Exercises to "Standard Model of Particle Physics II"

Winter 2018/19

Prof. Dr. Manfred Lindner and Dr. Werner Rodejohann   Sheet 6   November 21, 2018

Problem 12: *Fermion mass matrix diagonalization* [**10 Points**]

a) A hermitian $n \times n$ matrix $M = M^\dagger$ can always be diagonalized by a unitary transformation

$$UMU^\dagger = D = \text{diag}\,(m_1,\, m_2,\, \ldots,\, m_n)$$

where the eigenvalues m_i can be negative as well as positive. Show that one can choose an appropriate bi-unitary transformation $UMV^\dagger$ to diagonalize M so that all diagonal elements are non-negative.

b) Show that for any complex $n \times n$ matrix A one can find two unitary transformation matrices U and V such that $UAV^\dagger$ is diagonal with non-negative elements.

c) Now we know that the mass matrices for the SM fermions can be diagonalized by bi-unitary transformations with non-negative mass eigenvalues. The diagonalization in the charged lepton sector leads to a mixing for the charged leptons

$$e_L = U^\dagger e'_L\,, \qquad e_R = V^\dagger e'_R\,.$$

where the primed (unprimed) fields denote the fields in the flavour (mass) basis. Note that since the interactions with W and Z are diagonal in flavor basis, they would mix different mass eigenstates. Show that in the case of massless neutrinos this mixing for the charged leptons would have no physical effect.

d) In the case of quarks, both the up-type and the down-type quarks have a mass matrix, such that diagonalization leads to

$$u_L = U_u^\dagger u'_L\,, \qquad u_R = V_u^\dagger u'_R\,,$$

$$d_L = U_d^\dagger d'_L\,, \qquad d_R = V_d^\dagger d'_R\,.$$

Show that in this case the mixing of quarks *does* leave a physical effect in charged-current (coupling to W interactions). Give, in terms of the U and V matrices above, the 3-by-3 matrix which describes the coupling between u, c, and t quarks in their mass basis with respect to d, s and b quarks in their mass basis in charged-current interactions. This is the famous Cabibbo-Kobayashi-Maskawa (CKM) matrix.

Problem 13: *The Rho-parameter* [**10 Points**]

We encountered the Rho-parameter defined by

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2(\theta_W)}$$

when discussing the Higgs mechanism induced by a Higgs triplet with hypercharge 1 and found it to be 1/2 instead of 1 as predicted by the SM. Now we generalize our findings while keeping the above definition.

a) From representation theory we know that in a n-dimensional representation (i.e. n-plet, or spin-$(n-1)/2$ representation), an eigenbasis to the generators t^3 of SU(2) can be found, such that t_3 acts like $t^3 = \mathrm{diag}(j, j-1, \ldots, -j)$, where $j = (n-1)/2$. Show that if we expand a Higgs n-plet Φ with a hypercharge of y in this basis its components will be eigenstates of $Q = t^3 + \hat{Y}$, where $\hat{Y}\Phi = y\mathbb{1}\Phi = y\Phi$. What is the condition on y to ensure that there is a neutral component?

b) In this basis the raising and lowering operators defined from $t^\pm = (t^1 \pm it^2)$ can be descibed by their action on the orthonormal basis vectors e_m (labelled by their t^3 eigenvalues m):

$$t^+ e_m = \sqrt{j(j+1) - m(m+1)}\, e_{m+1}\,, \qquad t^- e_m = \sqrt{j(j+1) - m(m-1)}\, e_{m-1}\,.$$

Assume that the hypercharge is such that ϕ_m is neutral and that only this neutral component of Φ acquires a vev, $\vec{v} = v e_m$. Recall that the covariant derivative reads $D_\mu\Phi = (\partial_\mu - igt^i W_\mu^i - ig'\hat{Y}B_\mu)\Phi$, and, in analogy to exercise 8 on sheet 4, show that

$$m_W^2 = \frac{g^2}{2}\vec{v}^\dagger(t^+t^- + t^-t^+)\vec{v}\,.$$

c) Show that this leads to

$$m_W^2 = g^2 v^2 (j(j+1) - m^2)\,.$$

d) With the standard mixing angle $\tan\theta_W = g'/g$, derive the mass of Z and use it to show that

$$\rho = \frac{v^2(j(j+1) - y^2)}{2v^2y^2}\,.$$

e) Check that the Higgs doublet with hypercharge 1/2 satisfies the condition that $\rho = 1$. What would be the next combination of weak isospin and (rational) hypercharge that can satisify this bound?

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in and discussion of sheet:
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