Exercises to "Standard Model of Particle Physics II"

Winter 2019/20

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

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Problem 18: *Discrete symmetries* [**5 Points**]

A CP transformation takes a left-handed neutrino and transforms it into a right-handed anti-neutrino. Consider the probability $P(\nu_\alpha \to \nu_\beta,\, t)$ of a neutrino of flavor α to oscillate into a flavor β after time t of propagation.

a) How do the discrete symmetries CP, T, and CPT change the initial and final states of the process given above? That is, give

$$(\mathrm{CP})^{-1} P(\nu_\alpha \to \nu_\beta,\, t)(\mathrm{CP})\,,$$
$$(\mathrm{T})^{-1} P(\nu_\alpha \to \nu_\beta,\, t)(\mathrm{T})\,,$$
$$(\mathrm{CPT})^{-1} P(\nu_\alpha \to \nu_\beta,\, t)(\mathrm{CPT})$$

in terms of other transition probabilities.

b) The CPT theorem states that every QFT described by a Lorentz invariant, local Lagrangian is invariant under CPT transformations. How could the above result be used to test this theorem in Nature? (Hint: Consider the case $\alpha = \beta$, i.e. the survival probability of (anti-) neutrinos.)

Problem 19: *Neutrino Oscillations and CP violation* [**15 Points**]

Neutrinos are produced and detected in weak interactions, but the neutrino mass matrix is diagonal in a different basis. The unitary matrix U rotates the flavor basis into the mass eigenbasis. Consider a neutrino of flavor α produced at time $t = 0$,

$$|\nu(t=0)\rangle = |\nu_\alpha\rangle = U^*_{\alpha k}|\nu_k\rangle, \tag{1}$$

as a superposition of mass eigenstates $|\nu_j\rangle$.

a) Using approximations i) and ii) below, and that the time evolution of $|\nu_k\rangle$ is governed by the Schrödinger equation to derive the general expression for neutrino oscillations in vacuum,

$$P_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta,\, t) = \delta_{\alpha\beta} - 4\sum_{k>j} \mathrm{Re}\left(\mathcal{I}^{kj}_{\alpha\beta}\right)\sin^2\left(\frac{\Delta_{kj}}{2}\right) + 2\sum_{k>j}\mathrm{Im}\left(\mathcal{I}^{kj}_{\alpha\beta}\right)\sin\left(\Delta_{kj}\right), \tag{2}$$

where $\mathcal{I}^{kj}_{\alpha\beta} = U^*_{\alpha k}U_{\beta k}U^*_{\beta j}U_{\alpha j}$ and $\Delta_{kj} = \frac{m_k^2 - m_j^2}{2E}L$. To arrive at this expression, you have to make the following assumptions:

i) Neutrinos are highly relativistic, such that the distance travelled is $L \simeq t$ $(c = 1)$ and $E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i \left(1 + \frac{m_i^2}{2p_i^2}\right)$.

ii) The central momentum of the neutrino beam is given by $p_i \simeq p \approx E$ for all i.

b) Derive the expression for the CP asymmetry $\mathcal{A}_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta, t) - P(\overline{\nu}_\alpha \to \overline{\nu}_\beta, t)$. (Hint: How does the state (1) change under a CP transformation? Use this fact and Eq. (2)!)

c) Consider now the special case of two neutrino flavors, say e and μ, hence

$$U = \begin{pmatrix} \cos\theta & e^{-i\delta}\sin\theta \\ -e^{-i\delta}\sin\theta & \cos\theta \end{pmatrix}.$$

Calculate the CP asymmetry for $e - \mu$ oscillations. What are the conditions for CP violation to occur?

d) Consider now the special case of three neutrino flavors whose mixing in general can be parametrised as

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

where $c_{ab} \equiv \cos(\theta_{ab})$, $s_{ab} \equiv \cos(\theta_{ab})$, $0 \leq \theta_{ab} \leq \pi/2$, and $0 \leq \delta \leq 2\pi$. Calculate the CP asymmetry $\mathcal{A}_{e\mu}$ explicitly and show that it can only be non-vanishing if all of the following conditions are satisfied: 1. None of the mixing angles is 0 or $\pi/2$. 2. None of the mass-square differences (i.e. Δ_{ab}) are zero. 3. The phase δ is neither 0 nor π.

Problem 21: *A teaser for the holidays* **- Merry Christmas and happy new year!**

The Koide formula reads

$$Q = \frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2}.$$

a) Assuming only that the masses are positive, determine the range in which Q may take values.

b) Calculate Q for the current best-fit values according to PDG[1], $m_e = 0.510\,998\,946\,1\,\mathrm{MeV}$, $m_\mu = 105.658\,374\,5\,\mathrm{MeV}$, $m_\tau = 1776.86\,\mathrm{MeV}$.

c) The empirical value of Q lies very close to a rational number which has a special position in the range of possible Q-values. Can you find an explanation for this astonishing apparent coincidence?

[1] http://pdg.lbl.gov/2018/tables/rpp2018-sum-leptons.pdf