Exercises to “Standard Model of Particle Physics II”

Winter 2014/15

Dr. Werner Rodejohann — Sheet 7 — 10.12.14

Exercise 16: Neutrino oscillations I [**10 Points**]

The time evolution of the neutrino wave function in the mass basis is described by the expression

$$|\nu_\alpha(t)\rangle = U_{\alpha i} e^{-\mathrm{i}E_i t}|\nu_i\rangle,$$

where Greek (Latin) indices denote the neutrino fields in the flavour (mass) basis.
Since flavour and mass basis are distinct the oscillation probability is given by the square of the transition amplitude

$$\mathscr{A}(\nu_\alpha \to \nu_\beta;\ t) = \langle \nu_\beta|\nu_\alpha(t)\rangle.$$

a) Calculate the general expression for the oscillation probability and show that it can be expressed as

$$P(\nu_\alpha \to \nu_\beta;\ t) = \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2 + \sum_{i\neq j} U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}\ \ \mathrm{e}^{-\mathrm{i}(E_i - E_j)t}.$$

b) Consider neutrinos that move relativisticly and use $E_i \simeq p + \frac{m_i^2}{2p}$ as approximation for their energy. Note that in this approximation one assumes that the different mass states all have the same momentum. On the other hand one could assume that the different mass states have the same energy. Use $p_i \simeq E - \frac{m_i^2}{2E}$ and describe the spacial propagation of each mass state with a phase factor $e^{\mathrm{i}p_i x}$. Compare the oscillation probability for both approximations. Are the assumptions above justified?

c) Now we consider the case of two neutrino flavours, where the mixing matrix is parameterized as ($c \equiv \cos\theta_0,\ s \equiv \sin\theta_0$)

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.$$

Take $E_i \simeq p + \frac{m_i^2}{2p}$ as approximation for the neutrino energy and calculate $P(\nu_e \to \nu_\mu;\ t)$.
As result you should obtain:

$$P(\nu_e \to \nu_\mu;\ t) = \sin^2(2\theta_0)\sin^2\left(\frac{\Delta m^2}{4E}t\right) \quad \text{with} \quad \Delta m^2 \equiv m_2^2 - m_1^2.$$

Exercise 17: Neutrino oscillations II [**10 Points**]

Consider the case of three neutrino flavours. The mixing matrix then reads

$$\begin{pmatrix} \nu_{e\mathrm{L}} \\ \nu_{\mu\mathrm{L}} \\ \nu_{\tau\mathrm{L}} \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_{1\mathrm{L}} \\ \nu_{2\mathrm{L}} \\ \nu_{3\mathrm{L}} \end{pmatrix}$$

The oscillation has a CP-violating fraction $\Delta P_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta;\ t) - P(\overline{\nu}_\alpha \to \overline{\nu}_\beta;\ t)$.

a) Calculate the CP-violating fraction $\Delta P_{\alpha\beta}$. Which result do you obtain for $\Delta P_{\alpha\alpha}$ and what does it imply?

b) Use the general expression for the transition probability [Exercise 16, part a)] and show that

$$\Delta P_{\alpha\beta} = 4 \sum_{j>k} \mathrm{Im}\Big\{ U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} \Big\} \sin(\Delta_{jk} t) \quad \text{with} \quad \Delta_{jk} \equiv \frac{m_j^2 - m_k^2}{2E}.$$

c) Use the unitarity of U to show that $\mathrm{Im}\{U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k}\}$ is independent (up to a sign) of the choice of α and β $(\alpha \neq \beta)$.

   Note: The same invariant can be constructed for the CKM matrix and is a measure for CP-violation in the meson sector.

d) Discuss your results and the influence of the mixing parameters on CP-violation.

Enjoy your Holidays !!!

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Tutorials homepage: http://www.mpi-hd.mpg.de/manitop/StandardModel2/exercise.html

Hand-in and discussion of sheet:
during tutorial on Thursday, 18.12.14, 9.15 am, kHs, Philosophenweg 12