequency ω_2 is reinforced**. Not only the frequency, but also the direction of propagation and the polarization of the emitted photon are **imposed** in SCS by the external field. **Only the direction \mathbf{n} of the electron is arbitrary.**

WHAT SCS IS

Remarks on terminology:

Kapitza-Dirac effect is SCS with free electrons and standing waves ($\omega_2 = \omega_1$, $\beta = \pi$), see the title of *L. S. Bartell et al.*, PRL **14**, 851 (1965)

And other papers in IEEE ...

The name was used for the first left satellite in Laser-Assisted X-ray Photoionization, *A. Cionga et al.*, Phys. Rev.A **47**, 1830-1841 (1993).

Relation of SCS to CS

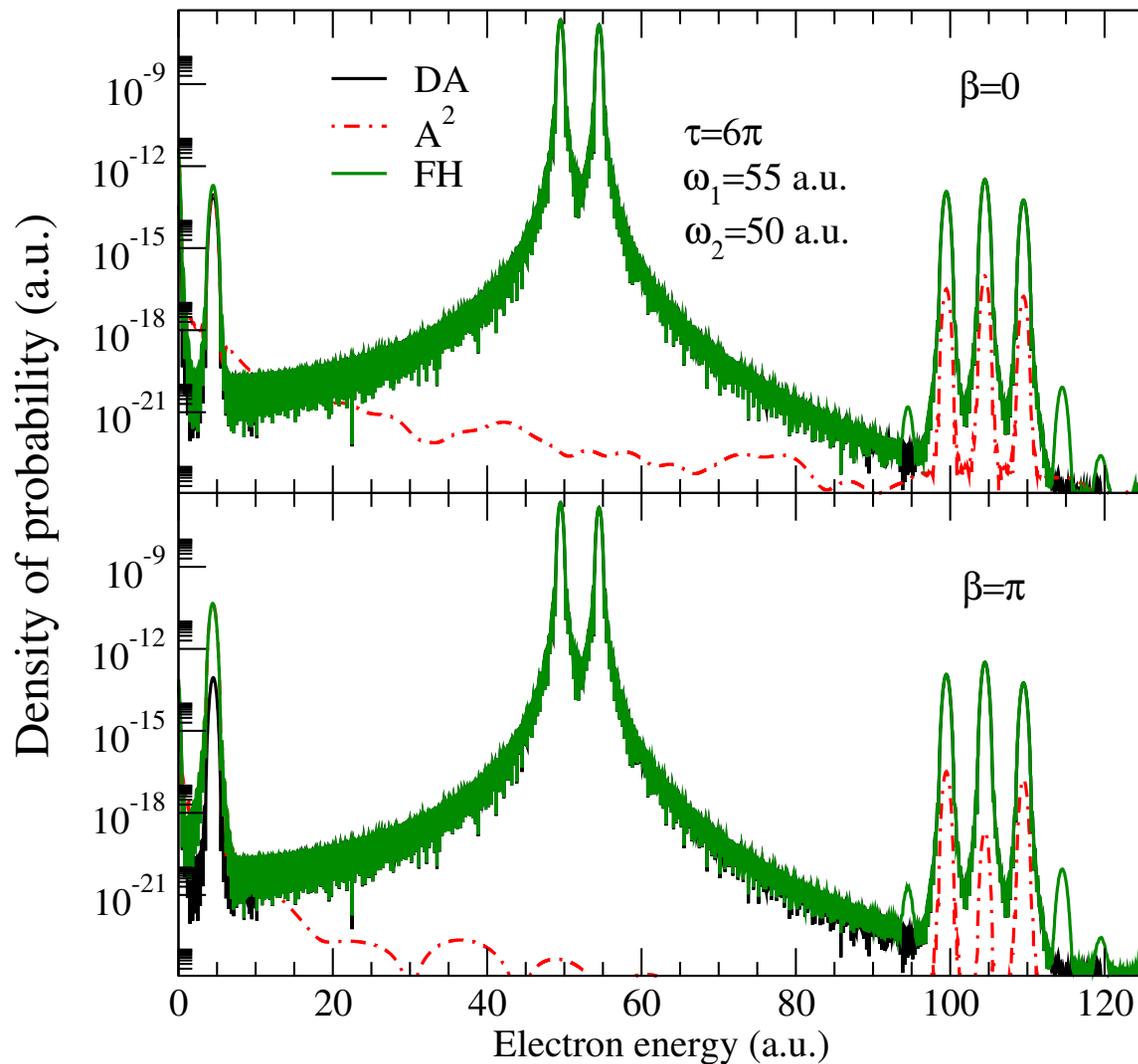
Analogy with stimulated/spontaneous emission of radiation, in which case the relation between Einstein differential coefficients for stimulated and spontaneous emission is valid,

$$b = \frac{8\pi^3 c^3}{\hbar\omega^3} a.$$

The rate of SCS is connected with the rate of CS at the frequency, direction of propagation and polarization of the lower frequency external field

$$\frac{d\Gamma_{SCS}}{d\Omega_{\mathbf{n}}} = \frac{8\pi^3 c^2}{\hbar\omega_2^3} I_2 \frac{d^3\Gamma_{CS}(\omega_2, \mathbf{n}_2, \epsilon_2)}{d\omega_2 d\Omega_2 d\Omega_{\mathbf{n}}}$$

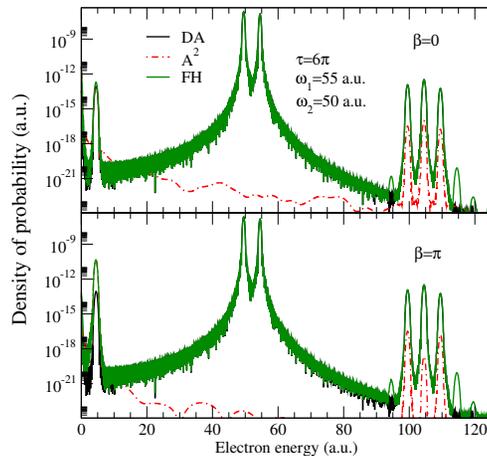
ELECTRON SPECTRUM I



Electron spectra for a peak intensity of 3.51×10^{16} W/cm² $\equiv I_0$, two pulses propagating in the same (upper panel) and opposite (lower panel) directions. The total pulse duration is 6π .

