On the effect of time-dependent inhomogeneous magnetic fields in Sauter-Schwinger pair production

Reinhard Alkofer & Christian Kohlfürst

Institute of Physics — Theoretical Physics
University Graz

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Outline

First calculation by Sauter

Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs.

Von **Fritz Sauter** in München.

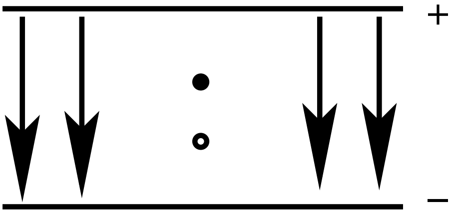
Mit 6 Abbildungen. (Eingegangen am 21. April 1931.)

Es werden die Lösungen der Diracgleichung mit dem Potential $V = vx$ angegeben und ihr Verhalten diskutiert. Zu dem Funktionsverlauf, der auch bei nichtrelativistischer Rechnung auftritt, kommt in der Diracschen Theorie noch ein Gebiet hinzu, in dem Elektronenimpuls und -geschwindigkeit entgegengesetztes Vorzeichen besitzen. Im Anschluß daran wird für ein Elektron die Wahrscheinlichkeit berechnet, aus dem Gebiet „positiven Impulses" in das mit „negativem Impuls" überzugehen. Es ergibt sich, daß die Durchgangswahrscheinlichkeit erst dann endliche Werte annimmt, wenn die Größe des Potentialanstieges auf einer Strecke gleich der Comptonwellenlänge vergleichbar wird mit der Ruheenergie des Elektrons. Die von O. Klein berechneten großen Werte für die Durchgangswahrscheinlichkeit durch einen Potentialsprung von der Größenordnung der doppelten Ruheenergie sind in dem Sinne als Grenzwerte im Falle unendlich steilen Potentialanstieges zu verstehen.

Vor einiger Zeit erschien eine interessante Arbeit von O. Klein* über

Schwinger's formula

1931: First calculation, *correct* interpretation [F.Sauter, Z.Phys. **69**(1931)742]
1932: Discovery of the positron [C.D. Anderson, Phys. Rev. **43** (1933) 491]
1936: Theoretical description of pair production from fields
[W. Heisenberg and H. Euler, Z. Phys. **98** (1936) 714 [arXiv:physics/0605038]]
1950/51: first quantum field theoretical calculation
[J. Schwinger, Phys. Rev. **82** (1951) 664.]



Static & spatially uniform electric field $\Longrightarrow$ "Vacuum" decays

Schwinger's formula

Full one-loop calculation in background of classical electric field for (boson/fermion) pair production provides
vacuum persistence probability / volume · time
(Schwinger's formula):

$$P_0 = \frac{(e\mathcal{E})^2}{4\pi^3 c\hbar^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\pi m_e^2 c^3/\hbar e\mathcal{E}}$$

- Due to tunneling of n e^+e^- pairs out of "vacuum".
- $n = 1$ term dominates ...

Schwinger's formula

Estimate of scales:

electron Compton wavelength $\frac{\lambda_e}{2\pi} = \frac{\hbar}{m_e c}$=386 fm

rest energy of e^+e^- pair $2m_ec^2$=1.022 MeV

work of field on charge e over Compton wavelength = rest energy

$$e\mathcal{E}_c \frac{\lambda_e}{2\pi} = m_e c^2 \qquad \Longrightarrow \mathcal{E}_c = \frac{m_e^2 c^3}{e\hbar} \approx 1.3 \cdot 10^{18} \frac{\mathrm{V}}{\mathrm{m}}$$

$\mathcal{E} \ll \mathcal{E}_c$: pair production is a quantum tunneling process

$\Longrightarrow$ amplitude $\propto \exp\left(-\pi\mathcal{E}_c/\mathcal{E}\right) \propto \exp\left(-\frac{1}{e}\right)$

is **NON-perturbative** for time-independent homogeneous $\mathcal{E}$.

NB: No essential singularity for finite temporal / spatial extent.

Theoretical tools

Aim:
Investigate pair production for e.m. pulses generated by laser beams.

