Exercises to "Standard Model of Particle Physics II"

Winter 2020/21

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in of solutions: January 13, 2021 - via e-mail, **before 14:00**

Discussion of solutions: January 13, 2021 - on zoom

Problem 13: *Running couplings* [**10 Points**]

a) In QED the coupling $\alpha = e^2/4\pi$ is a running coupling. For low energies one obtains $\alpha(\mu = m_e) = 1/137$. The running of the QED coupling with only one lepton is given by

$$\alpha_{\text{QED}}(\mu) = \frac{\alpha(m_e)}{1 - \frac{2\alpha(m_e)}{3\pi}\ln(\frac{\mu}{m_e})} \ . \tag{1}$$

At which scale does $\alpha_{\text{QED}}(\mu)$ become infinite? (If further charged leptons and quarks are considered, the coupling increases even faster.)

b) In QCD, with $\alpha_s = g_3^2/4\pi$, the running coupling

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{(33-2f)\alpha_s(\mu_0)}{6\pi}\ln(\frac{\mu}{\mu_0})} \tag{2}$$

is obtained, where f denotes the respective number of quark flavours with mass $2m_q \leq \mu$ in the considered energy range.

The experimental boundary conditions are $\alpha_s(\mu_0 = m_Z = 91 \text{ GeV}) = 0.12$, $m_t = 175$ GeV, $m_b = 4.8$ GeV, $m_c = 1.4$ GeV and $m_u \simeq m_d \simeq m_s \simeq 0$.

i) Determine the pole $\mu = \Lambda_{\text{QCD}}$ of eq. (2).

ii) For which μ does the coupling $\alpha_s(\mu)$ become very small (asymptotic freedom), and where does perturbation theory break down?

iii) Determine the value of $\alpha_s(\mu)$ in the different energy ranges ($2m_q \leq \mu \leq 2m_t$ etc.) at the thresholds.

iv) $\alpha_s(\mu)$ should be continuous. However, from the threshold values calculated in the last part we saw that this is not yet true. To make $\alpha_s(\mu)$ continuous, promote μ_0 to a function $\mu_0(f)$ with $\mu_0(f = 5) = 91$ GeV. Find a relation between $\mu_0(f)$ and $\mu_0(f-1)$ assuming that $\alpha_s(\mu_0(f)) = 0.12$ for all f. With this relation determine the values of $\mu_0(4)$ and $\mu_0(6)$.

c) Draw $\alpha_s^{-1}(\mu)$ and $\alpha_{\text{QED}}^{-1}(\mu)$ as functions of $\ln(\mu)$. What could the intersection of the curves indicate?

Problem 14: *Integrating out the W-boson* **[10 Points]**

Heavy particles decouple from a given theory when the relevant energy scale of a process is well below their mass. An important example are the $W^\pm$ bosons of the SM, which we will consider in this exercise. The relevant part of the SM Lagrangian reads

$$\mathcal{L}_W = -\frac{1}{2} W^-{}_{\mu\nu} W^{+\mu\nu} + m_W^2\, W^+{}_\mu W^{-\mu} + \frac{g}{\sqrt{2}}\, W^+{}_\mu J^{-\mu} + \frac{g}{\sqrt{2}}\, W^-{}_\mu J^{+\mu}\,,$$

with the electroweak charged currents acting as source terms,

$$J^+{}_\mu = \overline{e}_L \gamma_\mu \nu_L\,, \quad J^-{}_\mu = \overline{\nu}_L \gamma_\mu e_L\,.$$

a) Derive the classical equation of motion for the $W^\pm$ bosons and expand to lowest order in $\frac{p}{m_W}$, where p is the momentum of the W (off-shell), i.e. $\partial_\mu W^\pm{}_\nu = -ip_\mu W^\pm{}_\nu$.

b) Solve the equation to lowest order in $\frac{p}{m_W}$ ignoring non-linear interactions among the W's.

c) Show that this solution can be used to obtain the Lagrangian of Fermi's effective theory of weak interactions,

$$\mathcal{L}_{\text{Fermi}} = -2\sqrt{2} G_F J^+{}_\mu J^{-\mu}$$

and determine G_F in terms of the W mass, m_W, and the weak gauge coupling, g.

Problem 15: *A teaser for the holidays* **- Merry Christmas and happy new year! [5 Points]**

The Koide formula reads

$$Q = \frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2}\,.$$

a) Assuming only that the masses are positive, determine the range in which Q may take values.

b) Calculate Q for the current best-fit values according to PDG[1], $m_e = 0.510\,998\,946\,1\,\text{MeV}$, $m_\mu = 105.658\,374\,5\,\text{MeV}$, $m_\tau = 1776.86\,\text{MeV}$.

c) The empirical value of Q lies very close to a rational number which has a special position in the range of possible Q-values. Can you find an explanation for this astonishing apparent coincidence?

[1] http://pdg.lbl.gov/2018/tables/rpp2018-sum-leptons.pdf