

Exercises to “Standard Model of Particle Physics II”

Winter 2014/15

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Sheet 6

26.11.14

Exercise 14: Optical Theorem [10 Points]

The scattering of particles $a + b \rightarrow c + d$ is described by the amplitude

$$f(\theta) = \frac{1}{2ki} \sum_l (2l+1)(\eta_l e^{2i\delta_l} - 1) P_l(\cos \theta),$$

where P_l are the Legendre-polynomials, θ is the scattering angle, and δ_l and η_l are both real functions. δ_l denotes the phase difference and η_l was introduced to describe inelastic scattering. We have $\eta_l = 1$ for elastic and $\eta_l < 1$ for inelastic scattering.

The optical theorem states that the cross section in a forward scattering process is given by

$$\sigma_{\text{tot}} = \frac{4\pi}{k} \text{Im}\{f(0)\}.$$

a) Show with the help of the optical theorem that

$$\sigma_{\text{tot}} = \frac{2\pi}{k^2} \sum_l (2l+1)(1 - \eta_l \cos(2\delta_l)).$$

b) The differential cross section for elastic scattering is given by

$$\frac{d\sigma_{\text{el}}}{d\Omega} = |f(\theta)|^2$$

From this derive the following expression for the elastic scattering cross section

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_l (2l+1) |\eta_l e^{2i\delta_l} - 1|^2.$$

c) From a) and b) it follows that

$$\sigma_{\text{inel}} = \frac{\pi}{k^2} \sum_l (2l+1)(1 - \eta_l^2).$$

Show with this equation that for the reaction $\nu_\mu + e^- \rightarrow \mu^- + \nu_e$ we obtain the relation

$$\sigma(\nu_\mu + e^- \rightarrow \mu^- + \nu_e) \leq \frac{\pi}{2E_{\text{cm}}^2}, \quad (1)$$

where E_{cm} denotes the center-of-mass energy. Note that this is an $l = 0$ scattering process and that a spin factor $(2s + 1)$ should be taken into account.

d) In Fermi theory the cross section is given by

$$\sigma = \frac{G_F^2 s}{\pi}, \quad (2)$$

where G_F is Fermi's constant and \sqrt{s} denotes the invariant mass.

Use Eqs. (1) and (2) to find the energy at which Fermi theory breaks down.

Exercise 15: Vacuum energy [10 Points]

The energy-momentum tensor of a massive scalar field is given by

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}g_{\mu\nu}(\partial_\lambda\phi)(\partial^\lambda\phi) + \frac{1}{2}g_{\mu\nu}m^2\phi^2,$$

with the metric convention (+, -, -, -). We are interested in the vacuum expectation value of this quantity: $\langle T_{\mu\nu} \rangle \equiv \langle 0|T_{\mu\nu}|0 \rangle$ with the energy density $\rho = \langle T_{00} \rangle$ and the pressure $p = \langle T_{ii} \rangle$. In the operator representation the field ϕ reads

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} [\hat{a}(\vec{k})e^{-ik_\mu x^\mu} + \hat{a}^\dagger(\vec{k})e^{ik_\mu x^\mu}].$$

We assume that the vacuum is constant over time and empty (free of particles), which means that $\hat{a}(\vec{k})$ annihilates the vacuum state $|0\rangle$. The commutator of annihilation and creation operator is given by $[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = (2\pi)^3 2E_k \delta^{(3)}(\vec{k} - \vec{k}')$.

- a) Calculate the energy density and the pressure of the vacuum (as defined above), where you can use some cut-off scale Λ as regulator for the divergent momentum-integrals. Order the terms of your result by powers in Λ , where you can use the condition $\Lambda^2 \gg m^2$.
- b) In a Friedmann cosmology the local conservation of energy and momentum is given by the following formula (where a denotes the scale factor of the universe)

$$\partial_\mu T^{\mu 0} = \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p).$$

Investigate for each power in Λ whether the conservation of energy and momentum of the vacuum is fulfilled or not.

- c) Insert your findings for energy density and pressure of part a) into the second Friedmann equation:

$$3\frac{\ddot{a}}{a} = -4\pi G_N(\rho + 3p).$$

What does your result mean for the dynamics of the universe?

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Tutorials homepage: <http://www.mpi-hd.mpg.de/manitop/StandardModel2/exercise.html>

Hand-in and discussion of sheet:

during tutorial on Thursday, 04.12.14, 9.15 am, kHs, Philosophenweg 12