Reproduced from PRA **90** (2014)

ELECTRON SPECTRUM I



FH - TDSE with the approximate $\tilde{\mathcal{H}}$
DA - the integration with $H_0 + H_{DA}^{(1)}$
A² - integration with $H_0 + H_{ret}^{(2)}$

Processes revealed by the spectrum

The main peaks **one-photon ionization** due to each ω_2 and ω_1 dominated by $\mathbf{A} \cdot \mathbf{P}$; even DA approximation works well

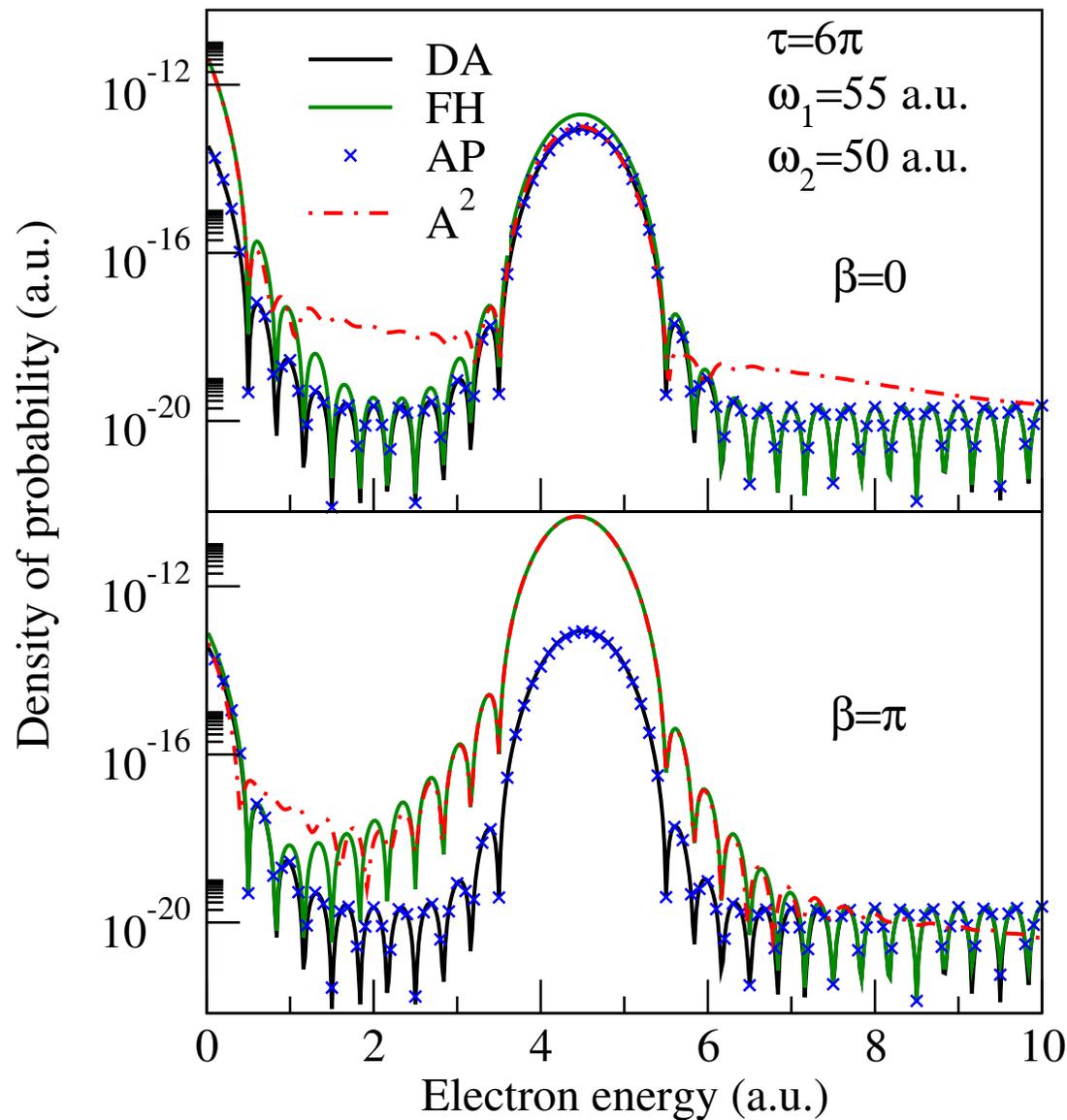
Multiphoton peaks at right: $2\omega_2$, $\omega_1 + \omega_2$, $2\omega_1$ and higher order

Stimulated Compton scattering SCS

peak at left near $\omega_1 - \omega_2 - E_B \approx 4.5$ a.u.

This presentation focuses on SCS.

