Exercises to "Standard Model of Particle Physics II"

Winter 2015/16

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Exercise 17: *Neutrino oscillations* [**10 Points**]

Consider the case of three neutrino flavours. The mixing matrix then reads

$$\begin{pmatrix} \nu_{e\mathrm{L}} \\ \nu_{\mu\mathrm{L}} \\ \nu_{\tau\mathrm{L}} \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \begin{pmatrix} \nu_{1\mathrm{L}} \\ \nu_{2\mathrm{L}} \\ \nu_{3\mathrm{L}} \end{pmatrix}$$

The oscillation has a CP-violating fraction $\Delta P_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta;\ t) - P(\overline{\nu}_\alpha \to \overline{\nu}_\beta;\ t)$.

a) Consider the general expression for the transition probability

$$P(\nu_\alpha \to \nu_\beta;\ t) = \sum_j |U_{\alpha j}|^2 |U_{\beta j}|^2 + \sum_{j \neq k} U_{\alpha j} U^*_{\beta j} U^*_{\alpha k} U_{\beta k}\ \mathrm{e}^{-i(E_j - E_k)t}.$$

Which result do you obtain for $\Delta P_{\alpha\alpha}$ and what does it imply?

b) Use the unitarity of U to show that $\Im m[U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k}]$ is independent (up to a sign) of the choice of α and β ($\alpha \neq \beta$).

Note: The same invariant can be constructed for the CKM matrix and is a measure for CP-violation in the meson sector.

c) Discuss your results and the influence of the mixing parameters on CP-violation.

Exercise 18: *Neutrino oscillations in matter* [**10 Points**]

The MSW effect describes oscillations of neutrinos in matter. Consider the case of two flavour oscillations in a realistic (non-constant) mass distribution.
The time evolution of the eigenfunctions of the Hamiltonian in matter is described by the following expression:

$$i\frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} |\nu_\mathrm{A}\rangle \\ |\nu_\mathrm{B}\rangle \end{pmatrix} = \begin{pmatrix} E_\mathrm{A}(t) & -i\dot{\theta}(t) \\ i\dot{\theta}(t) & E_\mathrm{B}(t) \end{pmatrix} \begin{pmatrix} |\nu_\mathrm{A}\rangle \\ |\nu_\mathrm{B}\rangle \end{pmatrix}$$

The eigenstates in matter are $|\nu\rangle = (|\nu_\mathrm{A}\rangle,\ |\nu_\mathrm{B}\rangle)^\mathrm{T}$ and to any given time they are connected to the flavor states via: $|\tilde{\nu}\rangle = \tilde{U}(t)|\nu\rangle$ with

$$|\tilde{\nu}\rangle = \begin{pmatrix} |\tilde{\nu}_e\rangle \\ |\tilde{\nu}_\mu\rangle \end{pmatrix} \qquad \tilde{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and the angle

$$\tan(2\theta) = \frac{\frac{\Delta m^2}{2E}\sin(2\theta_0)}{\frac{\Delta m^2}{2E}\cos(2\theta_0) - \sqrt{2}G_\mathrm{F}N_e(t)}, \tag{1}$$

where θ_0 is constant in time, and $N_e(t)$ denotes the electron number density in matter.
The effective Hamiltonian is not diagonal, since matter eigenstates and thus the mixing angle θ are not constant. If, however, $\dot{\theta}$ is small, i.e. the change in θ occurs slowly, we obtain the adiabatic approximation, where we can neglect $\dot{\theta}$. Suppose that at the time $t = t_i$ an electron neutrino is produced,

$$|\tilde{\nu}_\alpha(t_i)\rangle \equiv |\tilde{\nu}_e\rangle = \cos\theta_i\ |\nu_\mathrm{A}(t_i)\rangle + \sin\theta_i\ |\nu_\mathrm{B}(t_i)\rangle.$$

a) What is the expression for $|\tilde{\nu}_\alpha(t_f)\rangle$ in terms of $|\nu_\mathrm{A}(t_i)\rangle$ and $|\nu_\mathrm{B}(t_i)\rangle$ at some later time t_f in the adiabatic approximation?

b) Show that in the adiabatic approximation the result for the transition probability is given by

$$P(\nu_e \to \nu_\mu) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta_i)\cos(2\theta_f) - \tfrac{1}{2}\sin(2\theta_i)\sin(2\theta_f)\cos\Phi_\mathrm{AB}$$

with

$$\Phi_\mathrm{IJ} \equiv \int_{t_i}^{t_f} [E_\mathrm{I}(t) - E_\mathrm{J}(t)]\, \mathrm{dt},$$

where $\mathrm{I},\ \mathrm{J} \in \{\mathrm{A},\ \mathrm{B}\}$.

c) Use Eq. (1) to find an expression for $\sin^2(2\theta)$. The formula for $\sin^2(2\theta)$ describes a Breit-Wigner distribution. We define the resonance width at half height, Δr, as the width at which the curve of the distribution is larger than a half $[\sin^2(2\theta) > 1/2]$. Find an approximate expression for Δr.

d) The adiabatic parameter at the MSW resonance is

$$\gamma_r \equiv \frac{\sin^2(2\theta_0)\Delta m^2}{\cos(2\theta_0)2E} L \quad .$$

The scale $L = |\dot{N}_e/N_e|^{-1}_\mathrm{res}$ is the characteristic length scale where the electron density varies in the resonance region. Express γ_r in terms of Δr and the oscillation length given by

$$l_\mathrm{res} = \frac{2\pi}{|E_\mathrm{A} - E_\mathrm{B}|_\mathrm{res}} = \frac{4\pi E/\Delta m^2}{\sin(2\theta_0)}.$$

Discuss the adiabatic condition $\gamma_r \gtrsim 3$.

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Lecture webpage: http://www.mpi-hd.mpg.de/manitop/StandardModel2/index_WS15.html

Hand-in and discussion of sheet:
Tuesday, 08.12.15, 16.15 am, INF 501 / CIP R. 103.