Theoretical tools include:

- Semiclassical methods (WKB and generalizations thereof)
- Worldline formalism
- Quantum Kinetic Theory
- Dirac-Heisenberg-Wigner (DHW) formalism

Inhomogeneous Fields: **DHW**

Dirac-Heisenberg-Wigner Formalism

- Phase-space $\{\vec{x}, \vec{p}, t\}$ formulation of QM: **Wigner function**
  resp., Wigner quasiprobability distribution
- QFT: wave fcts. in Wigner fcts. $\rightarrow$ exp. values of **field operators**
  Kadanoff & Baym, *Quantum Statistical Mechanics* (1962)
- expectation values of products of Dirac field operators, *e.g.*
  $$\langle \Omega | [\psi_\alpha(\vec{x}, t), \psi^\dagger_\beta(\vec{x}', t)] | \Omega \rangle$$
  Dirac (1934); Heisenberg (1934)
- DHW formalism & 1st application to pair production in nineties
  I. Bialynicki-Birula, P. Gornicki and J. Rafelski, Phys. Rev. **D44** (1991) 1825
- Infinite hierarchy of n-point DHW functions: truncations necessary!
- **Hartree approximation:** Mean electromagnetic field
  well justified!

Dirac-Heisenberg-Wigner Formalism

Start: Covariant & gauge-invariant DHW operator

$$\widehat{W}(x,p) = \frac{1}{2}\int d^4y\, e^{ip\cdot y}\, U[A_\mu](x,y)\left[\bar{\psi}(x+y/2), \psi(x-y/2)\right]$$

Next: Choose suitable frame and project on equal time:

$$\widehat{\mathcal{W}}(\vec{x},\vec{p},t) = \int \frac{dp_0}{2\pi}\,\widehat{W}(x,p) = \frac{1}{4}\left(\mathbb{s} + i\gamma_5\mathbb{p} + \gamma_\mu\mathbb{v}^\mu + \gamma_\mu\gamma_5\mathbb{a}^\mu + \sigma_{\mu\nu}\mathbb{t}^{\mu\nu}\right)$$

Then: Derive e.o.m. in Hartree approximation and
take expectation value with background field in Dirac vacuum

$$\mathcal{W}(\vec{x},\vec{p},t) = \langle\Omega|\widehat{\mathcal{W}}(\vec{x},\vec{p},t)|\Omega\rangle$$

Dirac-Heisenberg-Wigner Formalism

- E.o.m.: Integro-differential equations
- *E.g.* for vanishing magnetic field $\vec{B} = 0$

$$D_t \mathcal{W}_{\alpha\beta} = -\frac{1}{2}\nabla \left[\gamma^0 \vec{\gamma}, \mathcal{W}\right]_{\alpha\beta} - i\left[m\gamma^0, \mathcal{W}\right]_{\alpha\beta} - i\left\{\gamma^0 \vec{\gamma}\vec{p}, \mathcal{W}\right\}_{\alpha\beta}$$

with pseudo-differential operator

$$D_t = \partial_t + e\int_{-1/2}^{1/2} d\lambda\, \vec{E}(\vec{x} + i\lambda\partial_p, t)\,\partial_p$$

F. Hebenstreit, PhD thesis, April 2011

F. Hebenstreit, R. A., H. Gies, Phys.Rev.Lett. **107** (2011) 180403

- Observables (particle density, charge density, . . .):
  Integrals over combinations of Wigner components $\mathbb{s}, \mathbb{v}^{0,i}, \ldots$

Dirac-Heisenberg-Wigner Formalism

C. Kohlfürst, PhD thesis, July 2015

- full equations including electric and magnetic fields
- 3+1, 2+1 and 1+1 dimensions
- selected symmetries as *e.g.* cylindrically symmetric fields
- Quantum Kinetic Theory in homogeneous limit
- most efficient numerical solution by pseudo-spectral methods (check for convergence at late time)
- calculations of observables from Wigner components straightforward

Model for electromagnetic field

- Superposition of left- and right-running pulses in 2+1 dim.:

$$\vec{A}(z,t) = \varepsilon\,\tau\left(\tanh\left(\frac{t}{\tau}+1\right)-\tanh\left(\frac{t}{\tau}-1\right)\right)\exp\!\left(-\frac{z^2}{2\lambda^2}\right)\,\vec{e}_x .$$

