Exercises to "Standard Model of Particle Physics II"

Winter 2019/20

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in of solutions: December 4, 2019 15:45, Philosophenweg 12, kHS
Discussion of solutions: December 11, 2019

Problem 15: *Fermion mass matrix diagonalization* [**10 Points**]

a) A hermitian $n \times n$ matrix $M = M^\dagger$ can always be diagonalized by a unitary transformation

$$UMU^\dagger = D = \mathrm{diag}\,(m_1,\, m_2,\, \ldots,\, m_n)$$

where the eigenvalues m_i can be negative as well as positive. Show that one can choose an appropriate bi-unitary transformation $UMV^\dagger$ to diagonalize M so that all diagonal elements are non-negative.

b) Consider an arbitrary complex $n \times n$ matrix A. Show that one can always find two unitary transformation matrices U and V such that $UAV^\dagger$ is diagonal with non-negative elements. Although it is not a formal requirement, you may assume that the eigenvalues of $AA^\dagger$ are non-zero, which will simplify the proof.

c) Now we know that the mass matrices for the SM fermions can be diagonalized by bi-unitary transformations with non-negative mass eigenvalues. The diagonalization in the charged lepton sector leads to a mixing for the charged leptons

$$e_L = U^\dagger e'_L\,, \qquad e_R = V^\dagger e'_R\,.$$

where the primed (unprimed) fields denote the fields in the flavour (mass) basis. Note that since the interactions with W and Z are diagonal in flavor basis, they would mix different mass eigenstates. Show that in the case of massless neutrinos this mixing for the charged leptons would have no physical effect.

d) In the case of quarks, both the up-type and the down-type quarks have a mass matrix, such that diagonalization leads to

$$u_L = U_u^\dagger u'_L\,, \qquad u_R = V_u^\dagger u'_R\,,$$

$$d_L = U_d^\dagger d'_L\,, \qquad d_R = V_d^\dagger d'_R\,.$$

Show that in this case the mixing of quarks *does* leave a physical effect in charged-current (coupling to W interactions). Give, in terms of the U and V matrices above, the 3-by-3 matrix which describes the coupling between u, c, and t quarks in their mass basis with respect to d, s and b quarks in their mass basis in charged-current interactions. This is the famous Cabibbo-Kobayashi-Maskawa (CKM) matrix.

Problem 16: *Custodial Symmetry and the* ρ*-Parameter* [**10 Points**]

The 1-loop correction to the ρ-parameter is given by

$$\Delta\rho = \frac{3G_F}{8\pi^2 2\sqrt{2}}\left(m_t^2 + m_b^2 - 2\frac{m_t^2 m_b^2}{m_t^2 - m_b^2}\log\frac{m_t^2}{m_b^2} - \frac{11}{9}m_Z^2 \sin^2\theta_W \log\frac{m_h^2}{m_Z^2}\right),$$

where m_t (m_b) is the top (bottom) quark mass, θ_W the Weinberg angle and m_h (m_Z) the mass of the Higgs (Z) boson.

a) Convince yourself that $\Delta\rho = 0$ for $m_t = m_b$ and if the hypercharge gauge coupling is zero.

b) As discussed in the lecture, the Higgs potential is invariant under $\mathrm{SU(2)_L \times SU(2)_R}$. Define

$$L = \begin{pmatrix} t \\ b \end{pmatrix}_L, \quad R = \begin{pmatrix} t \\ b \end{pmatrix}_R \quad \text{and } \Phi = \left(\phi, \tilde{\phi}\right),$$

with ϕ the Higgs doublet, $\widetilde{\phi} = i\sigma_2\phi^*$, and σ_2 the second Pauli matrix. Show that the Lagrangian containing the top and bottom Yukawa couplings g_t and g_b is invariant under the $\mathrm{SU(2)_L \times SU(2)_R}$ symmetry only if $g_t = g_b$.

c) Show by analyzing the kinetic term of the Higgs boson that it is invariant under $\mathrm{SU(2)_L \times SU(2)_R}$ symmetry only if the hypercharge gauge coupling is zero. Start by writing down the covariant derivative on Φ (taking into account the fact that ϕ and $\widetilde{\phi}$ have hypercharges 1 and -1 respectively).