

Exercises to “Standard Model of Particle Physics II”

Winter 2014/15

Dr. Werner Rodejohann

Sheet 2

29.10.14

Exercise 4: Goldstone theorem [15 Points]

The Goldstone–Theorem states: every generator of a continuous global symmetry, which does not annihilate the vacuum, corresponds to a massless scalar particle.

As “proof”, consider the Lagrangian $\mathcal{L}(\Phi_i, \partial_\mu \Phi_i)$ with real scalar fields Φ_i , invariant under the global transformation $\vec{\Phi} \rightarrow \exp(i\theta^a T^a) \vec{\Phi}$.

a) Which properties have the iT^a , if the Φ_i are real and if $\Phi^T \Phi$ is invariant under the transformation?

b) Show that from the conservation of the Noether current it follows that:

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} (iT^a)_{ij} \Phi_j + \frac{\partial \mathcal{L}}{\partial (\partial^\mu \Phi_i)} (iT^a)_{ij} \partial^\mu \Phi_j = 0.$$

c) Let $\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\Phi})^T (\partial^\mu \vec{\Phi}) - V(\vec{\Phi})$. Show with the help of the first two results that

$$\frac{\partial V}{\partial \Phi_i} (iT^a)_{ij} \Phi_j = 0.$$

d) Let \vec{v} be the minimum of $V(\vec{\Phi})$. Show with the results from above that $M^2 (iT^a) \vec{v} = 0$, where M_{ij}^2 is given by

$$(M^2)_{ij} := \left. \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} \right|_{\vec{\Phi}=\vec{v}}$$

e) The matrix M^2 is interpreted as a mass matrix. The broken vacuum \vec{v} is in general not invariant under the transformations $\exp(i\theta^a T^a)$. If $T^a \vec{v} = 0$, one has a *Wigner–Weyl realization* of the symmetry, if $T^a \vec{v} \neq 0$ one has a realization à la *Nambu–Goldstone*. With the result from the last point it follows that for all a with $T^a \vec{v} \neq 0$ the mass matrix has an eigenvector with eigenvalue 0.

Consider e.g. the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\Phi})^T (\partial^\mu \vec{\Phi}) - V(\vec{\Phi}^T \vec{\Phi})$ with the potential $V(\vec{\Phi}^T \vec{\Phi}) = \frac{1}{2}\mu^2 \vec{\Phi}^T \vec{\Phi} + \frac{1}{4}\lambda(\vec{\Phi}^T \vec{\Phi})^2$, where $\vec{\Phi}^T = (\phi_1, \phi_2, \phi_3)$ is a triplet of real and scalar particles. The coupling obeys $\lambda > 0$.

(i) Show that \mathcal{L} is invariant under an $SU(2)$ transformation $\vec{\Phi} \rightarrow \exp(i\theta^a T^a) \vec{\Phi}$. The 3 generators T^a of $SU(2)$ are written here as $(T^a)_{ij} = -i\epsilon_{aij}$ (this is called the adjoint representation of $SU(2)$).

(ii) For $\mu^2 < 0$ the potential has a minimum for $\vec{v}^T = (0, 0, v)$. How are v , μ^2 and λ connected? Show that the vacuum state \vec{v} is invariant under the transformation with T^3 , but not with T^1 and T^2 . That is, the vacuum state does *not* possess the symmetry of the Lagrangian (spontaneous symmetry breaking). Why do we speak in this case of spontaneous symmetry breaking from $SU(2)$ to $U(1)$?

(iii) The scalar fields will be expanded around the minimum: $\vec{\Phi}^T \equiv (\phi_1, \phi_2, v + \phi_3)$. Show by inserting into the above Lagrangian that there is one massive scalar field and 2 massless Goldstone bosons. What is the mass of ϕ_3 ? Convince yourself that the matrix M^2 from part d) is really the mass matrix.

Exercise 5: $SU(N)$ [5 Points]

Let $U \in SU(N)$, i.e. $\det U = 1$ and $U^\dagger U = \mathbb{1}$. Any element of $SU(N)$ can be written as $U = \exp(-i\theta^a T^a)$, where the T^a are generators of the group with normalization $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

- a) Show that the T^a are traceless hermitian matrices.
- b) How many linear independent generators are there?
- c) The structure constants, d_{abc} and f_{abc} , are defined through

$$[T^a, T^b] = i f_{abc} T^c, \quad \{T^a, T^b\} = \frac{1}{N} \delta_{ab} + d_{abc} T^c .$$

Show that

$$\begin{aligned} \text{Tr}(T^a T^b T^c) &= \frac{1}{4}(d_{abc} + i f_{abc}) , \\ \left[\sum_a T^a T^a, T^b \right] &= 0 . \end{aligned}$$

- d) Calculate the f_{abc} for

$$T^a = \frac{\sigma_a}{2} ,$$

where σ_a are Pauli matrices.

Tutor:

Pascal Humbert, email: `pascal.humbert@mpi-hd.mpg.de`

Tutorials homepage: <http://www.mpi-hd.mpg.de/manitop/StandardModel2/exercise.html>

Hand-in and discussion of sheet:

during tutorial on Thursday, 06.11.14, 9.15 am, kHs, Philosophenweg 12