

Exercises to “Standard Model of Particle Physics II”

Winter 2018/19

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Sheet 3

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Problem 6: Goldstone’s theorem [15 Points]

Goldstone’s theorem states:

Every generator of a continuous global symmetry, which does not annihilate the vacuum, corresponds to a massless scalar particle.

To demonstrate this, consider the Lagrangian $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ with n real scalar fields ϕ_i , invariant under the global transformation $\phi_i \rightarrow \{\exp(i\theta^a T^a)\}_{ij} \phi_j$. Defining $\Phi \equiv (\phi_1, \dots, \phi_n)^T$, we can write this transformation as $\Phi \rightarrow \exp(i\theta^a T^a) \Phi$.

- What are the properties of the iT^a , if the ϕ_i are real, and if $\Phi^T \Phi$ is invariant under the global transformation?
- Show that from the conservation of the Noether current, it follows that:

$$\frac{\partial \mathcal{L}}{\partial \phi_i} (iT^a)_{ij} \phi_j + \frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi_i)} (iT^a)_{ij} \partial^\mu \phi_j = 0.$$

- Let $\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^T (\partial^\mu \Phi) - V(\Phi)$. Show with the help of the first two results that

$$\frac{\partial V}{\partial \phi_i} (iT^a)_{ij} \phi_j = 0.$$

- Let \vec{v} be the minimum of $V(\Phi)$. Show with the results from above that $M^2 (iT^a) \vec{v} = 0$, where M_{ij}^2 is given by

$$(M^2)_{ij} \equiv \left. \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \right|_{\Phi=\vec{v}}.$$

- The matrix M^2 is interpreted as a mass matrix. The broken vacuum \vec{v} is in general not invariant under the transformations $\exp(i\theta^a T^a)$. If $T^a \vec{v} = 0$, one has a *Wigner–Weyl realization* of the symmetry, if $T^a \vec{v} \neq 0$ one has a realization à la *Nambu–Goldstone*. With the result from d) it follows that for all a with $T^a \vec{v} \neq 0$ the mass matrix has an eigenvector with eigenvalue 0.

As an explicit example, consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^T (\partial^\mu \Phi) - V(\Phi^T \Phi)$$

with the potential

$$V(\Phi^T \Phi) = \frac{1}{2} \mu^2 \Phi^T \Phi + \frac{1}{4} \lambda (\Phi^T \Phi)^2,$$

where $\Phi = (\phi_1, \phi_2, \phi_3)^T$ is a triplet of real and scalar particles. The coupling obeys $\lambda > 0$.

- Show that \mathcal{L} is invariant under an SU(2) transformation $\Phi \rightarrow \exp(i\theta^a T^a) \Phi$. The three generators T^a of SU(2) are written here as $(T^a)_{ij} = -i\epsilon_{aij}$ (this is called the adjoint representation of SU(2)).

- (ii) For $\mu^2 < 0$ the potential has a minimum $\vec{v}^T = (0, 0, v)$ with $v \neq 0$. Derive a relation between v , μ^2 , and λ . Show that the vacuum state \vec{v} is invariant under a transformation generated by T^3 , but not T^1 and T^2 . That is, the vacuum state does *not* possess the symmetry of the Lagrangian (spontaneous symmetry breaking). Why do we speak in this case of spontaneous symmetry breaking from $SU(2)$ to $U(1)$?
- (iii) The scalar fields will be expanded around the minimum, $\Phi^T \equiv (\phi_1, \phi_2, v + \phi_3)$, i.e. we treat ϕ_3 as a field fluctuating around v instead of 0. Show by inserting into the above Lagrangian that there is one massive scalar field and two massless Goldstone bosons. What is the mass of ϕ_3 ? Convince yourself that the matrix M^2 defined in d) is really the mass matrix.

Problem 7: Noether's theorem revised [5 Points]

Consider a Lagrange density of the general form $\mathcal{L} = \mathcal{L}_0(u, \partial_\mu u)$. From Noether's theorem we know that if the Lagrange density is left invariant by a symmetry transformation this leads to the conserved current

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)} \delta u.$$

Now imagine that we add a term \mathcal{L}_1 to the Lagrange density (so that $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$) that does not respect the symmetry. Show that in this case the new current is not conserved any more, that is,

$$\partial_\mu J^\mu \neq 0.$$

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Lecture webpage: <https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html>

Hand-in and discussion of sheet:

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