Exercises to “Standard Model of Particle Physics II”

Winter 2014/15

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Exercise 18: Neutrino oscillations in matter [**15 Points**]

The MSW effect describes oscillations of neutrinos in matter. Consider the case of two flavour oscillations in a realistic (non-constant) mass distribution.
The time evolution of the eigenfunctions of the Hamiltonian in matter is described by the following expression:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix}|\nu_\mathrm{A}\rangle\\|\nu_\mathrm{B}\rangle\end{pmatrix}=\begin{pmatrix}E_\mathrm{A}(t) & -\mathrm{i}\dot{\theta}(t)\\ \mathrm{i}\dot{\theta}(t) & E_\mathrm{B}(t)\end{pmatrix}\begin{pmatrix}|\nu_\mathrm{A}\rangle\\|\nu_\mathrm{B}\rangle\end{pmatrix}$$

The eigenstates in matter are $|\nu\rangle = (|\nu_\mathrm{A}\rangle,\ |\nu_\mathrm{B}\rangle)^\mathrm{T}$ and to any given time they are connected to the flavor states via: $|\tilde{\nu}\rangle = \tilde{U}(t)|\nu\rangle$ with

$$|\tilde{\nu}\rangle = \begin{pmatrix}|\tilde{\nu}_e\rangle\\|\tilde{\nu}_\mu\rangle\end{pmatrix} \qquad \tilde{U} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

and the angle

$$\tan(2\theta) = \frac{\frac{\Delta m^2}{2E}\sin(2\theta_0)}{\frac{\Delta m^2}{2E}\cos(2\theta_0) - \sqrt{2}G_\mathrm{F}N_e(t)}, \tag{1}$$

where θ_0 is constant in time, and $N_e(t)$ denotes the electron number density in matter.
The effective Hamiltonian is not diagonal, since matter eigenstates and thus the mixing angle θ are not constant. If, however, $\dot{\theta}$ is small, i.e. the change in θ occurs slowly, we obtain the adiabatic approximation, where we can neglect $\dot{\theta}$. Suppose that at the time $t = t_\mathrm{i}$ an electron neutrino is produced,

$$|\tilde{\nu}_\alpha(t_\mathrm{i})\rangle \equiv |\tilde{\nu}_e\rangle = \cos\theta_\mathrm{i}\,|\nu_\mathrm{A}(t_\mathrm{i})\rangle + \sin\theta_\mathrm{i}\,|\nu_\mathrm{B}(t_\mathrm{i})\rangle.$$

a) What is the expression for $|\tilde{\nu}_\alpha(t_\mathrm{f})\rangle$ in terms of $|\nu_\mathrm{A}(t_\mathrm{i})\rangle,\ |\nu_\mathrm{B}(t_\mathrm{i})\rangle$ at some later time t_f in the adiabatic approximation?

b) Show that in the adiabatic approximation the result for the transition probability is given by

$$P(\nu_e \to \nu_\mu) = \frac{1}{2} - \frac{1}{2}\cos(2\theta_\mathrm{i})\cos(2\theta_\mathrm{f}) - \frac{1}{2}\sin(2\theta_\mathrm{i})\sin(2\theta_\mathrm{f})\cos\Phi_\mathrm{AB}$$

with

$$\Phi_\mathrm{IJ} \equiv \int_{t_\mathrm{i}}^{t_\mathrm{f}} [E_\mathrm{I}(t) - E_\mathrm{J}(t)]\,\mathrm{dt},$$

where $\mathrm{I},\ \mathrm{J} \in \{\mathrm{A},\ \mathrm{B}\}$.

c) Use Eq. (1) to find an expression for $\sin^2(2\theta)$. The formula for $\sin^2(2\theta)$ describes a Breit-Wigner distribution. We define the resonance width at half height, Δr, as the width at which the curve of the distribution is larger than a half [$\sin^2(2\theta) > 1/2$]. Find an approximate expression for Δr.

d) The adiabatic parameter at the MSW resonance is

$$\gamma_\mathrm{r} \equiv \frac{\sin^2(2\theta_0)\Delta m^2}{\cos(2\theta_0)2E}L \quad .$$

The scale $L = |\dot{N}_e/N_e|^{-1}_{\text{res}}$ is the characteristic length scale where the electron density varies in the resonance region. Use the expression for the oscillation length $l_{\text{res}} = 2\pi/|E_A - E_B|_{\text{res}} = (4\pi E/\Delta m^2)/\sin(2\theta_0)$ and your expressions for Δr to express γ_r in another way, and discuss the adiabatic condition $\gamma_r > 3$.

Exercise 19: Neutrino masses [**5 Points**]

The best-fit values for the three-flavor oscillation parameters (without error) are given in Table 1.

parameter	normal ordering	inverted ordering
$\sin^2\theta_{12}$	0.304	0.304
θ_{12} [°]	33.48	33.48
$\sin^2\theta_{23}$	0.452	0.579
θ_{23} [°]	42.3	49.5
$\sin^2\theta_{13}$	0.0218	0.0219
θ_{13} [°]	8.50	8.51
δ_{CP} [°]	306	254
Δm^2_{21} [10^{-5} eV^2]	7.50	7.50
Δm^2_{3l} [10^{-3} eV^2]	+2.457	−2.449

Table 1: Three-flavor oscillation parameters (best-fit values without error). In the last line $\Delta m^2_{3l} \equiv \Delta m^2_{31} > 0$ for normal ordering and $m^2_{3l} \equiv \Delta m^2_{32} < 0$ for inverted ordering. Values taken from arXiv: 1409.5439.

Use the values from Table 1 to draw the following functions in double logarithmic plots as functions of the lightest neutrino mass from 10^{-4} eV to 2 eV:

a) the neutrino masses m_1, m_2 and m_3 for normal ordering,

b) the neutrino masses m_1, m_2 and m_3 for inverted ordering,

c) the electron neutrino mass m_β (as meassured from the β-decay spectrum endpoint energy e.g. by KATRIN) for both orderings.

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Tutorials homepage: `http://www.mpi-hd.mpg.de/manitop/StandardModel2/exercise.html`

Hand-in and discussion of sheet:
during tutorial on Thursday, 15.01.15, 9.15 am, kHs, Philosophenweg 12