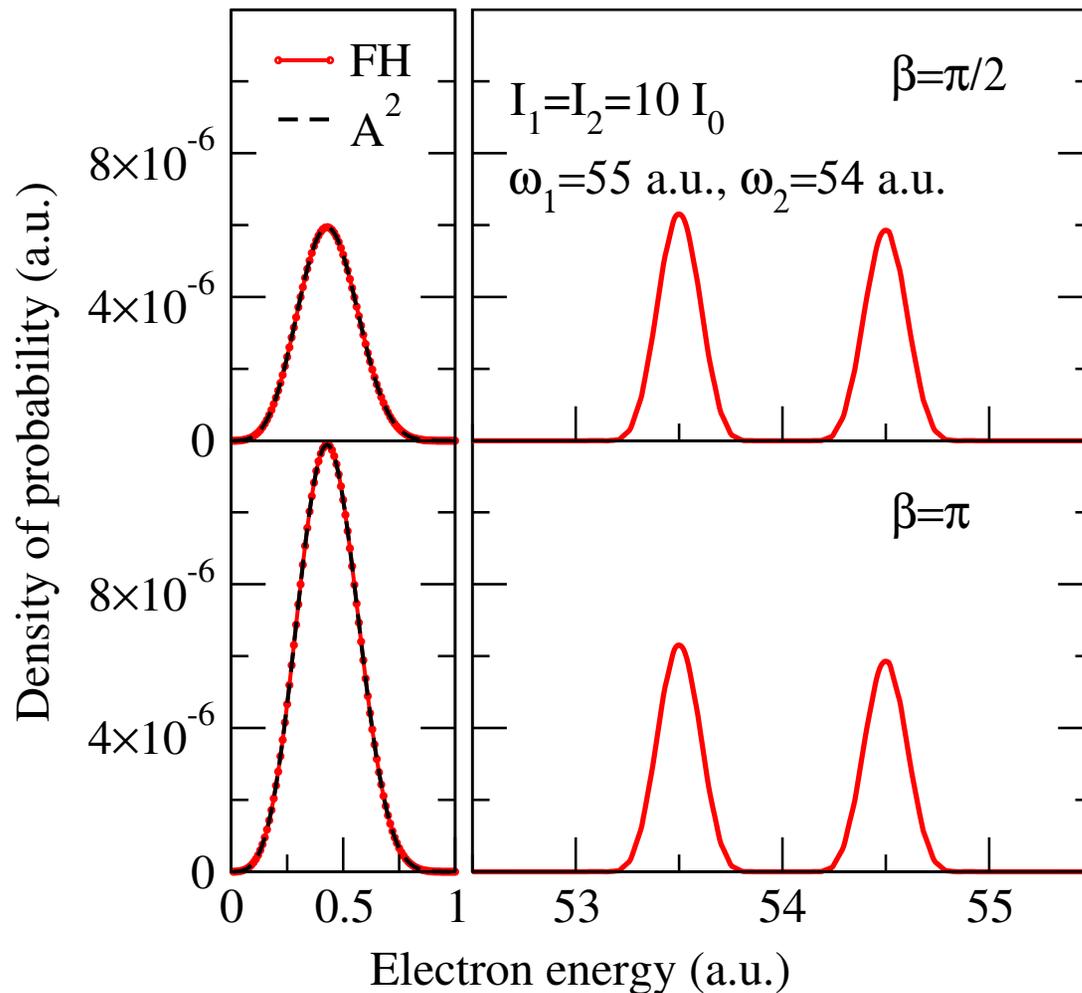
The SCS peak



Same as Fig. 1, zooming the SCS peak. A curve labeled AP is added, it refers to a calculation neglecting the contribution of \mathbf{A}^2 . Note that the AP and DA curves overlap.

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SCS vs one-photon ionization at $I=10 I_0$

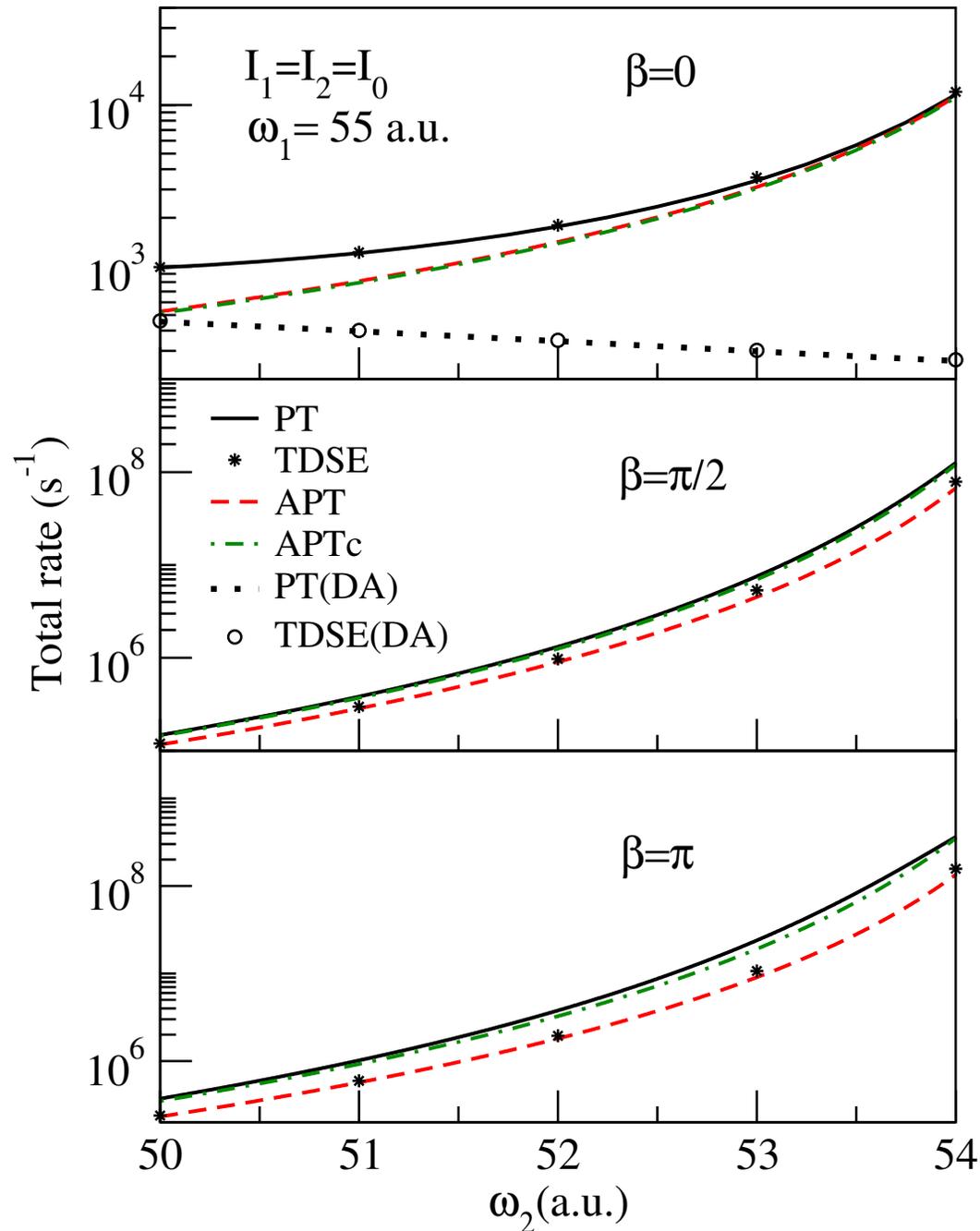


Photoelectron spectra near the SCS peak (left panels) and in the region of one-photon peaks (right panels), calculated for two pulses with a peak intensity of $3.51 \times 10^{17} \text{ W/cm}^2$ propagating in perpendicular (upper panels) and opposite (lower panels) directions. The total pulse duration is 12π . TDSE is integrated with the full Hamiltonian (FH label) or considering only the coupling term $\mathcal{H}_{RET}^{(2)}$ (label A^2).

