Exercises to "Standard Model of Particle Physics II"

Winter 2020/21

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Sheet 02 - November 11, 2020

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in of solutions:
November 18, 2020 - via e-mail, **before 14:00**

Discussion of solutions:
November 18, 2020 - on zoom

Problem 4: $SU(N)$ **[10 Points]**

Let $U \in \mathrm{SU}(N)$, i.e. $\det U = 1$ and $U^\dagger U = \mathbb{1}$. Any element of $\mathrm{SU}(N)$ can be written as $U = \exp(-i\theta^a T^a)$, where the T^a are generators of the group with normalization $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

a) Show that the T^a are traceless hermitian matrices.

b) How many linear independent generators are there?

c) The structure constants, d_{abc} and f_{abc}, are defined through

$$[T^a, T^b] = i f_{abc} T^c \,, \qquad \{T^a, T^b\} = \frac{1}{N}\delta_{ab} + d_{abc} T^c \,.$$

Show that

$$\mathrm{Tr}(T^a T^b T^c) = \frac{1}{4}(d_{abc} + i f_{abc}) \,,$$
$$\left[\sum_a T^a T^a, T^b\right] = 0 \,,$$
$$\Big[T^a, [T^b, T^c]\Big] + \Big[T^c, [T^a, T^b]\Big] + \Big[T^b, [T^c, T^a]\Big] = 0 \,.$$

d) Show that the structure constants form a representation of $\mathrm{SU}(N)$, i.e. take $(T^a)_{bc} = -i f_{abc}$ as a generator. This is the so-called adjoint representation.

e) Calculate the f_{abc} for

$$T^a = \frac{\sigma^a}{2} \,,$$

where a runs from 1 to 3, and σ^a are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Problem 5: *From QED to QCD* [**10 points**]

In **Problem 2** we showed that in QED $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is *invariant* under the U(1) gauge transformation $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha(x)$. For QCD, this gauge transformation is generalised to $A_\mu \to U(x)\left(A_\mu + \frac{i}{g}\partial_\mu\right)U(x)^\dagger$, while the quarks obey $\psi \to U(x)\psi$, where ψ carries one SU(N) index, $A_\mu \equiv A_\mu^a T^a$ is now matrix-valued, and $U(x) \in \mathrm{SU}(N)$.

a) Using the U(1) covariant derivative $D_\mu = \partial_\mu - ieA_\mu$, show that

$$[D_\mu, D_\nu]\,\psi = -ie\,F_{\mu\nu}\,\psi\,,$$

where ψ is the electron field.

b) For a set of parameters α^a, the transformation matrix of QCD is $U(x)_{ij} = \exp(i\alpha^a T^a)_{ij}$. Using the infinitesimal version of the gauge transformation of the gluon field, $A_\mu \to A_\mu + \frac{1}{g}(\partial_\mu\alpha^a)T^a + i\left[\alpha^a T^a, A_\mu^b T^b\right]$, show that

$$D_\mu\psi \to (1 + i\alpha^a T^a)D_\mu\psi\,,$$

with the QCD covariant derivative $D_\mu = \partial_\mu - igA_\mu^a T^a$.

c) In analogy to QED, we can define the QCD field strength matrix $F_{\mu\nu} = F_{\mu\nu}^a T^a$ via $[D_\mu, D_\nu]\,\psi = -igF_{\mu\nu}^a T^a\psi$, which is no longer invariant. Compute $F_{\mu\nu}^a$.

d) Show that the QCD Lagrangian is gauge-invariant.

$$\mathcal{L}_{\mathrm{QCD}} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}^a F^{a\,\mu\nu}\,.$$

Hint: This is a one-line proof if you use b) and c).