Exercises to “Standard Model of Particle Physics II”

Winter 2019/20

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

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Problem 12: *The Rho-parameter* [**10 Points**]

We encountered the Rho-parameter defined by

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2(\theta_W)}$$

when discussing the Higgs mechanism induced by a Higgs triplet with hypercharge 1 and found it to be 1/2 instead of 1 as predicted by the SM. Now we generalize our findings while keeping the above definition.

a) From representation theory we know that in a n-dimensional representation (i.e. n-plet, or spin-$(n-1)/2$ representation), an eigenbasis to the generators t^3 of SU(2) can be found, such that t_3 acts like $t^3 = \mathrm{diag}(j, j-1, \ldots, -j)$, where $j = (n-1)/2$. Show that if we expand a Higgs n-plet Φ with a hypercharge of y in this basis its components will be eigenstates of $Q = t^3 + \hat{Y}$, where $\hat{Y}\Phi = y\mathbb{1}\Phi = y\Phi$. What is the condition on y to ensure that there is a neutral component?

b) In this basis the raising and lowering operators defined from $t^{\pm} = (t^1 \pm i t^2)$ can be descibed by their action on the orthonormal basis vectors e_m (labelled by their t^3 eigenvalues m):

$$t^+ e_m = \sqrt{j(j+1) - m(m+1)}\, e_{m+1}\,, \qquad t^- e_m = \sqrt{j(j+1) - m(m-1)}\, e_{m-1}\,.$$

Assume that the hypercharge is such that ϕ_m is neutral and that only this neutral component of Φ acquires a vev, $\vec{v} = v e_m$. Recall that the covariant derivative reads $D_\mu \Phi = (\partial_\mu - i g t^i W_\mu^i - i g' \hat{Y} B_\mu)\Phi$, and, in analogy to exercise 8 on sheet 4, show that

$$m_W^2 = \frac{g^2}{2} \vec{v}^\dagger (t^+ t^- + t^- t^+) \vec{v}\,.$$

c) Show that this leads to

$$m_W^2 = g^2 v^2 (j(j+1) - m^2)\,.$$

d) With the standard mixing angle $\tan\theta_W = g'/g$, derive the mass of Z and use it to show that

$$\rho = \frac{v^2(j(j+1) - y^2)}{2v^2 y^2}\,.$$

e) Check that the Higgs doublet with hypercharge 1 satisfies the condition that $\rho = 1$. What would be the next combination of weak isospin and (rational) hypercharge that can satisify this bound?

Problem 13: *W-polarisation* [**5 Points**]

For a massive vector boson with four-momentum $k^\mu = (E,\, |k|\vec{n})$ propagation along the direction $\vec{n} = (\sin\theta,\, 0,\, \cos\theta)$, the polarisation vectors corresponding to the helicities $\lambda = 0, \pm 1$ can be written as

$$\epsilon^\mu_{\lambda=0} = m_W^{-1}\left(|k|,\, E\sin\theta,\, 0,\, E\cos\theta\right),$$
$$\epsilon^\mu_{\lambda=\pm 1} = \frac{1}{\sqrt{2}}\left(0,\, \mp\cos\theta,\, -i,\, \pm\sin\theta\right).$$

Check that the completeness relation holds, i.e. verify that

$$\sum_\lambda \epsilon^{\mu\,*}_\lambda \epsilon^\nu_\lambda = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m_W^2}.$$

Problem 14: *The invisible decay rate of the Z-boson* [**5 Points**]

Through its gauge coupling to fermions, the Z-boson decays into hadrons, charged leptons and neutrinos. In general, the decay rate of a particle A into final states $\{f\}$ is given by

$$\Gamma(A \to \{f\}) = \frac{1}{2m_A}\int \prod_{\{f\}} \frac{d^3 p_f}{(2\pi)^3\, 2E_f}(2\pi)^4 \delta^{(4)}(p_A - \{p_f\})|\mathcal{M}|^2, \qquad (1)$$

where p_f and E_f stand for the momentum and energy of each of the final states.
Compute the decay rate of Z-bosons to electron-neutrinos, i.e. $\Gamma(Z \to \bar{\nu}_e \nu_e)$, assuming that neutrinos are massless and going through the following steps:

a) Show that the magnitude squared of amplitude is given by

$$|\mathcal{M}|^2 = \frac{1}{3}\cdot\frac{g^2}{4\,\cos^2(\theta_w)}\left(6|\vec{p}_1||\vec{p}_2| - 2\,\vec{p}_1\cdot\vec{p}_2\right), \qquad (2)$$

where $\vec{p}_{1,2}$ stands for the momentum of the final neutrinos, g is the gauge coupling and θ_w is the Weinberg angle.

Hint 1: the relevant electroweak vertex is

$$\frac{-i}{2}\,\frac{g}{\cos(\theta_w)}\,\gamma_\mu\,(1-\gamma^5).$$

In the amplitude $\mathcal{M}$, the γ_μ from the vertex is contracted with the polarisation vector of the Z-boson, $\varepsilon^\mu_b(p_0)$.
Hint 2: in $|\mathcal{M}|^2$ the averaging over the polarisations of the Z-boson and summing over neutrino spins is implied.
Hint 3: for the sum over the polarisations of the Z-boson, use the relation from problem 13. Use similar relations for the sum of spinor products.

b) Insert the magnitude squared of the amplitude (2) into (1) and integrate using the delta functions and the rest frame of the Z-boson.