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SCS CAN NOT BE NEGLECTED

TDSE and LOPT - a comparison in the X-ray domain



TDSE results, marked by the symbol "*"

PT - LOPT (\mathbf{A}^2 with full retardation, $\mathbf{A} \cdot \mathbf{p}$ in DA)

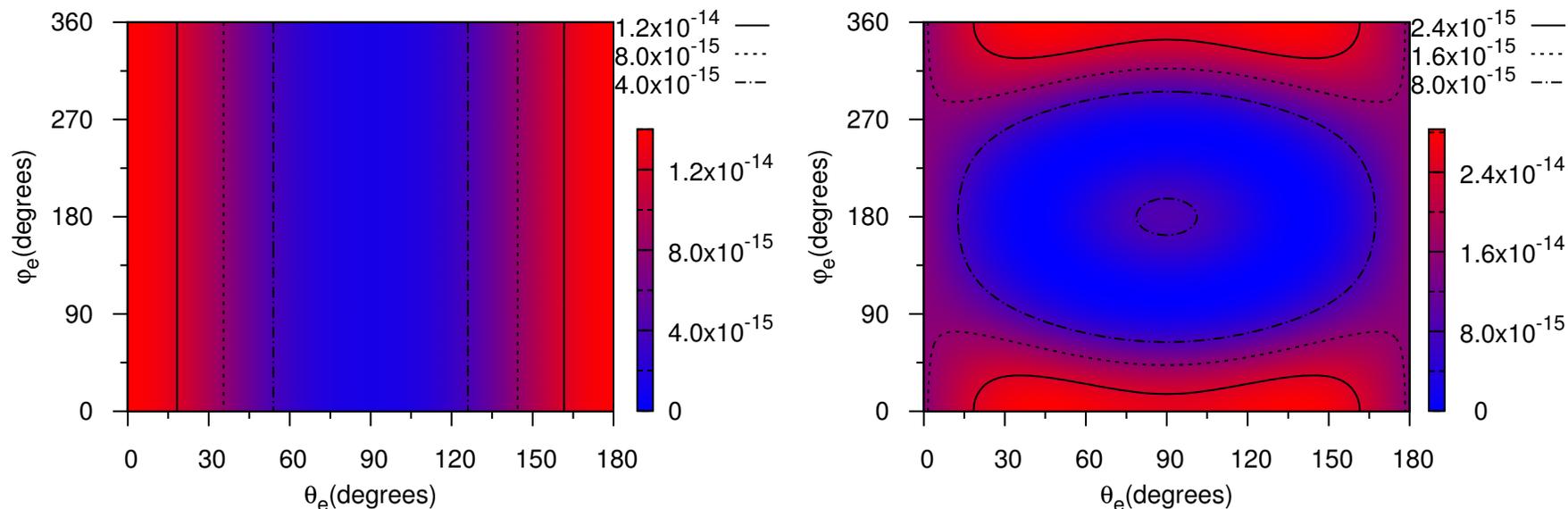
APT - only the contribution of the quadrupolar term in \mathbf{A}^2

APTc - includes the octupole contribution in \mathbf{A}^2

TDSE(DA) and PT(DA) - DA results, presented in the upper panel only.

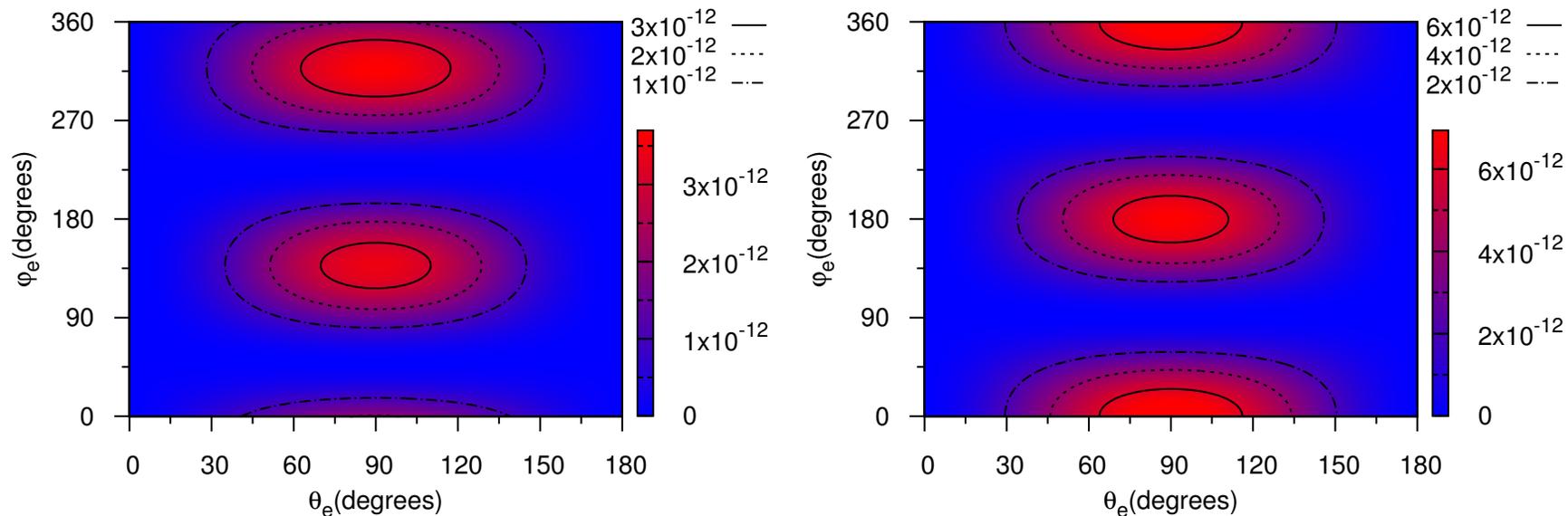
Retardation effects in all cases

PADS - retardation effects



The SCS electron angular distributions calculated from TDSE results for two pulses $\omega_1 = 55$ a.u., $\omega_2 = 50$ a.u. and a peak intensity of 3.51×10^{16} W/cm². The total pulse duration is 12π . The isoline values are indicated in the legend. (left) Calculations in DA; (right) **Angle $\beta = 0$.**

PADS - retardation effects



Same conditions as before

(left) Angle $\beta = \pi/2$. (right) Angle $\beta = \pi$.

Details on LOPT for CS and SCS - relevant equations

The standard version of PT refers to the *monochromatic fields*. In LOPT the transition rate for two-photon processes has a first order contribution from A^2 and a second order contribution from $\mathbf{A} \cdot \mathbf{P}$.

In DA the contribution of A^2 vanishes.

