Exercises to "Standard Model of Particle Physics II"

Winter 2014/15

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Exercise 4: Goldstone theorem [**15 Points**]

The Goldstone–Theorem states: every generator of a continuous global symmetry, which does not annihilate the vacuum, corresponds to a massless scalar particle.
As "proof", consider the Lagrangian $\mathcal{L}(\Phi_i, \partial_\mu \Phi_i)$ with real scalar fields Φ_i, invariant under the global transformation $\vec{\Phi} \to \exp(i\theta^a T^a)\,\vec{\Phi}$.

a) Which properties have the iT^a, if the Φ_i are real and if $\Phi^T\Phi$ is invariant under the transformation?

b) Show that from the conservation of the Noether current it follows that:

$$\frac{\partial \mathcal{L}}{\partial \Phi_i}\,(iT^a)_{ij}\,\Phi_j + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi_i)}\,(iT^a)_{ij}\,\partial^\mu \Phi_j = 0.$$

c) Let $\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\Phi})^T\,(\partial^\mu \vec{\Phi}) - V(\vec{\Phi})$. Show with the help of the first two results that

$$\frac{\partial V}{\partial \Phi_i}\,(iT^a)_{ij}\,\Phi_j = 0.$$

d) Let $\vec{v}$ be the minimum of $V(\vec{\Phi})$. Show with the results from above that $M^2\,(iT^a)\,\vec{v} = 0$, where M^2_{ij} is given by

$$(M^2)_{ij} := \left.\frac{\partial^2 V}{\partial \Phi_i\,\partial \Phi_j}\right|_{\vec{\Phi}=\vec{v}}$$

e) The matrix M^2 is interpreted as a mass matrix. The broken vacuum $\vec{v}$ is in general not invariant under the transformations $\exp(i\theta^a T^a)$. If $T^a\vec{v} = 0$, one has a *Wigner–Weyl realization* of the symmetry, if $T^a\vec{v} \neq 0$ one has a realization à la *Nambu–Goldstone*. With the result from the last point it follows that for all a with $T^a\vec{v} \neq 0$ the mass matrix has an eigenvector with eigenvalue 0.
Consider e.g. the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\vec{\Phi})^T(\partial^\mu\vec{\Phi}) - V(\vec{\Phi}^T\vec{\Phi})$ with the potential $V(\vec{\Phi}^T\vec{\Phi}) = \frac{1}{2}\mu^2\vec{\Phi}^T\vec{\Phi} + \frac{1}{4}\lambda(\vec{\Phi}^T\vec{\Phi})^2$, where $\vec{\Phi}^T = (\phi_1, \phi_2, \phi_3)$ is a triplet of real and scalar particles. The coupling obeys $\lambda > 0$.

(i) Show that $\mathcal{L}$ is invariant under an $SU(2)$ transformation $\vec{\Phi} \to \exp(i\theta^a T^a)\,\vec{\Phi}$. The 3 generators T^a of $SU(2)$ are written here as $(T^a)_{ij} = -i\epsilon_{aij}$ (this is called the adjoint representation of $SU(2)$).

(ii) For $\mu^2 < 0$ the potential has a minimum for $\vec{v}^T = (0, 0, v)$. How are v, μ^2 and λ connected? Show that the vacuum state $\vec{v}$ is invariant under the transformation with T^3, but not with T^1 and T^2. That is, the vacuums state does *not* possess the symmetry of the Lagrangian (spontaneous symmetry breaking). Why do we speak in this case of spontaneous symmetry breaking from $SU(2)$ to $U(1)$?

(iii) The scalar fields will be expanded around the minimum: $\vec{\Phi}^T \equiv (\phi_1, \phi_2, v + \phi_3)$. Show by inserting into the above Lagrangian that there is one massive scalar field and 2 massless Goldstone bosons. What is the mass of ϕ_3? Convince yourself that the matrix M^2 from part d) is really the mass matrix.

Exercise 5: SU(N) **[5 Points]**

Let $U \in \mathrm{SU}(N)$, i.e. $\det U = 1$ and $U^\dagger U = \mathbb{1}$. Any element of SU(N) can be written as $U = \exp(-i\theta^a T^a)$, where the T^a are generators of the group with normalization $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

a) Show that the T^a are traceless hermitian matrices.

b) How many linear independent generators are there?

c) The structure constants, d_{abc} and f_{abc}, are defined through

$$[T^a, T^b] = i f_{abc} T^c, \qquad \{T^a, T^b\} = \frac{1}{N}\delta_{ab} + d_{abc} T^c \ .$$

Show that

$$\mathrm{Tr}(T^a T^b T^c) = \frac{1}{4}(d_{abc} + i f_{abc}) \ ,$$
$$\left[\sum_a T^a T^a, T^b\right] = 0 \ .$$

d) Calculate the f_{abc} for

$$T^a = \frac{\sigma_a}{2} \ ,$$

where σ_a are Pauli matrices.

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Tutorials homepage: `http://www.mpi-hd.mpg.de/manitop/StandardModel2/exercise.html`

Hand-in and discussion of sheet:
during tutorial on Thursday, 06.11.14, 9.15 am, kHs, Philosophenweg 12