  - ε maximal electric field strength
  - τ temporal extent (difference of Sauter pulses)
  - λ spatial extent (Gaußian)
- homogeneous Maxwell eqs. fulfilled by construction
- electric field: double-peak structure, antisymmetric in time
- magnetic field: maximal strength $\varepsilon\tau/\lambda^2$
- Field energy in
  - electric field for $\tau/\lambda \ll 1$
  - magnetic field for $\tau/\lambda \gg 1$

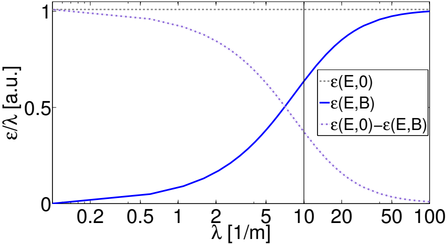
for more details and other model fields see:

Chapter 7 of C. Kohlfürst, PhD thesis, July 2015; C. Kohlfürst & RA, in preparation

Numerical Results for Model Field

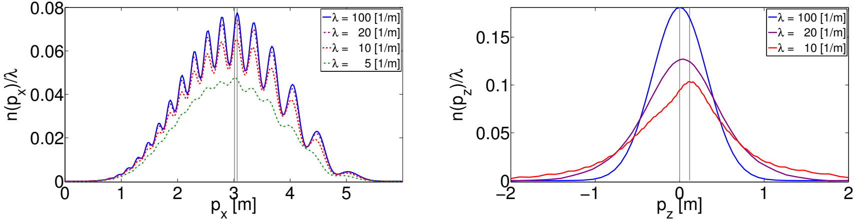
- pseudoscalar Lorentz invariant $\tilde{F}_{\mu\nu}F^{\mu\nu} \propto \vec{E}\vec{B} = 0$
- scalar Lorentz invariant $F_{\mu\nu}F^{\mu\nu} \propto \vec{E}^2 - \vec{B}^2 \;\binom{>}{<}\; 0$
- pair production only for regions in which $E^2(t,z) - B^2(t,z) > 0$

- Effectively available field energy
  $\mathcal{E}[\vec{E},\vec{B}] = \int dtdz\,\left(E^2(t,z) - B^2(t,z)\right) \Theta\left(E^2(t,z) - B^2(t,z)\right)$



Numerical Results for Model Field

Reduced particle density as function of p_x and p_z
(ε = 0.707 E_c, $\tau = 5/m$):



$\lambda \gg \tau, 1/m$: homogeneous limit

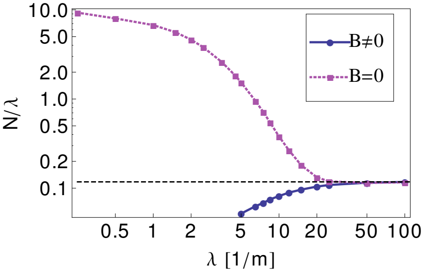
$\lambda \approx \tau$: deflection due to magnetic field

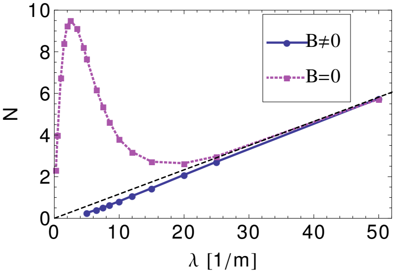
Suppression of pair production?

Numerical Results for Model Field

Comparison to calculation with magnetic field neglected

Reduced total particle number, resp., total particle number (ε = 0.707 E_c, $\tau = 10/m$):





Significantly overestimated particle number for small λ!

NB: Regions with negative “particle distribution”!

Summary and Outlook

- Included magnetic fields in DHW calculation for pair production (Homogeneous Maxwell equation fulfilled by construction!)
- For model field:
  - Weak magnetic fields negligible.
  - Strong magnetic field, resp., small spatial extent: (expected?) suppression of pair production
- Evidence that neglecting magnetic field introduces large errors!

Despite substantial progress,
calculation with "realistic" field parameters:
still a big numerical challenge . . .

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