The transition amplitude for Compton scattering (bound-free Kramers-Heisenberg matrix element)

$$\begin{aligned} \mathcal{M}_{CS} &= \langle E \mathbf{n} - | e^{\frac{i}{\hbar} \boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2) \cdot \mathbf{r}} | 1s \rangle \\ &- \frac{1}{m} \langle E \mathbf{n} - | e^{-\frac{i}{\hbar} \boldsymbol{\kappa}_2 \cdot \mathbf{r}} P_z G(-E_B + \hbar\omega_1 + i\epsilon) e^{\frac{i}{\hbar} \boldsymbol{\kappa}_1 \cdot \mathbf{r}} P_z | 1s \rangle \\ &- \frac{1}{m} \langle E \mathbf{n} - | e^{\frac{i}{\hbar} \boldsymbol{\kappa}_1 \cdot \mathbf{r}} P_z G(-E_B - \hbar\omega_2) e^{-\frac{i}{\hbar} \boldsymbol{\kappa}_2 \cdot \mathbf{r}} P_z | 1s \rangle \end{aligned}$$

$$\hbar\omega_1 - E_B = \hbar\omega_2 + p^2/2m_e, \quad \epsilon \rightarrow 0^+.$$

LOPT for CS and SCS

In contrast to the calculation of CS rate, which requires to treat the field responsible for emission as quantized, **the calculation of SCS rate can be done in semiclassical theory**, treating both fields classically, as the transition is induced (absorption and stimulated emission). The connection with CS rate follows from this calculation and the result is that mentioned before and coincides with one obtained from a quantum description of both fields, which describes also the CS. The explanation in the factor $\sqrt{N_2 + 1}$ from the action of the creation operator of the emitted photon.

LOPT for CS: the A^2 term \equiv the sea-gull term

For the **Coulomb field** the transition amplitude of CS has an analytic expression:

- The A^2 term, named the sea-gull term in the literature

DA - vanishes due to the orthogonality of ground and continuum states

Full retardation (exact NR results): F. Schnaidt, Ann. Phys. **21**, 89 (1934) and rederived many times in the literature

Quadrupole only: E. G. Drukarev, A. I. Mikhailov, and I. A. Mikhailov, Phys. Rev. A **82**, 023404 (2010) - justified in view of the neglected relativistic corrections

The exact result presents a spurious singularity which disappears if the relativistic energy conservation law is used (R.H. Pratt and coworkers)

The origin of A^2 term: the nonrelativistic limit of the intermediate positronic states contribution to the relativistic matrix element

LOPT for CS: $\mathbf{A} \cdot \mathbf{P}$

- $\mathbf{A} \cdot \mathbf{P}$: two terms ("absorption first" and "emission first")

Analytic results : [M. Gavrilu](#) -full retardation=exact NR results and DA
Phys. Rev. A 1972

Transition amplitude expressed in terms of hypergeometric functions:
Lauricella F_D of four variables when retardation is included, Appell
functions F_1 of two variables in DA.

There is a partial cancellation between the two $\mathbf{A} \cdot \mathbf{P}$ terms noticed in DA
by P. Eisenberg and P.M. Platzmann, Phys. Rev. A **2**, 415 (1970), which
explains the dominance of \mathbf{A}^2 in CS (and SCS) at low energies.

There is a infrared divergence for low emitted frequency.

It is possible to obtain analytic expressions valid at the limit of zero electron
energy. The results depend on the electron direction.

N.B. The individual terms have no physical meaning, the separation being
that valid in the Coulomb gauge.

Extracting transition rates from the electron spectrum

The procedure: The electron spectrum is integrated in the vicinity of the SCS peak located approximately at $\hbar\omega_1 - \hbar\omega_2 - E_B$,

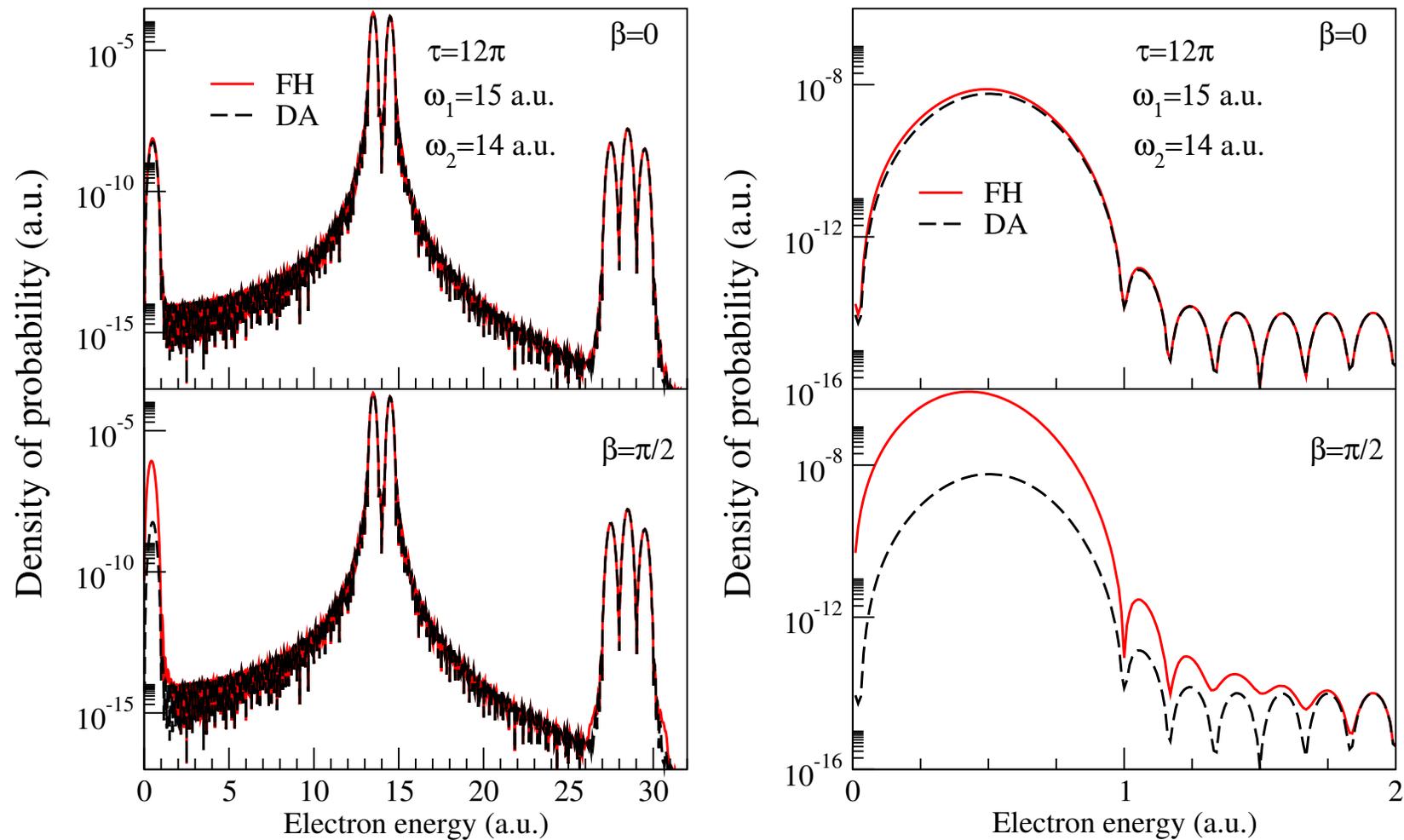
$$p_N^{\text{TDSE}} = \int_{\text{peak}} \mathcal{P}(E) dE,$$

