Exercises to “Standard Model of Particle Physics II”

Winter 2016/17

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Problem 13: *Fermion mass matrix diagonalization* [**5 Points**]

a) A hermitian $n\times n$ matrix $M = M^\dagger$ can always be diagonalized by a unitary transformation

$$UMU^\dagger = D = \mathrm{diag}\,(m_1,\, m_2,\, \ldots,\, m_n)$$

where the eigenvalues m_i can be negative as well as positive. Show that one can choose an appropriate bi-unitary transformation $UMV^\dagger$ to diagonalize M so that all diagonal elements are non-negative.

b) Show that for any complex $n \times n$ matrix A one can find two unitary transformation matrices U and V such that $UAV^\dagger$ is diagonal with non-negative elements.

c) Now we know that the mass matrices for the SM fermions can be diagonalized by bi-unitary transformations with non-negative mass eigenvalues. The diagonalization in the charged lepton sector leads to a mixing for the charged leptons

$$e_L = U^\dagger e'_L \qquad\qquad e_R = V^\dagger e'_R$$

where the primed (unprimed) fields denote the fields in the flavour (mass) basis. Show that in the case of massless neutrinos this mixing for the charged leptons would have no physical effect.

Problem 14: *Neutrino oscillations in matter* [**15 Points**]

The MSW effect describes oscillations of neutrinos in matter. We consider the case of two-flavour oscillations in a realistic (non-constant) mass distribution. The time evolution of the eigenfunctions of the Hamiltonian in matter is described by the following expression:

$$i\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix}|\nu_\mathrm{A}\rangle\\ |\nu_\mathrm{B}\rangle\end{pmatrix} = \begin{pmatrix}E_\mathrm{A}(t) & -i\dot\theta(t)\\ i\dot\theta(t) & E_\mathrm{B}(t)\end{pmatrix}\begin{pmatrix}|\nu_\mathrm{A}\rangle\\ |\nu_\mathrm{B}\rangle\end{pmatrix}$$

The *instantaneous* mass eigenstates in matter are $|\nu\rangle = (|\nu_\mathrm{A}\rangle,\ |\nu_\mathrm{B}\rangle)^\mathrm{T}$ and to any given time they are connected to the flavor states via $|\tilde\nu\rangle = \tilde U(t)|\nu\rangle$ with

$$|\tilde\nu\rangle = \begin{pmatrix}|\tilde\nu_e\rangle\\ |\tilde\nu_\mu\rangle\end{pmatrix} \qquad\qquad \tilde U(t) = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

and the *time-dependent* angle θ is defined via

$$\tan(2\theta) = \frac{\frac{\Delta m^2}{2E}\sin(2\theta_0)}{\frac{\Delta m^2}{2E}\cos(2\theta_0) - \sqrt{2}G_\mathrm{F}N_e(t)}, \tag{1}$$

where θ_0 is constant in time, and $N_e(t)$ denotes the electron number density in matter.
The effective Hamiltonian is not diagonal, since matter eigenstates and thus the mixing angle θ are not constant. If, however, $\dot\theta$ is small, i.e. the change in θ occurs slowly, we obtain the adiabatic approximation, where we can neglect $\dot\theta$. Suppose that at the time $t = t_i$ an electron neutrino is produced,

$$|\tilde\nu_\alpha(t_i)\rangle \equiv |\tilde\nu_e\rangle = \cos\theta_i\,|\nu_\mathrm{A}(t_i)\rangle + \sin\theta_i\,|\nu_\mathrm{B}(t_i)\rangle.$$

a) What is the expression for $|\tilde{\nu}_\alpha(t_f)\rangle$ in terms of $|\nu_\mathrm{A}(t_i)\rangle$ and $|\nu_\mathrm{B}(t_i)\rangle$ at some later time t_f in the adiabatic approximation?

b) Show that in the adiabatic approximation the result for the transition probability is given by

$$P(\nu_e \to \nu_\mu) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta_i)\cos(2\theta_f) - \tfrac{1}{2}\sin(2\theta_i)\sin(2\theta_f)\cos\Phi_\mathrm{AB}$$

with

$$\Phi_\mathrm{IJ} \equiv \int_{t_i}^{t_f} \left[E_\mathrm{I}(t) - E_\mathrm{J}(t)\right] \mathrm{dt},$$

where I, J $\in \{\mathrm{A},\ \mathrm{B}\}$.

c) Use Eq. (1) to find an expression for $\sin^2(2\theta)$. The formula for $\sin^2(2\theta)$ describes a Breit-Wigner distribution. We define the resonance width at half height, Δr, as the width at which the curve of the distribution is larger than a half [$\sin^2(2\theta) > 1/2$]. Find an approximate expression for Δr.

d) The adiabatic parameter at the MSW resonance is

$$\gamma_r \equiv \frac{\sin^2(2\theta_0)\Delta m^2}{\cos(2\theta_0)2E} L \quad .$$

The scale $L = |\dot{N}_e/N_e|^{-1}_\mathrm{res}$ is the characteristic length scale where the electron density varies in the resonance region. Express γ_r in terms of Δr and the oscillation length given by

$$l_\mathrm{res} = \frac{2\pi}{|E_\mathrm{A} - E_\mathrm{B}|_\mathrm{res}} = \frac{4\pi E/\Delta m^2}{\sin(2\theta_0)}.$$

Discuss the adiabatic condition $\gamma_r \gtrsim 3$.

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Hand-in and discussion of sheet:
Wednesday, 14:15, Phil12, R106