Exercises to "Standard Model of Particle Physics II"

Winter 2014/15

Dr. Werner Rodejohann — Sheet 6 — 26.11.14

<u>Exercise 14:</u> Optical Theorem [**10 Points**]

The scattering of particles $a + b \to c + d$ is described by the amplitude

$$f(\theta) = \frac{1}{2k\mathrm{i}} \sum_l (2l+1)(\eta_l \mathrm{e}^{2\mathrm{i}\delta_l} - 1) P_l(\cos\theta),$$

where P_l are the Legendre-polynomials, θ is the scattering angle, and δ_l and η_l are both real functions. δ_l denotes the phase difference and η_l was introduced to describe inelastic scattering. We have $\eta_l = 1$ for elastic and $\eta_l < 1$ for inelastic scattering.
The optical theorem states that the cross section in a forward scattering process is given by

$$\sigma_{\text{tot}} = \frac{4\pi}{k} \mathrm{Im}\{f(0)\}.$$

a) Show with the help of the optical theorem that

$$\sigma_{\text{tot}} = \frac{2\pi}{k^2} \sum_l (2l+1)(1 - \eta_l \cos(2\delta_l)).$$

b) The differential cross section for elastic scattering is given by

$$\frac{d\sigma_{\text{el}}}{d\Omega} = |f(\theta)|^2$$

From this derive the following expression for the elastic scattering cross section

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_l (2l+1) |\eta_l \mathrm{e}^{2\mathrm{i}\delta_l} - 1|^2.$$

c) From a) and b) it follows that

$$\sigma_{\text{inel}} = \frac{\pi}{k^2} \sum_l (2l+1)(1 - \eta_l^2).$$

Show with this equation that for the reaction $\nu_\mu + e^- \to \mu^- + \nu_e$ we obtain the relation

$$\sigma(\nu_\mu + e^- \to \mu^- + \nu_e) \leq \frac{\pi}{2E_{\text{cm}}^2}, \tag{1}$$

where E_{cm} denotes the center-of-mass energy. Note that this is an $l = 0$ scattering process and that a spin factor $(2s+1)$ should be taken into account.

d) In Fermi theory the cross section is given by

$$\sigma = \frac{G_{\text{F}}^2 s}{\pi}, \tag{2}$$

where G_{F} is Fermi's constant and $\sqrt{s}$ denotes the invariant mass.

Use Eqs. (1) and (2) to find the energy at which Fermi theory breaks down.

Exercise 15: Vacuum energy [**10 Points**]

The energy-momentum tensor of a massive scalar field is given by

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}g_{\mu\nu}(\partial_\lambda\phi)(\partial^\lambda\phi) + \frac{1}{2}g_{\mu\nu}m^2\phi^2,$$

with the metric convention (+, -, -, -). We are interested in the vacuum expectation value of this quantity: $\langle T_{\mu\nu}\rangle \equiv \langle 0|T_{\mu\nu}|0\rangle$ with the energy density $\rho = \langle T_{00}\rangle$ and the pressure $p = \langle T_{ii}\rangle$. In the operator representation the field ϕ reads

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3 2E_k}\left[\hat{a}(\vec{k})e^{-ik_\mu x^\mu} + \hat{a}^\dagger(\vec{k})e^{ik_\mu x^\mu}\right].$$

We assume that the vacuum is constant over time and empty (free of particles), which means that $\hat{a}(\vec{k})$ annihilates the vacuum state $|0\rangle$. The commutator of annihilation and creation operator is given by $[\hat{a}(\vec{k}),\ \hat{a}^\dagger(\vec{k}')] = (2\pi)^3 2E_k\delta^{(3)}(\vec{k}-\vec{k}')$.

a) Calculate the energy density and the pressure of the vacuum (as defined above), where you can use some cut-off scale Λ as regulator for the divergent momentum-integrals. Order the terms of your result by powers in Λ, where you can use the condition $\Lambda^2 \gg m^2$.

b) In a Friedmann cosmology the local conservation of energy and momentum is given by the following formula (where a denotes the scale factor of the universe)

$$\partial_\mu T^{\mu 0} = \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p).$$

Investigate for each power in Λ whether the conservation of energy and momentum of the vacuum is fulfilled or not.

c) Insert your findings for energy density and pressure of part a) into the second Friedmann equation:

$$3\frac{\ddot{a}}{a} = -4\pi G_{\mathrm{N}}(\rho + 3p).$$

What does your result mean for the dynamics of the universe?

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Tutorials homepage: `http://www.mpi-hd.mpg.de/manitop/StandardModel2/exercise.html`

Hand-in and discussion of sheet:
during tutorial on Thursday, 04.12.14, 9.15 am, kHs, Philosophenweg 12