obtaining a total **ionization probability**. If this quantity is proportional to the duration of the pulse and to I^2 , this means that PT could make sense.

The quantity to compare with perturbation theory in the case of a \cos^2 "long" pulse of radiation,

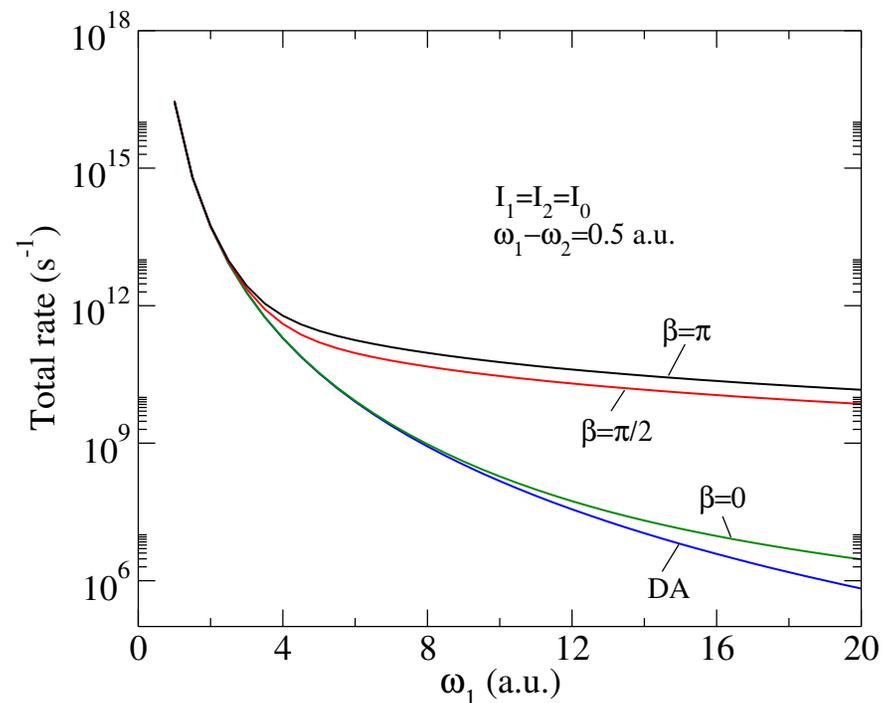
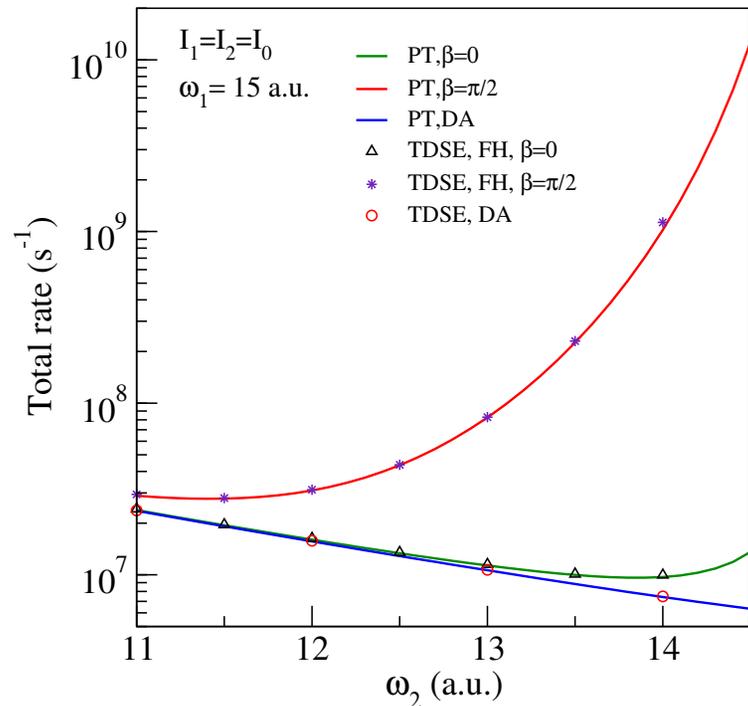
$$\Gamma_{\text{NSCS}}^{\text{TDSE}} = \frac{128}{35} \frac{p_N^{\text{TDSE}}}{\tau}.$$

EXTENSION TO THE SOFT X RAY REGION



(Left panels) Photoelectron spectra calculated with a peak intensity of 3.51×10^{16} W/cm² for two pulses propagating in the same (upper panel) and perpendicular (lower panel) directions. The total pulse duration is 12π . The calculations are performed with the full Hamiltonian (FH) and in the dipole approximation (DA). (Right panels) Same as left panels, zooming in on the SCS peak.

