Exercises to "Standard Model of Particle Physics II"

Winter 2020/21

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Sheet 03 - November 18, 2020

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in of solutions:
November 25, 2020 - via e-mail, **before 14:00**

Discussion of solutions:
November 25, 2020 - on zoom

Problem 6: *Goldstone's theorem* [**15 Points**]

Goldstone's theorem states:

> *Every generator of a continuous global symmetry, which does not annihilate the vacuum, corresponds to a massless scalar particle.*

To demonstrate this, consider the Lagrangian $\mathscr{L}(\phi_i, \partial_\mu \phi_i)$ with n real scalar fields ϕ_i, invariant under the global transformation $\phi_i \to \{\exp(i\theta^a T^a)\}_{ij}\, \phi_j$. Defining $\Phi \equiv (\phi_1, \ldots, \phi_n)^T$, we can write this transformation as $\Phi \to \exp(i\theta^a T^a)\, \Phi$.

a) What are the properties of the iT^a, if the ϕ_i are real, and if $\Phi^T \Phi$ is invariant under the global transformation?

b) Show that from the conservation of the Noether current, it follows that:

$$\frac{\partial \mathscr{L}}{\partial \phi_i}\, (iT^a)_{ij}\, \phi_j + \frac{\partial \mathscr{L}}{\partial(\partial^\mu \phi_i)}\, (iT^a)_{ij}\, \partial^\mu \phi_j = 0\,.$$

c) Let $\mathscr{L} = \frac{1}{2}(\partial_\mu \Phi)^T\, (\partial^\mu \Phi) - V(\Phi)$. Show with the help of the first two results that

$$\frac{\partial V}{\partial \phi_i}\, (iT^a)_{ij}\, \phi_j = 0\,.$$

d) Let $\vec{v}$ be the minimum of $V(\Phi)$. Show with the results from above that $M^2\, (iT^a)\, \vec{v} = 0$, where M^2_{ij} is given by

$$(M^2)_{ij} \equiv \left. \frac{\partial^2 V}{\partial \phi_i\, \partial \phi_j} \right|_{\Phi = \vec{v}}\,.$$

e) The matrix M^2 is interpreted as a mass matrix. The broken vacuum $\vec{v}$ is in general not invariant under the transformations $\exp(i\theta^a T^a)$. If $T^a \vec{v} = 0$, one has a *Wigner–Weyl realization* of the symmetry, if $T^a \vec{v} \neq 0$ one has a realization à la *Nambu–Goldstone*. With the result from d) it follows that for all a with $T^a \vec{v} \neq 0$ the mass matrix has an eigenvector with eigenvalue 0.

   As an explicit example, consider the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \Phi)^T (\partial^\mu \Phi) - V(\Phi^T \Phi)$$

   with the potential

$$V(\Phi^T \Phi) = \frac{1}{2}\mu^2 \Phi^T \Phi + \frac{1}{4}\lambda(\Phi^T \Phi)^2\,,$$

   where $\Phi = (\phi_1, \phi_2, \phi_3)^T$ is a triplet of real and scalar particles. The coupling obeys $\lambda > 0$.

(i) Show that $\mathscr{L}$ is invariant under an SU(2) transformation $\Phi \to \exp(i\theta^a T^a)\,\Phi$. The three generators T^a of SU(2) are written here as $(T^a)_{ij} = -i\epsilon_{aij}$ (this is called the adjoint representation of SU(2)).

(ii) For $\mu^2 < 0$ the potential has a minimum $\vec{v}^T = (0, 0, v)$ with $v \neq 0$. Derive a relation between v, μ^2, and λ. Show that the vacuum state $\vec{v}$ is invariant under a transformation generated by T^3, but not T^1 and T^2. That is, the vacuum state does *not* possess the symmetry of the Lagrangian (spontaneous symmetry breaking). Why do we speak in this case of spontaneous symmetry breaking from SU(2) to U(1)?

(iii) The scalar fields will be expanded around the minimum, $\Phi^T \equiv (\phi_1, \phi_2, v + \phi_3)$, i.e. we treat ϕ_3 as a field fluctuating around v instead of 0. Show by inserting into the above Lagrangian that there is one massive scalar field and two massless Goldstone bosons. What is the mass of ϕ_3? Convince yourself that the matrix M^2 defined in d) is really the mass matrix.

Problem 7: *Noether's theorem revised* [5 **Points**]

Consider a Lagrange density of the general form $\mathscr{L} = \mathscr{L}_0(u, \partial_\mu u)$. From Noether's theorem we know that if the Lagrange density is left invariant by a symmetry transformation this leads to the conserved current

$$J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu u)} \delta u \,.$$

Now imagine that we add a term $\mathscr{L}_1$ to the Lagrange density (so that $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1$) that does not respect the symmetry. Show that in this case the new current is not conserved any more, that is,

$$\partial_\mu J^\mu \neq 0 \,.$$