LOPT and TDSE- comparison in the soft x-ray region

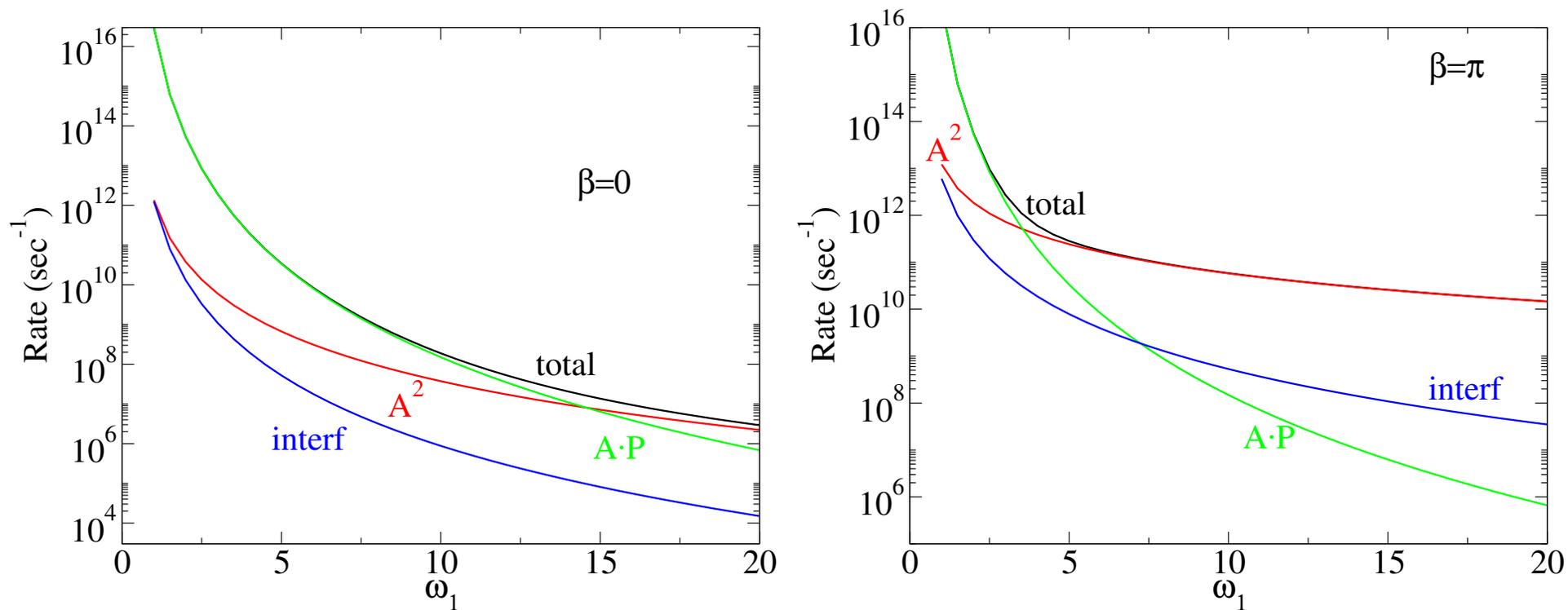


(Left) SCS ionization rates as function of ω_2 : i) TDSE in DA and with the full Hamiltonian and ii) in perturbation theory (PT). The two pulses have a peak intensity of 3.51×10^{16} W/cm² and propagate in the same ($\beta = 0$) or perpendicular ($\beta = \pi/2$) directions.

A good agreement between TDSE results and LOPT. Retardation effect : In DA, the SCS rates decrease monotonically with ω_2 . With retardation the rates reach a minimum (whose position depends on geometry), then they start to increase strongly, this behaviour is enhanced in the vicinity of the threshold.

(Right) SCS ionization rates calculated in LOPT at the limit $p = 0$ ($\omega_1 - \omega_2 = E_B$).

LOPT predictions at threshold



LOPT: SCS ionization rate **at threshold** as function of ω_1 .

Angle between the photons $\beta = 0$ (left), $\beta = \pi$ (right).

Details on the numerical solving of TDSE

We have used a spectral method based on the expansion of the wave function

$$\psi(\mathbf{r}, t) = \sum_{n, l, m} e^{-\frac{i}{\hbar} E_{nl} t} c_n^{(l, m)}(t) u_{nlm}(\mathbf{r}) \quad (1)$$

in a discrete basis of \mathcal{H}_{at} eigenfunctions $\{u_{nlm}(\mathbf{r}) = X_{nl}(r)/r Y_{lm}(\mathbf{r}/r)\}$. The eigenvalues E_{nl} of \mathcal{H}_{at} , and the radial eigenfunctions $X_{nl}(r)$ are determined by solving numerically the radial Schrödinger equation in a basis of B-spline functions ^{*)}. The initial state is the ground state with energy $E_1 \equiv E_{10}$. The functions $u_{nlm}(\mathbf{r})$ are normalized to 1.

In a typical calculation we use 5 angular momenta l within the range $l = 0 - 4$, with $|m| \leq l$, and a basis of 2400 B-spline functions of order $k = 7$, distributed linearly inside a box of length $b = 600$ a.u. The box radius b is chosen such that the probability to find the photoelectron outside the box at the end of the pulse is negligible. The convergence of the calculations is checked by varying the box size and/or the number of basis functions and angular momenta.

The coefficients $c_n^{(l, m)}(t)$ satisfy a system of coupled equations. Besides the couplings ($l' = l \pm 1$; $m' = m$), specific to DA (due to the term $\mathcal{H}_{DA}^{(1)}$), there are new couplings, not conserving the magnetic quantum number m : $\mathcal{H}_{RET}^{(1)}$ and $\mathcal{H}_{RET}^{(2)}$ couple, respectively, states with ($l' = l, l \pm 2$; $m' = m \pm 1$) and ($l' = l \pm 1$; $m' = m \pm 1$).

^{*)} H. Bachau, E. Cormier, P. Decleva, J. E. Hansen and F. Martín, Rep. Prog. Phys. **64**, 1815 (2001).

Electron distributions from the wave function

The transition amplitude $\mathcal{A}(E, \mathbf{n})$ is obtained by projecting the wave function, at $t = t_f$ (the end of the pulse), $\mathcal{A}(E, \mathbf{n}) \equiv \langle \mathbf{p} - | \psi(t_f) \rangle$, onto the incoming continuum Coulomb solutions $|\mathbf{p} - \rangle$, orthonormalized on the momentum scale, $\langle \mathbf{p} - | \mathbf{p}' - \rangle = \delta(\mathbf{p} - \mathbf{p}')$.

The electron energy and angular distribution

$$\mathcal{P}(E, \mathbf{n}) = m_e p |\mathcal{A}(E, \mathbf{n})|^2,$$

where $\mathbf{n} = \mathbf{p}/p$ is the direction of the asymptotic momentum \mathbf{p} of the photoelectron of energy $E = p^2/2m$.

Integrating $\mathcal{P}(E, \mathbf{n})$ we get the photoelectron angular distribution

$$\mathcal{P}(\mathbf{n}) = \int m_e p |\mathcal{A}(E, \mathbf{n})|^2 dE,$$

and the photoelectron energy spectrum

$$\mathcal{P}(E) = m_e p \int |\mathcal{A}(E, \mathbf{n})|^2 d\Omega_{\mathbf{n}} = m_e p \sum_{l, m} |\mathcal{A}_{lm}(p)|^2,$$

The LAURICELLA functions

The Lauricella functions depend on four variables. The type we meet

$$F_D(a; j_1 - n, j_1 - n, j_2 + n, j_2 + n; a + 1; \xi, \eta, \xi', \eta') \equiv F_D \quad (2)$$

j_1 and j_2 are non-negative integers, a a function of the photon frequencies, different for the two terms of the $\mathbf{A} \cdot \mathbf{P}$ contribution to CS (and SCS) transition amplitude. $n = -i \frac{\alpha Z m c}{p}$, an imaginary quantity. We evaluate these functions using the integral representation for the electron momentum $p \neq 0$,

$$F_D = a \int_0^1 \rho^{a-1} [1 - (\xi + \eta) \rho + \xi \eta \rho^2]^{-j_1} [1 - (\xi' + \eta') \rho + \xi' \eta' \rho^2]^{-j_2} \times \left(\frac{1 - (\xi + \eta) \rho + \xi \eta \rho^2}{1 - (\xi' + \eta') \rho + \xi' \eta' \rho^2} \right)^n d\rho.$$

The first correction to the DA, linear in photon momenta, can be extracted.

The LAURICELLA functions. The limit of ZERO ELECTRON ENERGY

In the limit $p = 0$ the variables have a finite limit, but also we have $n \rightarrow \infty$, which requires the first correction in p to the variables. With these corrections the Lauricella function becomes

$$F_D \approx a \int_0^1 \rho^{a-1} w(\rho)^{-j_1-j_2} [1 + p v(\rho)]^{-i\lambda/p} d\rho.$$

When $p \rightarrow 0$ the last factor under the integral goes to an exponential,

$$f_{a,j} = \lim_{p \rightarrow 0} F_D = a \int_0^1 \rho^{a-1} w(\rho)^{-j} e^{-i\lambda v(\rho)} d\rho.$$

In the DA case, treated previously by Gavrilu, using the series expansion of the Appell functions F_1 of two variables, our integral representation allows the identification of a Humbert function Φ_1 and we are lead to the expression given by Gavrilu for the scattering amplitudes in DA for $p = 0$.

This shows that the two limits $p = 0$ and $\kappa_i = 0$ ($i = 1, 2$) can be interchanged.

CONCLUSIONS

- Theoretical study, based on two methods (TDSE and LOPT), of two-color ionization of hydrogen atom by a coherent superposition of two x-ray or soft x-ray pulses
- Focus on SCS peak at an electron energy close to $\hbar(\omega_1 - \omega_2) - E_B$
- Important retardation effects, in the X-ray range \mathbf{A}^2 dominates.
- The SCS ionization probability is maximum for counterpropagating fields; SCS contribution larger than one-photon ionization at the high intensity $I=10 I_0$ and frequencies (55 a.u, 54a.u.).
- The SCS ionization rates increase as $\omega_1 - \omega_2$ decreases, they vary by several orders of magnitude over the energy range of ω_2 .
- The SCS ionization rates display a good agreement with perturbation theory predictions. In the x-ray case the difference is mainly due to the octupole contribution in \mathbf{A}^2 , not included in the TDSE calculation.
- PADS for $\omega_1 = 55$ a.u. and $\omega_2 = 50$ a.u. and various relative propagation directions fully differ from the DA distributions and they strongly depend on the angle β .
- SCS rates at zero electron energy based on LOPT analytic formulas.

It seems that it would be a challenge to detect SCS at XFEL facilities...

THANKS TO THE ORGANIZERS AND TO THE AUDIENCE !

Collaboration (2010-present)

This presentation: the recent results of a collaboration (2010-present)

- started in the framework of the **COST Action CM0702 (CUSPFEL)** (2008-2012), director Fernando Martin
- included later in the ESF Research Networking Program **SILMI** (2009 -2014), director Charles Joachain
- recent work connected with the **COST Action CM1204 (Xlic)** (2013-2017), director Manuel Alcamí

The University of Bordeaux provided access to the Mesocentre de Calcul Intensif Aquitain (MCIA).

The ANCS through CNCS-UEFISCDI (Romania) has granted the Contract No. 1 RNP/2012 for three years.

Extracting transition rates from the electron spectrum

For a peak in the electron spectrum situated approximately at the position corresponding to N photon ionization, $N\hbar\omega - E_B$, one calculates a total **ionization probability**

$$p_N^{\text{TDSE}} = \int_{\text{peak}} \mathcal{P}(E) dE. \quad (3)$$

If this quantity is proportional to the duration of the pulse and to I^N , the **total ionization rate** is $\Gamma_N^{\text{TDSE}} \equiv \frac{p_N^{\text{TDSE}}}{\tau}$.

The case of a \cos^2 "long" pulse of radiation,

$$\mathcal{E}(t) = E_M \cos^2 \frac{\pi t}{\tau} \quad |t| \leq \tau/2, \quad I(t) \propto \mathcal{E}^2(t), \quad I_M \propto E_M^2, \quad (4)$$

In LOPT, for a **monochromatic** radiation with constant intensity I , $\Gamma_N^{\text{LOPT}} = \gamma_N I^N$. For the pulse, this leads to a total ionization probability

$$p_N^{\text{LOPT}} = \gamma_N \int_{-\tau/2}^{\tau/2} I^N(t) dt = \Gamma_N^{\text{LOPT}}(I_M) \frac{\tau}{C_N}, \quad \frac{1}{C_N} = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \cos^{4N} \frac{\pi t}{\tau} dt$$

If PT is valid, one should have $p_N^{\text{TDSE}} = p_N^{\text{LOPT}}$

$$\tilde{\Gamma}_N^{\text{TDSE}} \equiv \Gamma_N^{\text{LOPT}}(I_M) = C_N \frac{p_N^{\text{TDSE}}}{\tau}, \quad N = 2, \quad C_2 = 128/35.$$

In the **two-color** case with the two pulses of the same intensity and duration and a ionization process involving $|N_1|$ and $|N_2|$ photons, with the final electron energy $E = N_1 \hbar \omega_1 + N_2 \hbar \omega_2 - E_B > 0$,

$$\Gamma_{N_1 N_2} = C(N) \frac{p_{N_1 N_2}}{\tau}, \quad N = |N_1| + |N_2|.$$